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C_0 COARSE GEOMETRY

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Abstract

In this paper we introduce an alternative form of coarse geometry on proper metric spaces, which is more delicate at infinity than the bounded metric coarse structure. This new structure is called the C_0 coarse structure. This is a more refined form of coarse geometry on metric spaces. The coarse geometry of a proper coarse space X is studied analytically via the Roe algebra. This is an algebra of operators on a Hilbert space carrying a representation of $C_0(X)$; specifically, in the case of the C_0 coarse structure, it is the completion of the algebra of locally compact operators with propagation tending to zero at infinity.

Motivated by a construction of Roe for the usual bounded coarse structure, if X is a complete Riemannian manifold, then we define a 'higher index' for Dirac type operators on bundles over X. This index lies in the K-theory of the C_0 version of the Roe algebra. The first major result of the thesis is a vanishing theorem for the index. Assuming the manifold has an unbounded component, then for any Dirac-type operator with no essential spectrum we show that the C_0 index vanishes. We can relate this to positive scalar curvature. For the spinor Dirac operator on an open spin manifold, if the scalar curvature $\kappa(x)$ tends to infinity as x tends to infinity, then D has no essential spectrum, and we can apply the vanishing theorem. As a corollary to these results we show that if X is a compact spin manifold with a metric of uniformly positive scalar curvature, then there is a bound R such that every metric uniformly close to the given one has at least one point of scalar curvature less than R.

The coarse Baum-Connes conjecture can also be formulated in the C_0 context. We show that for any finite dimensional locally finite simplicial complex, with an appropriate metric, the C_0 coarse Baum-Connes conjecture holds. We then use this to reformulate the coarse Baum-Connes conjecture for a space W equipped with the *bounded* coarse structure, in terms of a forgetful functor, which coarsens from the C_0 structure to the bounded structure on some space X built out of W. This reformulation works for any bounded geometry space W, however given stronger hypotheses, namely that W also has finite asymptotic dimension, then we can use the reformulation to give a new proof of the theorem of Yu: the coarse Baum-Connes conjecture holds for spaces of finite asymptotic dimension.

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Chapter 1

Introduction

This thesis introduces a new coarse structure for metric spaces. The notion of a coarse (or bornologous) map between metric spaces was introduced in [18] and in [7] this was generalized to define an abstract coarse structure. In these terms, to a metric space there is a naturally associated abstract coarse structure, called the bounded metric coarse structure. Here we will associate a second coarse structure to a metric space — the C_0 coarse structure. This is a refinement of the bounded structure, and as such it is better suited to the study of existence problems for metrics satisfying certain scalar curvature conditions. Indeed the original motivation for introducing the C_0 structure was to investigate obstructions to *properly positive scalar curvature* on open manifolds. By definition the scalar curvature κ is properly positive, if it is a proper function having range in the interval $[\kappa_0, \infty)$ for some $\kappa_0 \in \mathbb{R}$. We will show that the K-theory group of the Roe algebra associated to the C_0 coarse structure is the natural receptor for a 'C_0 higher index,' and that this index, when non-zero, is an obstruction to properly positive scalar curvature.

This thesis also discusses another significant application of the C_0 coarse structure. As this coarse structure is more refined than the bounded metric structure, the K-theory groups mentioned above are more topological in nature, and correspondingly they are easier to compute than in the bounded case. For any abstract coarse structure there is a conjectural isomorphism between these K-theory groups and the coarse K-homology[1]. This is the coarse Baum-Connes conjecture. For any coarse structure the coarse K-homology is defined by a topological construction, and hence is fairly easy to compute. We are therefore able to show for a wide class of metric spaces that the coarse Baum-Connes conjecture holds for the C_0 coarse structure. This result applies in sufficient generality that the coarse K-homology for the *bounded* coarse structure can be identified with the K-theory of the Roe algebra for the C_0 coarse structure. Thus the version of the conjecture which is of most interest in applications — that is the coarse Baum-Connes conjecture for the bounded coarse structure — can be interpreted as a relation between the K-theory for the C_0 and bounded coarse structures. The thesis concludes by establishing that this relation is true, and hence the coarse Baum-Connes conjecture holds, in the case of spaces of finite asymptotic dimension. This gives a new proof of the results of [25].

In chapter 2 we will deal with the preliminaries. We begin by defining an abstract coarse space axiomatically. We can then say when two maps between coarse spaces are *coarse* maps, and when and two maps into a coarse space are *close*. We then proceed to define the specific class of coarse structures in which we will be interested, namely the C_0 *coarse structure* on a metric space. We will note that this satisfies the axioms for a coarse space. Throughout we will denote a metric space X equipped with the C_0 coarse structure by X_0, to distinguish from the bounded metric structure. Chapter 2 concludes with the definition of the *Roe algebra* C^*X for a coarse space X. For a Hilbert space $\mathfrak{H}$

[1]It should be noted that this conjecture is now known to be false for certain classes of examples.

carrying a representation of $C_0(X)$, the Roe algebra is the completion of the algebra of operators which are locally compact and whose support is controlled for the given coarse structure. We will show that the Roe algebra is covariantly functorial in a certain sense. To be precise its K-theory is a coarse invariant, and coarse maps induce homomorphism on the K-theory group.

The remainder of the thesis deals with two topics. Firstly it addresses the question of how these K-theory groups can be computed. Secondly if we know the K-theory groups, and if we can identify the elements in K-theory corresponding to certain operators, then we can make geometric deductions. In chapter 3 we will set up the framework for calculating the groups. We will introduce the two fundamental techniques for calculating them, namely the 'Eilenberg swindle,' and a coarse Mayer-Vietoris exact sequence. The former of these is used to make the first explicit calculations of the K-theory groups for the C_0 coarse structure, for example it can be used to show that the groups $K_*(C^*\mathbb{R}_0^+)$ vanish. We will also carry out a 'hands on' calculation of the groups in the case of a uniformly discrete space (by contrast with the bounded coarse structure, the C_0 coarse type of a uniformly discrete space depends only on its cardinality). Having made these direct calculations, we will see that the K-theory groups for a number of spaces can then be computed using the Mayer-Vietoris sequence. The framework set up in this chapter will be used in chapter 4 to calculate the C_0 higher index, and in chapter 5 to tackle the coarse Baum-Connes conjecture.

In chapter 4 we begin by briefly recapitulating the definition and properties of the bounded coarse index appearing in [19]. We then recall the definition and basic properties of a Dirac type operator. It is for such an operator that the C_0 coarse index will be defined. In fact the construction can be generalized to elliptic operators, however in the geometric applications it is the Dirac type operators that arise. For X a complete Riemannian manifold, and D a Dirac type operator on a bundle over X, we then proceed to define the C_0 *index*, $\mathrm{Index}\, D$, in the group $K_j(C^*X_0)$, where $j = 0$ if the bundle is graded and $j = 1$ if not. Having defined the index we turn to a discussion of metrics of properly positive scalar curvature. We observe that for X a spin manifold, and D the corresponding spinor Dirac operator, if X has properly positive scalar curvature then D has no essential spectrum, i.e D has discrete spectrum with finite dimensional eigenspaces. This spectral condition is the hypothesis we then use to study the K-theory element $\mathrm{Index}\, D$. Let us assume that D has no essential spectrum. We show that under this assumption, the index of D lies in the image of the group $K_j(\mathfrak{K})$ under the map induced by the inclusion $\mathfrak{K} \hookrightarrow C^*X_0$. Then in the ungraded case the index therefore vanishes as $K_1(\mathfrak{K}) = 0$, while in the graded case if X has a non-compact component then we show that the map $\mathbb{Z} = K_0(\mathfrak{K}) \to K_0(C^*X_0)$ factors through $K_0(C^*\mathbb{R}_0^+) = 0$ and again the index vanishes. We conclude chapter 4 by computing the index in some specific examples where it is non-zero. As a corollary we obtain the following result: Whenever X is a compact spin manifold equipped with a metric of uniformly positive scalar curvature, there is a bound R such that every metric uniformly close to the given one has at least one point of scalar curvature less than R.

In chapter 5 we study the coarse Baum-Connes conjecture. Many of the constructions involved in stating and proving the conjecture involve the study of metric simplicial complexes, and we begin by defining the *uniform spherical metric* on a simplicial complex, and discussing the geometry of this.

We then define the *coarse K-homology* $KX_*(X)$ of a coarse space X, and the assembly map $\mu\colon KX_*(X) \to K_*(C^*X)$. The *coarse Baum-Connes conjecture* asserts that μ is an isomorphism. We show that both $KX_*(X)$ and $K_*(C^*X)$ are covariant functors on the category of coarse spaces, and each has certain homological properties. Specifically the Mayer-Vietoris exact sequence mentioned above also holds for $KX_*(X)$, and both functors are invariant under *coarse homotopy equivalence.* Based on experience of homology uniqueness in the context of topology we therefore expect (in favourable circumstances) to be able to prove inductively that μ is an isomorphism for a space X by building it out of simpler pieces. For a finite dimensional simplicial complex equipped with a uniform spherical metric we show that these homological methods allow the reduction to the 0-dimensional case, that it the uniformly discrete case. In other words, to prove the C_0 coarse Baum-Connes conjecture for such a complex, it will suffice to prove it for uniformly discrete spaces. This however is straightforward in the C_0 case; as noted above the C_0 coarse type depends only on cardinality, and hence we give a single calculation to complete the proof of the coarse Baum-Connes conjecture for a simplicial complex with uniform spherical metric, equipped with the C_0 coarse structure.

However in many applications such as the Novikov conjecture it is not the C_0 version of the conjecture, but the bounded version that is of interest. We therefore turn our attention to applying the C_0 results to deduce cases of the conjecture in the bounded case. Given a discrete metric space W (any metric space with bounded structure is coarsely equivalent to some discrete space W) we will construct a new space X called the *coarsening space* of W built out of a sequence of simplicial complexes. We can compute the K-theory of C^*X_0 by an Eilenberg swindle, indeed the K-theory vanishes.

We show that there are ideals I_0 and I_h in the algebras C^*X_0 and C^*X respectively, for which $KX_*(W) \cong K_*(I_0)$ and $K_*(C^*W) \cong K_*(I_h)$. Moreover $I_0 \subseteq I_h$ and we show that identifying $KX_*(W)$ and $K_*(C^*W)$ with the K-theory of these ideals, the inclusion of I_0 into I_h induces the assembly map μ on K-theory. Finally we show that in the case where W has finite asymptotic dimension, there is another coarse structure on X, denoted X_h and called the *hybrid coarse structure*, such that C^*X_h is a subalgebra of C^*X containing I_h as an ideal. If W has finite asymptotic dimension then the space X can be constructed in such a way that the K-theory of C^*X_h vanishes by a further Eilenberg swindle. Hence the K-theory long exact sequence gives isomorphisms $K_{*+1}(C^*X_0/I_0) \cong K_*(I_0)$ and $K_{*+1}(C^*X_h/I_h) \cong K_*(I_h)$. Again with the assumption of finite asymptotic dimension, the space X may be built out of subsets Y such that for J_0 and J_h the corresponding ideals of C^*Y_0, C^*Y_h, the algebras C^*Y_0/J_0 and C^*Y_h/J_h are isomorphic. Using a further homology uniqueness argument in this context we show that $K_*(C^*X_0/I_0) \cong K_*(C^*X_h/I_h)$. Hence we obtain a new proof of the theorem of Yu (see [25]): for W with finite asymptotic dimension the assembly map μ is an isomorphism, i.e the coarse Baum-Connes conjecture holds for W.

Chapter 2

Coarse Geometry & Controlled Operators

2.1 Abstract coarse structures

An abstract coarse structure on a set is defined either by a collection of *entourages*, or by a *closeness* relation on maps into the set. Abstract coarse structures were first introduced in [7]. A treatment based on the notion of closeness is developed in [9]. We shall take the former approach here. We shall deal with spaces equipped with a topology and the coarse structure will satisfy conditions of compatibility with this. Usually the spaces will even be equipped with a metric, and the topology and coarse structure will both derive from this; these concrete examples will appear in the following section.

Let X be a set, and let $\mathcal{E}$ be a collection of subsets of $X \times X$. Members of $\mathcal{E}$ will be called *entourages* and are said to be *$\mathcal{E}$-controlled*, or, and when the collection $\mathcal{E}$ is understood we will just say they are *controlled.*

Definition 2.1. A collection $\mathcal{E}$ of entourages defines a *unital coarse structure* on a set X if it satisfies the following axioms:

1. The diagonal of $X \times X$ is controlled.

2. If A and B are controlled then $A \cup B$ is controlled.

3. If A is controlled and B is a subset of A then B is controlled.

4. The union of all controlled sets is $X \times X$. Equivalently every singleton $\{(x, y)\}$ is controlled.

5. Transposition: If A is controlled then

$$A^T = \{(y, x) \in X \times X \mid (x, y) \in A\}$$

is controlled.

6. Composition: If A and B are controlled then

$$A \circ B = \{(x, z) \in X \times X \mid \exists y \in X \text{ such that } (x, y) \in A, (y, z) \in B\}$$

is controlled.

For X a Hausdorff topological space the coarse structure is *proper* if additionally:

7. There is a controlled open neighbourhood of the diagonal of $X \times X$.

8. If $K \subseteq X$ is compact and A is controlled then

$$\{x \in X \mid \exists y \in K \text{ with } (x, y) \in A \text{ or } (y, x) \in A\}$$

is relatively compact.

An equivalent definition of a proper structure is established in lemma 2.5 below.

Definition 2.2. A coarse structure $\mathcal{E}'$ *coarsens* a structure $\mathcal{E}$ if every $\mathcal{E}$ controlled set is $\mathcal{E}'$-controlled.

If $\mathcal{E}, \mathcal{E}'$ are coarse structures, and each structure coarsens the other then they are equal.

Definition 2.3. Given any collection $\mathcal{C}$ of subsets of $X \times X$, the *coarse structure* $\mathcal{E}$ *generated by* $\mathcal{C}$ is the minimal coarse structure containing $\mathcal{C}$. In other words $\mathcal{E}$ is the collection of all sets A, such that A is controlled for every coarse structure containing $\mathcal{C}$.

The following definition describes coarsely the familiar metric properties of boundedness and uniform boundedness. One view of an abstract coarse space is as a space equipped with a notion of uniform boundedness.

Definition 2.4. A subset A of a coarse space X is *bounded* if $A \times A$ is controlled. A collection $\mathcal{C}$ of subsets of X is *uniformly bounded* if there is a controlled set which contains $A \times A$ for each A in $\mathcal{C}$.

Lemma 2.5. *Let X be a Hausdorff topological space equipped with a coarse structure. This latter is proper if and only if it is generated by open sets, and is such that the properties of boundedness and relative compactness coincide.*

Note that a metric space is proper if and only if the properties of relative compactness and *metric* boundedness agree.

Proof. Suppose $\mathcal{E}$ is a coarse structure generated by open sets. Then every controlled set is a subset of a controlled open set. As the diagonal is controlled, so is an open neighbourhood of this, i.e. axiom 7 holds. Conversely for a coarse structure $\mathcal{E}$ satisfying axiom 7, let U be the given controlled neighbourhood of the diagonal. Consider the collection $\mathcal{C}$ of sets produced by composing the elements of $\mathcal{E}$ by U on both sides. Certainly these compositions are open. As $U \circ A \circ U$ contains A for each A in $\mathcal{E}$, the structure generated by $\mathcal{C}$ coarsens $\mathcal{E}$. On the other hand $U \circ A \circ U$ is always $\mathcal{E}$-controlled by the composition axiom. Thus the coarse structure generated by $\mathcal{C}$ is $\mathcal{E}$.

We will now assume that the coarse structure is generated by open sets, and we must show under this assumption that axiom 8 is equivalent to the condition on boundedness. Suppose that X has a proper coarse structure i.e. axiom 8 holds. All compact sets are bounded as the generating sets form an open cover of $X \times X$ hence a finite number will cover a compact rectangle. A relatively compact set lies within a compact and bounded set so is bounded. On the other hand if K is bounded then $K \times K$ is controlled. Applying axiom 8 to the compact set $\{x\}$, for some $x \in K$ shows that K is relatively compact.

Conversely suppose that relatively compact sets and bounded sets coincide. Let A be controlled and K compact. By the transposition and union axioms we may assume that A is symmetric i.e. $(x, y) \in A \iff (y, x) \in A$. Then we need to show that $K' = \{x \in X \mid \exists y \in K, (x, y) \in A\}$ is relatively compact. Let $A_K = A \cap (X \times K)$. The composition $A_K \circ (K \times K) \circ A_K^T$ gives $K' \times K'$. As A is controlled and K compact and so bounded by assumption, each term in the composition is controlled. Thus by the composition axiom $K' \times K'$ is controlled, i.e. K' is bounded. Thus K' is relatively compact as required and axiom 8 holds. □

Definition 2.6. A coarse space is *separable* if there exists a countable uniformly bounded cover.

Remark 2.7. For a proper space this is equivalent to the existence of a countable uniformly bounded *open* cover, and if additionally the underlying topology is metrizable then it is equivalent to the topological definition of separability.

To give a full description of the coarse category we must also describe the morphisms. These are 'proper' maps taking controlled sets to controlled sets.

Definition 2.8. Let X, Y be equipped with coarse structures. A map $\alpha\colon X \to Y$ is a *coarse map* if:

1. For each controlled $A \subseteq X \times X$ the image $(\alpha \times \alpha)(A)$ is controlled for Y.

2. For each bounded $K \subseteq Y$ the pre-image $\alpha^{-1}(K)$ is bounded in X.

It is immediate from the definition that a composition of coarse maps is coarse.

In terms of uniform boundedness, the first condition asserts that the image of any uniformly bounded collection of subsets of X is uniformly bounded. For both X and Y proper coarse spaces the second condition is that the pre-image of a relatively compact set is relatively compact. Thus if the map is also continuous then it is topologically proper.

For a single set equipped with two coarse structures $\mathcal{E}, \mathcal{E}'$, the identity map from $(X, \mathcal{E})$ to $(X, \mathcal{E}')$ is a coarse map if and only if $\mathcal{E}'$ coarsens $\mathcal{E}$ and the notions of boundedness coincide.

The following definition relates the entourage description of coarse geometry to the closeness description.

Definition 2.9. Let S be any set, and X a coarse space. Let α, β be maps from S to X. Then α and β are *close* if the image $\{(\alpha(s), \beta(s)) \mid s \in S\}$ of (α, β) is controlled.

From the composition axiom we can make the following deductions. Composing a pair of close maps with a coarse map, produces close maps. Similarly for α a coarse map and β close to α, it follows that β maps controlled sets to controlled sets. For such α, β, and for K bounded, consider $K' = \beta^{-1}(K)$. Then $\beta(K')$ is bounded, and again by composition, so also is $\alpha(K')$. But then $\alpha^{-1}\alpha(K')$ is bounded, and this contains K', so the preimage of a bounded set under β is bounded. Thus every map close to a coarse map is also coarse.

It is sometimes convenient to restrict to consideration of Borel coarse maps. The following lemma allows us to do this.

Lemma 2.10. *Let X, Y be a topological spaces equipped with coarse structures. Suppose X is proper and separable. Then every map $X \to Y$ is close to a Borel map. If the given map is coarse then the Borel map will be coarse as well.*

Proof. Let α be a map from X to Y. Let $\{U_i\}$ be a countable uniformly bounded open cover of X, and pick x_i in U_i for each i. Define $\gamma\colon X \to X$ by $\gamma(U_i \setminus \bigcup_{i'<i} U_{i'}) = \{x_i\}$. Let $\beta = \alpha \circ \gamma$, which is Borel as the preimage of any set is a countable union of sets of the form $U_i \setminus \bigcup_{i'<i} U_{i'}$. As $\{U_i\}$ is uniformly bounded γ is close to the identity, and hence β is close to α as required. As β is close to α, if α is coarse then so is β. □

We are now in a position to define coarse equivalence of a pair of spaces.

Definition 2.11. Let X, Y be equipped with coarse structures. Then X and Y are *coarsely equivalent* if there exist coarse maps $\alpha\colon X \to Y$ and $\beta\colon Y \to X$ such that $\beta \circ \alpha$ is close to the identity on X and $\alpha \circ \beta$ is close to the identity on Y. The maps α, β are called *coarse equivalences*.

Later we will define an algebra of operators associated to a coarse space. This will be functorial at the level of K-theory, i.e. $X \mapsto K_0(C^*X), K_1(C^*X)$ will define functors. It will then follow from functoriality that a coarse equivalence of spaces provides natural isomorphisms of these these groups.

2.2 The C_0 coarse structure

The main objects of study here are coarse structures that derive from a metric. The following definition describes the unique coarse structure such that the metric and coarse definitions of uniform boundedness coincide.

Definition 2.12. Let (X, d) be a metric space. The *standard coarse structure on* X is

$$\mathcal{E} = \{A \subseteq X \times X \mid \text{there exists } R \in \mathbb{R}_+ \text{ such that } A \subseteq U_R\}$$

where

$$U_R = \{(x, y) \in X \times X \mid d(x, y) < R\}.$$

To measure 'propagation near infinity' we use an alternative coarse structure. Roughly speaking a set will be controlled if it lies inside a 'pinched tube' about the diagonal, having width tending to zero at infinity. We replace the constant R of the preceding definition with a function R of t, where t is a parameter measuring the distance from a base-point x_0 in X. The choice of base-point will not affect the structure. Instead of using a distance parameter t and a function R of the distance, we could alternatively use a function r on $X \times X$.

Definition 2.13. Let (X, d) be a metric space. The C_0 *coarse structure* on X is

$$\mathcal{E} = \{A \subseteq X \times X \mid \text{there exists } R \in C_0(\mathbb{R}_+), R(t) > 0 \text{ such that } A \subseteq U_R\}$$

where

$$U_R = \{(x, y) \in X \times X \mid d(x, y) < R(d(x, x_0) + d(y, x_0))\}.$$

We denote the space X equipped with this coarse structure by X_0 to distinguish from the standard coarse structure.

Definition 2.14. The *proper* C_0 *coarse structure* on X is defined to be:

$$\mathcal{E} = \{A \subseteq X \times X \mid \exists r \in C_0(X \times X),\ r(x, y) = r(y, x) \geq 0 \text{ such that } A \subseteq U_r\}$$

where

$$U_r = \{(x, y) \in X \times X \mid d(x, y) < r(x, y)\}.$$

With this structure we denote the space by X_0^P.

Lemma 2.17 below establishes that the C_0, and proper C_0 structures will coincide for *proper* metric spaces.

Note that replacing C_0 by C_b in either of the above definitions would produce the standard coarse structure. The standard structure coarsens the C_0 structure. A collection of subsets of X all of the same diameter is uniformly bounded in the standard coarse structure, but will be C_0 uniformly bounded only if they all lie within some metric ball. Indeed for a collection of subsets to be C_0 uniformly bounded, for each $\varepsilon > 0$ the subsets of diameter greater than ε must all lie within some metric ball.

These structures may be described in terms of closeness as follows. Let S be a set, and let α, β be maps from S to X. For the standard coarse structure these are close if the uniform distance $\sup_{s\in S} d(\alpha(s), \beta(s))$ between the maps is finite. For the C_0 structure we additionally require that the distance tends to zero as $\alpha(s), \beta(s)$ tend to infinity. More precisely we require that for all $\varepsilon > 0$ the set of pairs $(\alpha(s), \beta(s))$ with $d(\alpha(s), \beta(s)) > \varepsilon$ is metrically bounded in the case of the C_0 structure, and is relatively compact in that case of the proper C_0 structure.

Definition 2.15. Two metrics d_1, d_2 on a space X are *coarsely equivalent* if the identity map from (X, d_1) to (X, d_2) is a coarse equivalence for the bounded coarse structure. Two metrics d_1, d_2 on X are C_0*-coarsely equivalent* if the identity map from (X, d_1) to (X, d_2) is a coarse equivalence for the C_0 coarse structure.

Lemma 2.16. *The proper C_0 structure is proper for any locally compact metric space (X, d).*

Proof. If K is a compact subset of X then certainly it is metrically bounded so to be coarsely bounded for the proper C_0 structure we need only have a C_0 function on $X \times X$ which is greater than this bound on $K \times K$. This will exist by Urysohn's lemma.

Conversely suppose that K is coarsely bounded for the proper C_0 structure. Then there exists $r \in C_0(X \times X)$ with $d(x, y) < r(x, y)$ for x, y in K. Choose $\epsilon > 0$ less than the diameter of K (if K has diameter 0 it is trivially compact). Consider $\{(x, y) \mid r(x, y) > \epsilon\}$. This is open and has non-empty intersection with $K \times K$. It is relatively compact as r

is C_0, so it follows that there exists x_0 in K and a relatively compact open ball in X about x_0. Then for δ sufficiently small, K lies in:

$$\{x \mid d(x, x_0) < \delta\} \cup \{x \mid r(x, x_0) \geq \delta\}.$$

This is relatively compact, and so K is relatively compact as required. □

In fact it is always true that all coarsely bounded sets of X_0^P are relatively compact. The requirement of local compactness in the lemma is to ensure that that all compact sets are coarsely bounded.

For the C_0 structure the notion of boundedness coincides with the metric definition[1], hence by lemma 2.5 this structure is proper if and only if the metric is proper. (The same is true for the standard coarse structure.) As X_0^P is always proper, for it to agree with X_0 it is necessary that the metric is proper. The following lemma shows that this also sufficient.

Lemma 2.17. *Let (X, d) be a metric space. Let $\mathcal{E}$ denote the C_0 coarse structure, and let $\mathcal{E}^P$ be the proper C_0 structure, as defined in 2.13.*

If the metric is proper then $\mathcal{E}$ is equivalent to the coarse structure $\mathcal{E}^P$. In general $\mathcal{E}$ coarsens $\mathcal{E}^P$.

Proof. Given $R > 0$ in $C_0(\mathbb{R}_+)$ let $r(x, y) = R(d(x, x_0) + d(y, y_0))$. For all $k > 0$, $K = \{(x, y) \mid d(x, x_0) + d(y, x_0) \leq k\}$ is closed and bounded, hence is compact if the

[1]Metric boundedness of a set K gives a diameter bound as required by the standard coarse structure, and also provides a bound on the parameter $d(x, x_0) + d(y, x_0)$ so we may take a C_0 (or even C_c) function R controlling $K \times X$.

metric is proper. Thus $d(x, x_0) + d(y, x_0) \to \infty$ and hence $r(x, y) \to 0$ as $(x, y) \to \infty$. For $r(x, y)$ so defined, r is symmetric, non-negative, and lies in $C_0(X \times X)$, so in the proper case every $\mathcal{E}$ entourage is also an $\mathcal{E}^P$ entourage.

Conversely given $U_r \in \mathcal{E}^P$ as in definition 2.13, define R_1 by

$$R_1(t) = \sup\{r(x, y) \mid d(x, x_0) + d(y, x_0) \geq t\}.$$

As r is C_0 it is bounded and so also is R_1. Further, for each $\epsilon > 0$ there exists a compact subset K of X with $r < \epsilon$ outside $K \times K$. Let $t_0 = 2 \sup\{d(x, x_0) \mid x \in K\}$, which is finite by compactness. For $t > t_0$ it follows that

$$R_1(t) \leq \sup\{r(x, y) \mid (x, y) \notin K \times K\} \leq \epsilon.$$

Thus $R_1(t) \to 0$ as $t \to \infty$. (Note that we did not require the metric to be proper in this argument.) As R_1 is bounded and tends to zero at infinity, we may find $R \geq R_1$, *continuous* and tending to zero at infinity. Then

$$U_r \subseteq \{(x, y) \in X \times X \mid d(x, y) < R(d(x, x_0) + d(y, x_0))\} \in \mathcal{E}.$$

Thus $\mathcal{E}$ always coarsens $\mathcal{E}^P$, and so the two are equivalent when the metric is proper. □

The following lemma gives examples of C_0 coarse maps. It also illustrates that the C_0 structure is much more closely related to the underlying topology than the bounded structure is.

Lemma 2.18. *Let X, Y be proper metric spaces, and let $\alpha\colon X \to Y$ be (topologically) proper and uniformly continuous. Then $\alpha\colon X \to Y$, is C_0 coarse.*

Proof. As α is continuous and topologically proper, it is coarsely proper. Thus it suffices to show for $A \subseteq X \times X$, C_0-controlled, that $B = \alpha \times \alpha(A)$ is again C_0-controlled. By uniform continuity, for all $\varepsilon > 0$ there exists $\delta > 0$, such that $d(\alpha(x), \alpha(x')) < \varepsilon$ whenever $d(x, x') < \delta$. The condition on A implies that there exists $K \subseteq X \times X$ compact, such that if $(x, x') \in A \setminus K$ then we have $d(x, x') < \delta$, so $d(\alpha(x), \alpha(x')) < \varepsilon$. Hence B lie in the union of $\alpha \times \alpha(K)$, which is compact, with the set of pairs (y, y') with $d(y, y') < \varepsilon$. As ε can be chosen arbitrarily small B is C_0-controlled as required. ☐

For a sequence of metric spaces we generalize the C_0 coarse structure to the disjoint union.

Definition 2.19. Let $X = \bigsqcup X_n$ be a disjoint union of proper metric spaces. The C_0 *disjoint union* structure is given by entourages which are the union of a relatively compact subset of $X \times X$, and a sequence of C_0 controlled subsets A_n of X_n, with $\sup\{d(x, y) \mid (x, y) \in A_n\}$ tending to zero as $n \to \infty$.

Note that if we also have a proper metric on the disjoint union which agrees with the metrics on each X_n, then the C_0 structure obtained from this metric is the same as the C_0 disjoint union structure. As a simple example if each X_n is the closed unit interval with the usual metric, then the structure on the disjoint union is the same as the C_0 structure on $\bigcup_n [2n, 2n+1]$ obtained from the usual metric.

We will be particularly interested in the C_0 structure on Riemannian manifolds, and we make the convention that if the manifold is not connected, then the C_0 structure on this is the C_0 disjoint union.

2.3 Representations and the Roe algebra

In this section we define the coarse C^*-algebra or Roe algebra of a coarse space. For the metric coarse structure this is discussed in [19], and for abstract coarse structures it was introduced in [7]. We will establish functoriality at the level of K-theory. Throughout this section X, Y will denote separable locally compact Hausdorff topological spaces equipped with coarse structures.

Definition 2.20. Let A be a C^*-algebra, $\mathfrak{H}$ a separable Hilbert space, and $\rho\colon A \to \mathcal{B}(\mathfrak{H})$ a faithful non-degenerate representation. That is ρ is an injective *-homomorphism from A to $\mathcal{B}(\mathfrak{H})$ such that $\rho[A]\mathfrak{H}$ is a dense subspace of $\mathfrak{H}$. Then ρ is *ample* if $\rho[A] \cap \mathfrak{K}(\mathfrak{H}) = \{0\}$, i.e. if the composition of ρ with the projection to the Calkin algebra $\mathcal{Q}(\mathfrak{H})$ is also injective.

We will be interested in ample representations of the C^*-algebra $C_0(X)$. For X without isolated points, equipped with a reasonable measure (specifically a measure assigning positive mass to all non-empty open sets) the standard example is $\mathfrak{H} = L^2(X)$ and ρ the representation by pointwise multiplication.

Definition 2.21. Suppose a Hilbert space $\mathfrak{H}$ and an ample representation $\rho\colon C_0(X) \to \mathcal{B}(\mathfrak{H})$ are given. Let T be an operator in $\mathcal{B}(\mathfrak{H})$. We will define the *support* of T with respect to the given representation, and denote this by $\operatorname{Supp} T$.

Let U, V be open subsets of X. The support of T fails to meet $U \times V$ if $\rho(f)T\rho(g) = 0$ for every pair $f \in C_0(U)$ and $g \in C_0(V)$. By definition $X \times X \setminus \operatorname{Supp} T$ is the union of all such open rectangles $U \times V$.

Taking complements of the above, a point $(x, y) \in X \times X$ lies in the support of T if and only if for every open rectangle $U \times V$ about (x, y) there exist $f \in C_0(U)$, $g \in C_0(V)$ with $\rho(f)T\rho(g) \neq 0$.

There is an obvious generalization to an operator $T\colon \mathfrak{H}_X \to \mathfrak{H}_Y$ where there are representations of $C_0(X), C_0(Y)$ on $\mathfrak{H}_X, \mathfrak{H}_Y$. We repeat the above with U an open subset of Y, and V open in X.

In the example of X equipped with a measure, consider T an integral operator $T\colon h \mapsto \int_{x \in X} k(\,\cdot\,, x)h(x)$ on $L^2(X)$. In this case the definition of support reduces to the usual definition of the support of the kernel k.

Note that by the Borel calculus, a representation of $C_0(X)$ may be extended to a representation of the C^*-algebra of bounded Borel functions. The complement of the support will then contain an open rectangle $U \times V$ if and only if $\rho(f)T\rho(g)$ vanishes for f, g the characteristic functions of U, V. Vanishing of $\rho(f)T\rho(g)$ is clearly sufficient; to prove necessity we observe that as f is the characteristic function of an open set, it is a monotone limit of continuous functions f_n supported in U, and similarly $g = \lim g_m$. In the Borel calculus $\rho(f), \rho(g)$ are then the weak limits of the sequences $\rho(f_n)$ and $\rho(g_m)$ respectively, so $\rho(f)T\rho(g)$ is the weak limit of operators $\rho(f_n)T\rho(g_m)$, which vanish if $U \times V$ does not meet the support. This shows necessity.

Lemma 2.22. *The support of ST is contained in the composition* $\operatorname{Supp}(S) \circ \operatorname{Supp}(T)$.

Proof. We want to show that if (x, y) is not in $A = \operatorname{Supp}(S) \circ \operatorname{Supp}(T)$ then it is not in $\operatorname{Supp}(ST)$. As X is assumed to be locally compact it will suffice to show that if U, V are relatively compact open sets such that $\overline{U} \times \overline{V}$ does not meet A then $U \times V$ does not meet $\operatorname{Supp}(ST)$. Let $W_1 = \{x \in X \mid \text{there exists } y \in \overline{U} \text{ with } (y, x) \in \operatorname{Supp}(S)\}$ and let $W_2 = \{x \in X \mid \text{there exists } y \in \overline{V} \text{ with } (x, y) \in \operatorname{Supp}(T)\}$. Compactness of $\overline{U}, \overline{V}$ implies that W_1, W_2 are closed, hence we get $\rho(\chi_{X \setminus W_2}) T \rho(\chi_V) = 0$ and $\rho(\chi_U) S \rho(\chi_{X \setminus W_1}) = 0$ by the above observations about the Borel calculus. If $\overline{U} \times \overline{V}$ does not meet A then W_1, W_2 are disjoint, thus we see that $T\rho(\chi_V) = \rho(\chi_{W_2}) T \rho(\chi_V) = \rho(\chi_{X \setminus W_1}) T \rho(\chi_V)$. We conclude that $\rho(\chi_U) ST \rho(\chi_V) = \rho(\chi_U) S \rho(\chi_{X \setminus W_1}) T \rho(\chi_V) = 0$, and as required, $U \times V$ does not meet $\operatorname{Supp}(ST)$. ☐

Lemma 2.23. *If T_n is a sequence of operators on $\mathfrak{H}$ with $T_n \to T$ weakly, then* $\operatorname{Supp}(T) \subseteq \overline{A}$ *where* $A = \operatorname{LimSup}_n \operatorname{Supp}(T_n)$.

Proof. Let $A = \operatorname{LimSup}_n \operatorname{Supp}(T_n)$. We need to show that if $(x, y) \notin \overline{A}$ then $(x, y) \notin \operatorname{Supp}(T)$. Thus suppose $(x, y) \notin \overline{A}$. In other words, suppose there is an open set $U \times V$ containing (x, y), such that for some n_0, the intersection $U \times V \cap \operatorname{Supp}(T_n)$ is empty for all $n > n_0$. For any functions f, g supported respectively in U, V the product fT_ng must vanish for all $n > n_0$. Thus $\langle fTgv, w\rangle = \lim_n \langle fT_ngv, w\rangle = 0$ for all f, g supported in U, V, and for all $v, w \in \mathfrak{H}$. Hence $fTg = 0$ for all f, g supported in U, V, so $U \times V$ does not meet the support of T. we conclude that $(x, y) \notin \operatorname{Supp}(T)$ as required. ☐

Definition 2.24. For X a metric space, the propagation of an operator is defined to be $\operatorname{Prop}(T) = \sup\{d(x, y) \mid (x, y) \in \operatorname{Supp}(T)\}$.

Definition 2.25. Let T be an operator in $\mathcal{B}(\mathfrak{H})$. If $\operatorname{Supp} T$ is controlled then we say T is *controlled.* If $\rho(f)T$ and $T\rho(f)$ are compact for all $f \in C_0(X)$, then T is *locally compact.* If the commutator $[\rho(f), T]$ is compact for all f in $C_0(X)$ then we say T is *pseudolocal.*

The following standard result is stated in [11]. For a proof see [9].

Lemma 2.26 (Kasparov's Lemma). *An operator T in $\mathcal{B}(\mathfrak{H})$ is pseudolocal if and only if $\rho(f)T\rho(g)$ is compact for all disjointly supported pairs of functions f, g in $C(X^+)$.*

□

Definition 2.27. The *Roe algebra* of a coarse space X, denoted $C^*_\rho X$, is the norm closure in $\mathcal{B}(\mathfrak{H})$ of the algebra of controlled and locally compact operators. Analogously we define an algebra $D^*_\rho X$ as the norm closure of the controlled and pseudolocal operators.

Remark 2.28. The collection of controlled operators is a *-subalgebra of $\mathcal{B}(\mathfrak{H})$. It is closed under composition by the composition axiom for a coarse structure, and is closed under the involution by the transposition axiom. The controlled and pseudolocal operators form a *-subalgebra of this which in turn contains the controlled and locally compact operators. These last will be a *-ideal in the controlled operators if the coarse structure is proper - indeed axiom 7 suffices. To show that an operator is locally compact it suffices to consider multiplication by C_c functions as these are dense. If T is controlled and $f \in C_c(X)$ then $T\rho(f)$ is compactly supported so equals $\rho(g)T\rho(f)$ for a bump function g in $C_c(X)$. It follows that for S locally compact $ST\rho(f) = S\rho(g)T\rho(f)$ is compact. Compactness of $\rho(f)ST$ holds for any T, so we have a right ideal. Closure under involution is clear hence it is a *-ideal. Without the additional assumption of properness, the controlled and locally compact operators form a *-ideal in the controlled and pseudolocal operators.

Example 2.29. Let X be a point. Then an ample representation of $C_0(X) = \mathbb{C}$ is a separable infinite dimensional Hilbert space $\mathfrak{H}$ with the representation by scalar multiplication. All operators on this are controlled (as every subset of $X \times X$ is controlled), and the locally compact operators are just compact operators. Thus C^*X is just the algebra of compact operators. Indeed for any compact topological space equipped with its unique proper coarse structure the same argument will apply as the constant functions in $C_0(X)$ must act by scalar multiplication.

Lemma 2.30. *For any proper coarse space X, the compact operators form an ideal in $C^*_\rho X$.*

Proof. We will show that every compact operator is the limit of a sequence of compact operators each of which is compactly supported. Let f_λ in $C_c(X)$ be an approximate unit for $C_0(X)$. Then $\rho(f_\lambda)$ converges strongly to the identity as ρ is nondegenerate. Hence for any compact operator K, the net $\rho(f_\lambda)K$ converges to K in norm. For fixed λ, we also have $\lim_\mu \rho(f_\mu)K\rho(f_\lambda) = K\rho(f_\lambda)$ and so we may approximate K in norm by $\rho(f_\mu)K\rho(f_\lambda)$ which is compact and compactly supported. □

The Roe algebra apparently depends on the choice of the representation ρ. Indeed for a given coarse space, though the Roe algebras obtained from different ample representations are isomorphic there is no canonical choice of isomorphism. However for proper spaces there is a natural isomorphism at the level of K-theory.

Definition 2.31. Let $\alpha\colon X \to Y$ be a coarse map. Let X, Y be equipped with ample representations $\rho_X\colon C_0(X) \to \mathfrak{H}_X$, and $\rho_Y\colon C_0(Y) \to \mathfrak{H}_Y$. A bounded operator $V\colon \mathfrak{H}_X \to \mathfrak{H}_Y$ *covers* the map α, if $\{(y, \alpha(x)) \mid (y, x) \in \operatorname{Supp} V\}$ is controlled.

This provides a notion of controlled operators between $\mathfrak{H}_X, \mathfrak{H}_Y$. Indeed an operator covering α is precisely an operator which is controlled for the coarse mapping space defined below. The following construction also appears in [21].

We will define a coarse mapping space $X \cup_\alpha Y$. Topologically we take this to be the disjoint union, which is locally compact for X, Y locally compact. Define a map $\tilde{\alpha}\colon X \cup_\alpha Y \to Y$ by $\tilde{\alpha}(x) = \alpha(x)$ for $x \in X$, and $\tilde{\alpha}(y) = y$ for $y \in Y$. We then define the controlled sets for $X \cup_\alpha Y$ to be those subsets A of $X \cup_\alpha Y \times X \cup_\alpha Y$ such that $(\tilde{\alpha} \times \tilde{\alpha})(A)$ is controlled for Y. We take the direct sum representation $\rho = \rho_X \oplus \rho_Y$ on $\mathfrak{H} = \mathfrak{H}_X \oplus \mathfrak{H}_Y$. Then V covers α if and only if the operator $\begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}$ is controlled.

By construction, the inclusion of X into the mapping space and the map $\tilde{\alpha}$ from the mapping space to Y are both coarse. The composition of $\tilde{\alpha}$ with the inclusion of Y into $X \cup_\alpha Y$ is close to the identity on the mapping space, while the inclusion followed by $\tilde{\alpha}$ is the identity on Y. Hence the mapping space is coarsely equivalent to Y. The idea of the construction is to be able to take a controlled operator on X, and to regard it as a controlled operator on the mapping space which is coarsely equivalent to Y.

Note that for $\beta\colon X \to Y$ close to α, the α and β mapping spaces are coarsely equivalent, specifically the identity map $X \cup_\alpha Y \to X \cup_\beta Y$ is a coarse equivalence. The construction is also compatible with compositions. For $\alpha\colon X \to Y$, and $\beta\colon Y \to Z$ coarse maps, the obvious inclusions from $X \cup_\alpha Y$, $Y \cup_\beta Z$, and $X \cup_{\beta\circ\alpha} Z$, into $X \cup_\alpha Y \cup_\beta Z$ are coarse, and the latter two are coarse equivalences.

Lemma 2.32. *Suppose X, Y have proper coarse structures, and α, V as above. Then Ad_V maps $C^*_{\rho_X}X$ into $C^*_{\rho_Y}Y$.*

Proof. Consider $\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Ad}_V(T) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & V^* \\ 0 & 0 \end{pmatrix}$ in $\mathcal{B}(\mathfrak{H}_X \oplus \mathfrak{H}_Y)$. Without loss of generality we may assume T is in the dense subalgebra of controlled operators. Then $T \oplus 0$ is controlled and as V covers α, it follows that $0 \oplus \mathrm{Ad}_V(T)$ is controlled for $X \cup_\alpha Y$. Thus $\mathrm{Ad}_V(T)$ is controlled for Y.

If X and Y have proper coarse structures we must show $\mathrm{Ad}_V(T)$ to be locally compact. It suffices to show this for $0 \oplus \mathrm{Ad}_V(T)$. The mapping space $X \cup_\alpha Y$ also satisfies axiom 7,[2] so the locally compact operators form an ideal in the controlled operators by remark 2.28. As $T \oplus 0$ is locally compact and $\begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}$ is controlled, $0 \oplus \mathrm{Ad}_V(T)$ is locally compact as required.

Thus for proper spaces we have established that Ad_V maps $C^*_{\rho_X}X$ into $C^*_{\rho_Y}Y$. ☐

We have shown that given a covering isometry we obtain a map of the coarse C^*-algebras. Indeed this is a $*$-homomorphism, so induces a map on K-theory. We will establish that there always exist isometries covering a given coarse map, and further that the map on K-theory is independent of the choices made. In fact we may state this independence as the fact that operators T and $\mathrm{Ad}_V(T)$ regarded as elements of

[2] This follows as both image and pre-image of a bounded set are bounded for the coarse map α, so boundedness in the mapping space again coincides with relative compactness.

$C^*(X \cup_\alpha Y)$ define the same element of K-theory. For existence we need to find a controlled operator $\tilde{V}$ in $\mathcal{B}(\mathfrak{H})$, such that $\tilde{V}^*\tilde{V}$ is the projection onto $\mathfrak{H}_X$, and $\tilde{V}\tilde{V}^*$ is a projection with range in $\mathfrak{H}_Y$.

Proposition 2.33. *Suppose X, Y have proper separable coarse structures, and α is a coarse map between them. Then there exists an isometry from $\mathfrak{H}_X$ to $\mathfrak{H}_Y$ covering α. Moreover for any given uniformly bounded open cover $\{U_i\}$ of Y, there is a covering isometry with support contained in $\bigcup_i \tilde{\alpha}^{-1}(\overline{U_i}) \times \tilde{\alpha}^{-1}(\overline{U_i})$.*

Proof. As close maps define the same coarse mapping space, we may assume α is Borel, by lemma 2.10.

Suppose there is a countable uniformly bounded Borel partition $\{Y_i\}$ of Y with each Y_i having non-empty interior. Let $X_i = \alpha^{-1}(Y_i)$. Then by definition $\{X_i \cup Y_i\} = \{\tilde{\alpha}^{-1}(Y_i)\}$ is a uniformly bounded Borel partition of $X \cup_\alpha Y$. Let f_i, g_i be the characteristic functions of X_i, Y_i. These form a partition of unity, and extending ρ by the Borel calculus they provide orthogonal projections with sum converging strongly to 1 (as ρ is non-degenerate).

From the assumption that each Y_i has non-empty interior, as ρ is ample it follows that the projections $\rho(g_i)$ have infinite dimensional range. Thus there exist isometries V_i from $\rho(f_i)\mathfrak{H}$ into $\rho(g_i)\mathfrak{H}$. Let $\tilde{V} = \sum_i V_i$ in $\mathcal{B}(\mathfrak{H})$. Then $\tilde{V}^*\tilde{V} = \sum_i V_i^* V_i = \sum_i \rho(f_i)$, the projection onto $\mathfrak{H}_X$. Likewise $\tilde{V}\tilde{V}^*$ is a projection with range in $\mathfrak{H}_Y$. Finally each V_i commutes with each $\rho(f_j + g_j)$ so as these have sum 1 (strongly), $\tilde{V} = \sum_i \rho(f_i + g_i)\tilde{V} \sum_j \rho(f_j + g_j) = \sum_i \rho(f_i + g_i)\tilde{V}\rho(f_i + g_i)$, which is controlled by $\bigcup_i \tilde{\alpha}^{-1}(Y_i) \times \tilde{\alpha}^{-1}(Y_i)$.

We now need to establish the existence of such a Borel cover. Let $\{U_i\}$ be the uniformly bounded open cover of Y provided either by separability or by hypothesis for the latter part of the statement. Note that as Y is proper, whenever A is controlled, so is $\overline{A}$ as this lies in $U \circ A \circ U$ for U any controlled open neighbourhood of the diagonal. Thus $\{\overline{U_i}\}$ is also uniformly bounded. Let $Y_1 = \overline{U_1}$, and inductively define $Y_{i+1} = \overline{U_{i+1}} \setminus \bigcup_{i' \leq i} Y_{i'}$. If Y_{i+1} has empty interior then every point of U_{i+1} must be a limit point of $Y_1 \cup \cdots \cup Y_i$, and so also is every point of $\overline{U_{i+1}}$. But $Y_1 \cup \cdots \cup Y_i$ is closed, so whenever Y_i has empty interior, it is empty. As Y_i lies in $\overline{U_i}$, we have $\{Y_i\}$ uniformly bounded as required. Passing to the subsequence of those Y_i which are non-empty gives the required cover. □

The final part of the above proposition ensures that we can find isometries which cover 'arbitrarily closely' a given map α. Specifically, for a proper metric space equipped with a proper coarse structure, for any $\varepsilon > 0$ there exist uniformly bounded open covers all of whose sets have diameter at most ε. This guarantees that there exist covering isometries for which $d(y, \alpha(x)) < \varepsilon$ whenever (y, x) is in the support. We will refer to coverings with this additional property as ε-coverings.

We can now establish functoriality.

Theorem 2.34. *Modulo natural isomorphism, the K-theory of the Roe algebra is independent of the choice of representation. It gives a functor on the category of proper separable coarse spaces and coarse maps. For $\alpha\colon X \to Y$ a coarse map, the induced map α_* is given by $\mathrm{Ad}_{V*}\colon K_*(C^*_{\rho_X} X) \to K_*(C^*_{\rho_Y} Y)$, where V is any isometry covering α.*

Proof. We have already established that such an isometry V always exists, and that Ad_V maps $C^*_{\rho_X}X$ to $C^*_{\rho_Y}Y$. We must show that the induced map on K-theory is well-defined, i.e. does not depend on the choice of isometry. Given this we deduce functoriality as follows.

Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be a composition of coarse maps, and $\mathfrak{H}_X \xrightarrow{V} \mathfrak{H}_Y \xrightarrow{W} \mathfrak{H}_Z$ a composition of covering isometries. As a composition of controlled operators is controlled, WV is a covering isometry for $\beta \circ \alpha$, and it follows that $(\beta \circ \alpha)_* = \beta_* \alpha_*$. The identity map from X to X is covered by the identity operator, provided that both copies of X are equipped with the same representation.

We may interpret this as functoriality on the category of pairs (X, ρ), with coarse maps as morphisms. Suppose X is given two different representations. The identity map $(X, \rho_1) \to (X, \rho_2)$ then induces a natural isomorphism at the level of K-theory, the inverse being induced by the identity $(X, \rho_2) \to (X, \rho_1)$. Thus the K-theory does not depend on the choice of representation, and in fact we have functoriality on the category of proper coarse spaces and coarse maps.

All that remains is to establish that the induced map is well defined. If $T \in C^*_{\rho_X}X$, and V_1, V_2 are two covering isometries for $\alpha\colon X \to Y$, then define

$$T_1 = \begin{pmatrix} \mathrm{Ad}_{V_1}(T) & 0 \\ 0 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Ad}_{V_2}(T) \end{pmatrix}, \text{ and } U = \begin{pmatrix} 1 - V_1V_1^* & V_1V_2^* \\ V_2V_1^* & 1 - V_2V_2^* \end{pmatrix}.$$

Then U is an involution and Ad_U interchanges T_1 and T_2. As V_1, V_2 cover α, for $i, j = 1, 2$, the operator $0 \oplus V_iV_j^*$ is controlled for $X \cup_\alpha Y$, so $V_iV_j^*$ is controlled for Y. By remark 2.28, $C^*_{\rho_Y}Y$ is an ideal in the algebra of controlled operators. Hence as inner automorphisms of an algebra induce the identity on the K-theory of an ideal,[3] $T \mapsto \mathrm{Ad}_{V_1}(T) \oplus 0$ and $T \mapsto 0 \oplus \mathrm{Ad}_{V_2}(T)$ induce the same isomorphism $K_*(C^*_{\rho_X} X) \to K_*(M_2(C^*_{\rho_Y} Y))$.Thus $\mathrm{Ad}_{V_1}, \mathrm{Ad}_{V_2}$ induce the same maps as required. □

Remark 2.35. The argument showing independence of the choice of covering applies in greater generality. For example suppose A, B are subalgebras of $\mathcal{B}(\mathfrak{H}_X), \mathcal{B}(\mathfrak{H}_Y)$, containing C^*X and C^*Y as ideals, and V_1, V_2 are two unitaries (for simplicity) covering $\alpha\colon X \to Y$. If $\mathrm{Ad}_{V_1}, \mathrm{Ad}_{V_2}$ both map A into B, and further $V_1V_2^*$ multiplies B into itself, then they induce the same maps from $K_*(A)$ to $K_*(B)$, and also from $K_*(A/C^*X)$ to $K_*(B/C^*Y)$. The argument is as above, the given condition ensuring that B is an ideal in the algebra generated by $V_1V_2^*$ along with B. This generalization will be useful later.

[3] Given J an ideal in an algebra A, the K-theory of the ideal injects into the K-theory of the algebra $D = \{(a, a + j) \mid a \in A, j \in J\}$, while a unitary u in A gives a unitary (u, u) in D which induces the identity on K-theory.

Chapter 3

K-theory for C_0 Coarse Geometry

In this chapter we will give some specific calculations for the K-theory groups of the Roe algebra defined in the chapter 2. Throughout the chapter X will be a proper metric space and X_0 will denote X equipped with the C_0 coarse structure. We will compute $K_*(C^*X_0)$ in some simple cases, and demonstrate some general methods by which it is possible to calculate these groups for more complicated spaces. In fact the general results will hold for any proper separable coarse space. The techniques developed in this chapter to deal with some straightforward examples will also be used in chapter 4 in the discussion of the coarse Baum-Connes conjecture, as will some of the specific calculations.

3.1 Rays and cones

In this section we will show some conditions on a space X for the K-theory of C^*X_0 to vanish. We will make use of this in the following sections to calculate $K_*C^*X_0$ for other spaces by putting together the groups arising from simpler pieces.

The following technical lemma gathers together sufficient conditions on the algebra for it to have trivial K-theory. It may be regarded as giving conditions under which there is a homotopy from the identity to 'the constant map at infinity.' We will make many applications of this result.

Definition 3.1. Let X be a proper separable coarse space, and let α_k be a sequence of coarse maps $X \to X$. Then the sequence α_k

- is *properly supported* if for any bounded set K, the intersection $K \cap \operatorname{Range} \alpha_k$ is non-empty for only finitely many k;
- is *uniformly controlled* if for every controlled set A there is a controlled set B_A such that $(x, x') \in A$ implies that $(\alpha_k(x), \alpha_k(x')) \in B$ for all k;
- has *uniformly close steps* if there is a controlled set C such that $(\alpha_k(x), \alpha_{k+1}(x)) \in C$ for all k and for all $x \in X$.

Lemma 3.2 (Eilenberg swindle). *Let X be a proper separable coarse space. Let α_k be a sequence of coarse maps $X \to X$ with α_0 the identity. If α_k is properly supported, uniformly controlled, and has uniformly close steps, then $K_*(C^*X) = 0$.*

Proof. Let $\mathfrak{H}$ be the given representation space on which elements of C^*X act. We will show that there is a sequence V_k of covering isometries for α_k such that for any $T \in C^*X$ (or in a matrix algebra over C^*X) with T a projection or unitary defining an element $[T] \in K^*(C^*)$, there are well defined operators $T \oplus \mathrm{Ad}_{V_1}(T) \oplus \mathrm{Ad}_{V_2}(T) \oplus \ldots$ and $\mathrm{Ad}_{V_1}(T) \oplus \mathrm{Ad}_{V_2}(T) \oplus \mathrm{Ad}_{V_3}(T) \oplus \ldots$ on $\mathfrak{H}^\infty = \mathfrak{H} \oplus \mathfrak{H} \oplus \ldots$ which are equal at the level of K-theory. As the inclusion of $\mathfrak{H}$ into $\mathfrak{H}^\infty$ on the first component is a covering isometry for the identity, this will imply that $[T] = 0$ in $K_*(C^*X)$ and as $[T]$ is arbitrary it will follows that the groups vanish.

Fix $\Delta \subseteq X \times X$ a controlled open neighbourhood of the diagonal, and let V_k be a covering isometry for α_k supported in

$$\{(x, x') : (x, \alpha_k(x')) \in \Delta\}.$$

As α_0 is the identity we may choose V_0 to be the identity. Suppose $T \in C^*X$ is controlled and let A be its support. Let B_A be the set provided by the uniform control hypothesis. Then $\mathrm{Ad}_{V_k} T = V_k T V_k^*$ is supported in $\Delta \circ B_A \circ \Delta^T$ for all k, hence $\bigoplus \mathrm{Ad}_{V_k}$ is controlled.

For a bounded set K let $K' = \{x : \exists (x, x') \in \Delta^T, x' \in K\}$. Then K' is bounded, and if it does not meet the range of α_k then the support of $\mathrm{Ad}_{V_k} T$ does not meet $K \times K$. Thus it follows that for any bounded set K, only finitely many of the terms $\mathrm{Ad}_{V_k} T$ have support meeting $K \times K$. Hence as each term is locally compact, the sum $\bigoplus \mathrm{Ad}_{V_k} T$ is also locally compact, and thus lies in C^*X for the representation on $\mathfrak{H}^\infty$. As an arbitrary element of C^*X is a limit of controlled locally compact elements, it follows that $\bigoplus \mathrm{Ad}_{V_k} T$ lies in C^*X for any $T \in C^*X$.

From $[T] \in K^*(C^*X)$, we have obtained elements $[T \oplus \mathrm{Ad}_{V_1}(T) \oplus \mathrm{Ad}_{V_2}(T) \oplus \dots]$ and $[\mathrm{Ad}_{V_1}(T) \oplus \mathrm{Ad}_{V_2}(T) \oplus \mathrm{Ad}_{V_3}(T) \oplus \dots]$ as claimed. It remains to show that these are equal. We now define a sequence of unitaries

$$U_k = \begin{bmatrix} V_{k+1}V_k^* & 1 - V_{k+1}V_{k+1}^* \\ 1 - V_k V_k^* & V_k V_{k+1}^* \end{bmatrix}.$$

It is not hard to check that each U_k is unitary and that composition with U_k maps $\left[\begin{smallmatrix} V_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} V_{k+1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Thus for $U = U_0 \oplus U_1 \oplus \dots$, $S = T \oplus \mathrm{Ad}_{V_1}(T) \oplus \dots$ and $S' =$

$\mathrm{Ad}_{V_1}(T) \oplus \mathrm{Ad}_{V_2}(T) \oplus \ldots$ we find that Ad_U takes $\left[\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} S' & 0 \\ 0 & 0 \end{smallmatrix}\right]$. We note that for each k the operator $V_k V_k^*$ is supported in $\Delta \circ \Delta^T$, and the operator $V_{k+1} V_k^*$ is supported in $\Delta \circ C \circ \Delta^T$, where C is provided by the hypothesis that α_k has uniformly close steps. Thus U is a controlled operator, and hence lies in the multiplier algebra of $M_2(C^* X)$. As inner automorphisms of the multiplier algebra induce the identity on K-theory it we conclude that the elements $[S], [S']$ are equal and hence $[T] = 0$ as required. □

Corollary 3.3. *For the ray $\mathbb{R}^+$ with the standard coarse structure, the K-theory groups $K_*(C^*\mathbb{R}^+)$ vanish.*

Proof. Let $\alpha_k \colon \mathbb{R}^+ \to \mathbb{R}^+$ be the right translation $\alpha_k(t) = k + t$. It is clear that the sequence is properly supported. To show that the sequence is uniformly controlled we note that a controlled set is simply one which lies within some finite distance R of the diagonal, and it is clear that if $d(t, t') < R$ then also $d(\alpha_k(t), \alpha_k(t')) < R$. Finally note that for all k the distance $d(\alpha_k(t), \alpha_{k+1}(t))$ is 1 for all t which shows that the sequence has uniformly close steps. □

In fact this may be generalized to a product of a ray with any other space. The argument is the same, with α_k taken to be as before in the direction of the ray, and to be the identity in the transverse direction.

The proof for the C_0 structure is similar.

Corollary 3.4. *For the ray $\mathbb{R}^+$ with the C_0 coarse structure, the K-theory groups $K_*(C^*\mathbb{R}_0^+)$ vanish.*

Proof. Let $\alpha_k \colon \mathbb{R}^+ \to \mathbb{R}^+$ be the right translation $\alpha_k(t) = \max\{\log k, t\}$. Again the proper support hypothesis is clear.

For the uniform control hypothesis, given a controlled set A there is a function $R \in C_0(\mathbb{R}^+)$ such that $d(t,t') < R(t+t')$ for all $(t,t') \in A$. Without loss of generality we may assume that R is decreasing. Then for $(t,t') \in A$ certainly

$$d(\alpha_k(t), \alpha_k(t')) \leq d(t,t') < R(t+t').$$

If $\log k \leq m = \max\{t,t'\}$ then, $\log k, t, t' \leq m$ and hence $\alpha_k(t), \alpha_k(t') \leq m$. But $t + t' + R(0) \geq 2m$, hence $t + t' \geq 2m - R(0) \geq \alpha_k(t) + \alpha_k(t') - R(0)$, and so as R is decreasing

$$d(\alpha_k(t), \alpha_k(t')) < R(t+t') \leq R(\alpha_k(t) + \alpha_k(t') - R(0)).$$

On the other hand if $\log k > \max\{t,t'\}$ then $d(\alpha_k(t), \alpha_k(t')) = 0$. Hence for all t, t' and for all k, the pair $(\alpha_k(t), \alpha_k(t'))$ lies in the C_0 controlled set $B_A = \{(t,t') \mid d(t,t') < R(t + t' - R(0))\}$.

Finally note that for all k the distance $d(\alpha_k(t), \alpha_{k+1}(t))$ bounded by $\log(k+1) - \log k < 1/k$ and is zero if $\log k \geq t$. Hence whenever $t > \log k_0$ then $d(\alpha_k(t), \alpha_{k+1}(t)) < 1/k_0$ for all k which gives uniformly close steps. $\square$

This may again be generalized to certain products. Note that there are several possible choices of metric on a product space $X \times Y$, for example $d_p((x,y),(x',y')) = (d(x,x')^p + d(y,y')^p)^{1/p}$ where $p \in [1,\infty)$, and $d_\infty((x,y),(x',y')) = \max(d(x,x'), d(y,y))$. For sequences (or nets) (x_i, y_i) and (x_i', y_i') in $X \times Y$, the product metrics $d_p((x_i,y_i),(x_i',y_i'))$ are bounded (respectively tend to zero) if and only if both $d(x_i,x_i')$ and $d(y_i,y_i')$ are bounded (respectively tend to zero). Hence the bounded coarse type of the product and the C_0 coarse type of the product, do not depend on which metric is chosen. In terms of operators, T is C_0 controlled if for any $\varepsilon > 0$, there is a compact subset K of $X \times Y$, outside of which T has propagation at most ε in either direction; that is if $((x,y),(x',y')) \in \operatorname{Supp} T$ lies outside $K \times K$, then $d(x,x'), d(y,y') < \varepsilon$.

Definition 3.5. We will say that a compact metric space X is *cone-like* if there exists a sequence β_i of maps $X \to X$ such that:

1. For each i there is a constant $\lambda_i < 1$ such that for all x, y in X, we have the inequality $d(\beta_i(x), \beta_i(y)) < \lambda_i d(x,y)$.

2. The sequence β_i converges uniformly to the identity as i tends to infinity.

Note in particular that for a compact space Y contained in a Hilbert space, and for X a closed cone on Y within the Hilbert space of one dimension higher, X is cone-like. The following proposition generalizes the result for a ray, to the product of a ray with a cone-like space. In fact we will use this in the following section to show a generalization to the product of a ray with any compact contractible space.

The following proposition is a first example of the more general notion of 'coarse homotopy equivalence.' This will be discussed in greater detail in chapter 4.

Proposition 3.6. *If X is a cone-like space, then the K-theory of $C^*(\mathbb{R}_+ \times X)_0$ vanishes.*

Proof. We will again use lemma 3.2. Let β_i, λ_i be given by the hypothesis on X. We would like to assume that $\lambda_1 \ldots \lambda_i \to 0$. It is clear that for some sequence of indices n_i the products $\lambda_1^{n_1} \ldots \lambda_i^{n_i}$ tend to zero, and hence if each β_i and λ_i is repeated n_i times then the limit will be zero as required. Let α_k be as in 3.4. Let $\gamma_k(t,x) =$

$(\alpha_k(t), \beta_k \circ \cdots \circ \beta_{i+1} \circ \beta_i(x))$ for $i = \lfloor t \rfloor \leq k$, and $\gamma_k(t,x) = (\alpha_k(t), x)$ if $\lfloor t \rfloor > k$. We will show that the sequence γ_k satisfies the conditions of the lemma. As usual the first hypothesis is immediate.

Let $d_t((t,x),(t',x')) = |t - t'|$ and let $d_x((t,x),(t',x')) = d(x,x')$. Given a C_0 controlled set A, for any $\varepsilon > 0$ we may write $A = K_\varepsilon \cup A_\varepsilon$ where K_ε is a bounded set, and the distance functions d_t, d_x are less than ε on A_ε. Let

$$B_A = \{(\gamma_k(t,x), \gamma_k(t',x')) : ((t,x),(t',x')) \in A, k = 1, 2, \dots\}.$$

Clearly d_t, d_x are bounded by ε on the pair $(\gamma_k(t,x), \gamma_k(t',x'))$ for all k and all $((t,x),(t',x')) \in A_\varepsilon$. As K_ε is bounded choose be an upper bound i for t, t' with $((t,x),(t',x')) \in K_\varepsilon$. Then for all $k \geq i$, we have $d_x < \lambda_k \dots \lambda_i \operatorname{Diam} X$ on pairs $(\gamma_k(t,x), \gamma_k(t',x'))$ for $((t,x),(t',x')) \in K_\varepsilon$. For k sufficiently large, we have $\lambda_k \dots \lambda_i \operatorname{Diam} X < \varepsilon$ and $\alpha_k(t) = \alpha_k(t')$ for all $((t,x),(t',x')) \in K_\varepsilon$. Hence for some k_0, if $k \geq k_0$ then d_x is bounded by ε, and $d_t = 0$ on each pair $(\gamma_k(t,x), \gamma_k(t',x'))$ with $((t,x),(t',x')) \in K_\varepsilon$. Thus B_A is contained in the union of a set on which $d_t, d_x < \varepsilon$, with a finite number of of images under γ_k for $k < k_0$ of K_ε. As each such image is bounded it follows that B_A is controlled.

Certainly $d_t(\gamma_k(t,x), \gamma_{k+1}(t,x)) \to 0$ as $t \to \infty$; we established this in the proof of 3.4. Given $\varepsilon > 0$ there is a k_0 such that for all $k \geq k_0$ and for all x we have $d(\beta_k(x), x) < \varepsilon$. For $t \geq k_0$, we have $d_x(\gamma_k(t,x), \gamma_{k+1}(t,x)) = d(y, \beta_{k+1}(y)) < \varepsilon$ if $y = \beta_k \circ \cdots \circ \beta_{i+1} \circ \beta_i(x)$ and $k \geq i = \lfloor t \rfloor$ or $y = x$ and $k+1 = \lfloor t \rfloor$. On the other hand if

$k+1 < \lfloor t \rfloor$ then $d_x(\gamma_k(t,x), \gamma_{k+1}(t,x)) = d(x,x) = 0$. Thus $d_x(\gamma_k(t,x), \gamma_{k+1}(t,x)) \to 0$ as $t \to \infty$ giving the third hypothesis. $\square$

3.2 Homology properties

In this section we will establish a Mayer-Vietoris sequence for K_*C^*X. This is based on a corresponding result for the bounded coarse structure appearing in [10]. We will proceed to show that on the category of compact metrizable spaces X, we obtain a generalized homology theory given by $K_*C^*(X \times \mathbb{R}_+)_0$.

Definition 3.7. For $A \subseteq X \times X$ controlled and $Y \subseteq X$ closed, define the A-neighbourhood of Y to be:

$$Y_A = \{x \in X \mid \text{ there exists } y \in Y \text{ with } (x,y) \in A \text{ or } x = y\}$$

Definition 3.8. If $B \subseteq X \times X$ is controlled, we say it is *near* to Y if there exists $A \subseteq X \times X$, with $B \subseteq Y_A \times Y_A$.

For any $A \subseteq X \times X$ controlled, the inclusion $Y \hookrightarrow Y_A$ is a coarse equivalence. There exist retractions $Y_A \to Y$ which are close to the identity, indeed we may define a map $r(y) = y$ for $y \in Y$, and define $r(x) = y$ for any $y \in Y$ with $(x,y) \in A$, when $x \in Y_A \setminus Y$. Any such choice of r is then an inverse to $Y \hookrightarrow Y_A$ modulo closeness, hence the inclusion is a coarse equivalence as claimed.

Note that for X a proper metric space equipped with the C_0 coarse structure, a set is near to Y if and only if it is contained in a set

$$Y_{R,x_0} = \{x \in X \mid \exists y \in Y \text{ such that } d(x,y) \leq R(d(x_0,x) + d(x_0,y))\},$$

where $R\colon \mathbb{R}_+ \to \mathbb{R}_+$ is C_0 and $x_0 \in X$.

Definition 3.9. Suppose a representation ρ of $C_0(X)$ on $\mathfrak{H}$ is given, and take the induced representations of $C_0(Y_A)$ on $\rho(\chi_{Y_A})\mathfrak{H}$, where ρ is extended to bounded Borel functions via the functional calculus. Define the ideal of C^*X *supported by* Y to be $\varinjlim_{A \in \mathcal{A}} C^*Y_A$, where $\mathcal{A}$ is the collection of controlled open sets containing the diagonal of $Y \times Y$, directed by inclusion. Denote this by I_Y.

Note that this *is* an ideal, as for $T \in C^*X$ with $\operatorname{Supp} T \subseteq B$ controlled, $T \cdot C^*Y_A \subseteq C^*Y_{B \circ A}$, and $C^*Y_A \cdot T \subseteq C^*Y_{A \circ B}$. Note also that for A a controlled open neighbourhood of $\{(y,y) \mid y \in Y\}$, the induced representation of $C_0(Y_A)$ is ample. As such A generate the coarse structure, I_Y is equally the direct limit over all controlled sets A, of the algebra of locally compact controlled operators with respect to the (not necessarily ample) induced representation of $C_0(Y_A)$.

For any controlled A, as $Y \hookrightarrow Y_A$ is a coarse equivalence, $K_*C^*Y \xrightarrow{\cong} K_*C^*Y_A$, and hence for $A, B \in \mathcal{A}$ with $A \subseteq B$, we have $K_*C^*Y \xrightarrow{\cong} K_*C^*Y_A \xrightarrow{\cong} K_*C^*Y_B$. Thus by continuity of K-theory under direct limits $K_*C^*Y \xrightarrow{\cong} K_*C^*I_Y$, the map being induced by any isometry covering $Y \hookrightarrow X$.

Definition 3.10. Suppose that $X = Y \cup Z$, with Y, Z closed in X. If for all controlled $A \subseteq X \times X$ there exists $B \subseteq X \times X$ controlled, such that $Y_A \cap Z_A \subseteq (Y \cap Z)_B$, then the decomposition is *coarsely excisive.*

Remark 3.11. In the cases of the C_0 and bounded coarse structures on a *path* metric space X, any decompostion of X into closed sets is excisive.

Lemma 3.12. *If $X = Y \cup Z$ is a coarsely excisive decomposition, then $I_{Y \cap Z} = I_Y \cap I_Z$.*

Proof. Certainly $(Y \cap X)_A \subseteq Y_A, Z_A$ for all A, so $I_{Y \cap Z} \subseteq I_Y, I_Z$. Conversely given T in $I_Y \cap I_Z$, we may write T as $T_1 T_2$ in $I_Y I_Z = I_Y \cap I_Z$. We may then approximate T_1, T_2 by T_3, T_4, with T_3 in $C^* Y_A$, and T_4 in $C^* Y_{A'}$ for some open controlled A, A'. Note

$$\operatorname{Supp} T_3 T_4 \subseteq \operatorname{Supp} T_3 \circ (Z_{A'} \times Z_{A'}) \cap (Y_A \times Y_A) \circ \operatorname{Supp} T_4 \subseteq (Y_B \cap Z_B) \times (Y_B \cap Z_B)$$

for $B \supseteq \operatorname{Supp} T_3 \circ A' \cup A \circ \operatorname{Supp} T_4$. Now as the decomposition is excisive, and X is proper, there exists D open and controlled, with $Y_B \cap Z_B \subseteq (Y \cap Z)_D$. Thus $T_3 T_4$ lies in $C^*(Y \cap Z)_D \subseteq I_{Y \cap Z}$, so $I_{Y \cap Z}$ is dense in and hence is equal to $I_Y \cap I_Z$. □

We now obtain a Mayer-Vietoris sequence for coarsely excisive decompositions.

Theorem 3.13. *For a coarsely excisive decomposition $X = Y \cup Z$, there is a cyclic Mayer Vietoris exact sequence:*

$$\cdots \to K_*(C^*(Y \cap Z)) \xrightarrow{i_{1*} \oplus i_{2*}} K_*(C^* Y) \oplus K_*(C^* Z) \xrightarrow{j_{1*} - j_{2*}} K_*(C^* X)$$
$$\xrightarrow{\partial} K_{*-1}(C^*(Y \cap Z)) \to \dots$$

where i_1, i_2 are respectively the inclusions of $Y \cap Z$ into Y, Z, and j_1, j_2 are respectively the inclusions of Y, Z into X.

Proof. Given the above observations it suffices to establish an exact sequence:

$$\dots K_*(I_{Y\cap Z}) \to K_*(I_Y) \oplus K_*(I_Z) \to K_*(C^*X) \to K_{*-1}(I_{Y\cap Z}) \to \dots$$

By the lemma $I_{Y\cap Z} = I_Y \cap I_Z$. Letting $P = \rho(\chi_Y)$ for ρ the representation of $C_0(X)$, we will show that for controlled T in C^*X, TP lies in I_Y. Certainly it is locally compact, and as $A = \operatorname{Supp} T$ is controlled we have $\operatorname{Supp} TP \subseteq Y_A \times Y$ near Y. Likewise $T(1-P) \in I_Z$, so $I_Y + I_Z$ is dense in C^*X, and hence equals this.

We now use a general construction to show that these conditions imply exactness of the sequence. Define

$$J = S(C^*X) = \{f \in C[0,1] \otimes C^*X \mid f(0) = f(1) = 0\}$$

$$A = \{f \in C[0,1] \otimes C^*X \mid f(0) \in I_Y, f(1) \in I_Z\}.$$

Note that the *-homomorphism $A \to I_Y \oplus I_Z$ given by $f \mapsto f(0) \oplus f(1)$ has kernel J and is onto, so $A/J \cong I_Y \oplus I_Z$. By homotopy invariance, the inclusion of $I_{Y\cap Z}$ into $C[0,1] \otimes I_{Y\cap Z}$ as constant functions, induces an isomorphism on K-theory. The image of $C[0,1] \otimes I_{Y\cap Z}$ is an ideal in A, and as $I_Y + I_Z = C^*X$, the quotient is given by

$$A/C[0,1] \otimes I_{Y\cap Z} = C_0[0,1) \otimes I_Y/C_0[0,1) \otimes I_{Y\cap Z} \oplus C_0(0,1] \otimes I_Z/C_0(0,1] \otimes I_{Y\cap Z}.$$

Again by homotopy invariance, this has trivial K-theory, so by the cyclic exact sequence in K-theory, $I_{Y\cap Z} \hookrightarrow A$ also induces an isomorphism on K-theory. The suspension isomorphism arising from the short exact sequence $0 \to J \to C_0[0,1) \otimes C^*X \to C^*X \to 0$, identifies K_*C^*X with $K_{*-1}J$, and hence we have identified the terms of the K-theory exact sequence arising from $0 \to J \to A \to A/J \to 0$, with those of the claimed exact sequence.

That the map $K_*(I_{Y\cap Z}) \to K_*(I_Y) \oplus K_*(I_Z)$ is induced by the diagonal inclusion follows from the fact that this inclusion is the composition $I_{Y\cap Z} \hookrightarrow A \to A/J \cong I_Y \oplus I_Z$. For the map $K_*(I_Y) \to K_*(C^*X)$, consider the following commutative ladder:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & \{f \in C[0,1) \otimes C^*X \mid f(0) \in I_Y\} & \longrightarrow & I_Y & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & C[0,1) \otimes C^*X & \longrightarrow & C^*X & \longrightarrow & 0 \end{array}$$

The map under consideration is given by the boundary map in K-theory for the first sequence, composed with the suspension isomorphism arising from the second. Naturality of the boundary map implies that this is the same as the map induced from $I_Y \hookrightarrow C^*X$. For $I_Z \hookrightarrow C^*X$ we get the same answer except for a change of sign due to the different suspension isomorphism arising from the cone $C(0,1] \otimes C^*X$. $\square$

We will now discuss further the case of $C^*(X \times \mathbb{R}_+)_0$, where X is a compact metrizable space.

If X, Y are compact, then every continuous map from X to Y is topologically proper and uniformly continuous, and the same is true for a product $\alpha \times \mathrm{id}$ from $X \times \mathbb{R}_+$ to $Y \times \mathbb{R}_+$. Hence lemma 2.18 implies that such maps are C_0 coarse, so it is correct to talk about compact *metrizable* spaces X, Y in this context rather than compact metric spaces, as the coarse type of $(X \times \mathbb{R}_+)_0$ is a homeomorphism invariant of X.

We will now show that these spaces behave well with respect to excision, indeed any decomposition which is a product at infinity is excisive.

Lemma 3.14. *Suppose X is compact and metrizable, and that Y, Z are closed subsets of X with $X = Y \cup Z$. Let W be a proper metric space. If $Y \cap Z$ is non-empty, then $Y \times W \cup Z \times W$ is a coarsely excisive decomposition of $(X \times W)_0$.*

Proof. Let $Y' = Y \times W, Z' = Z \times W$. We need to show that for all $R\colon \mathbb{R}_+ \to \mathbb{R}_+$ tending to zero at infinity, there exists $R'\colon \mathbb{R}_+ \to \mathbb{R}_+$ again C_0, such that $Y'_R \cap Z'_R \subseteq (Y' \cap Z')_{R'}$, where the notation Y_R etc. is as above. Equivalently we need to show that $(x,t) \mapsto d(x, Y \cap Z)$ is C_0 on $Y'_R \cap Z'_R$.

Now let (x_n, t_n) be a sequence in $Y'_R \cap Z'_R$, with $t_n \to \infty$. As $Y \cap Z$ is non-empty such sequences exist unless W is compact in which case the result is trivial. As $(x_n, t_n) \in Y'_R$ it follows that $d(x_n, Y) \to 0$ as $n \to \infty$, and similarly for $d(x_n, Z)$. Consider $d(x_n, Y \cap Z)$, and note that all subsequences of x_n have convergent subsubsequences, which from the above must tend to a point of $Y \cap Z$. It follows that $d(x_n, Y \cap Z) \to 0$ for any such sequence (x_n, t_n), and hence $(x,t) \mapsto d(x, Y \cap Z)$ is C_0 on $Y'_R \cap Z'_R$ as required. □

We have now shown enough to deduce that $X \mapsto K_*(C^*(X \times \mathbb{R}_+)_0)$ defines a generalized reduced homology theory on the category of compact metrizable spaces and continuous maps. This can be stated abstractly as follows.

Definition 3.15. Let $\mathcal{C}$ be a category whose objects are topological spaces, and with all continuous maps as morphisms. A *decomposition in* $\mathcal{C}$ of an object X is a pair of subsets Y, Z of X, with $X = Y \cup Z$, and such that $Y, Z, Y \cap Z$ are all (homeomorphic to) objects.

Lemma 3.16. *Let $\mathcal{C}$ be a category whose objects are topological spaces, and with all continuous maps as morphisms, and let $\tilde{h}_*$ be a covariant functor from $\mathcal{C}$ to the category of abelian groups. Suppose that for any object X of $\mathcal{C}$*

- *the product $X \times [0,1]$, the closed cone $\operatorname{Cone} X = X \times [0,1)^+$ and the suspension $SX = X \times (0,1)^+$ are objects of X;*
- *$\tilde{h}_* \operatorname{Cone} X$ is the trivial group;*
- *for every decomposition of X in $\mathcal{C}$ as $Y \cup Z$, there is a Mayer-Vietoris exact sequence*

$$\cdots \to \tilde{h}_*(Y \cap Z) \to \tilde{h}_*(Y) \oplus \tilde{h}_*(Z) \to \tilde{h}_*(X) \to \tilde{h}_{*-1}(Y \cap Z) \to \dots$$

  where: the homomorphisms $\tilde{h}_*(Y \cap Z)$ to $\tilde{h}_*(Y), \tilde{h}_*(Z)$ are induced from the inclusions of $Y \cap Z$ into Y, Z; the homomorphisms $\tilde{h}_*(Y), \tilde{h}_*(Z)$ to $\tilde{h}_*(X)$ are respectively $+/-$ the maps induced from the inclusions of Y, Z into X; the boundary map is natural.

Then $\tilde{h}_*$ is homotopy invariant, that is for any continuous map $\eta\colon X \times [0,1] \to Y$, and for $\eta_i(x) = \eta(x,i)$, we have $\eta_{0*} = \eta_{1*}$.

Proof. Let ϕ_i be the inclusions of X into $X \times [0,1]$ given by $\phi_i(x) = (x,i)$. Since $\eta_{i*} = (\eta \circ \phi_i)_* = \eta_* \phi_{i*}$, it suffices to show that ϕ_{0*} and ϕ_{1*} are equal. The projection

$\pi\colon X \times [0,1] \to X$ induces a left inverse to both ϕ_{0*} and ϕ_{1*}, so it suffices to prove that ϕ_{0*}, ϕ_{1*} are isomorphisms, as then both give the inverse to π_*.

We begin by identifying $\mathrm{S}X$ with $(X \times (-1,2))^+$. We decompose this as $Y \cup Z$ where $Y = (X \times (-1,1])^+$ and $Z = (X \times [0,2))^+$. Now let $Y' = (X \times (-1,0])^+$ and observe that $\mathrm{S}X$ can also be decomposed as $Y' \cup Z$. We have Mayer-Vietoris sequences for each decomposition, and as Y' includes into Y we get a commutative ladder:

$$\begin{array}{ccccccc}
\tilde h_{*+1}(Y') \oplus \tilde h_{*+1}(Z) & \longrightarrow & \tilde h_{*+1}(\mathrm{S}X) & \xrightarrow{\partial} & \tilde h_*(X \times \{0\}) & \longrightarrow & \tilde h_*(Y') \oplus \tilde h_*(Z) \\
\downarrow & & \| & & \downarrow{\scriptstyle \phi_{0*}} & & \downarrow \\
\tilde h_{*+1}(Y) \oplus \tilde h_{*+1}(Z) & \longrightarrow & \tilde h_{*+1}(\mathrm{S}X) & \xrightarrow{\partial} & \tilde h_*(X \times [0,1]) & \longrightarrow & \tilde h_*(Y) \oplus \tilde h_*(Z)
\end{array}$$

But $\tilde h_*(Y), \tilde h_*(Y'), \tilde h_*(Z)$ are all zero, and so the boundary maps are isomorphisms. This implies ϕ_{0*} is also an isomorphism as required, and similarly so is ϕ_{1*}. $\square$

Let us denote $K_*(C^*(X \times \mathbb{R}_+)_0)$ as $\tilde{\mathcal{K}}_{*-1}(X)$; it will be convenient to introduce this dimension shift.

Corollary 3.17. *The functor $\tilde{\mathcal{K}}_*$ is homotopy invariant.*

Proof. Let $\mathcal{C}$ be the category of (non-empty) compact metrizable spaces, and continuous maps. The objects $\mathrm{Cone}\, X$ and $\mathrm{S}X$ are the usual topological constructions. We have functoriality, vanishing for cone-like spaces, and we have a Mayer-Vietoris sequence for closed intersecting subsets Y, Z with $Y \cup Z = X$. We can construct $\mathrm{Cone}\, X$ by embedding X in a codimension one subspace of a Hilbert space and then building the cone in this Hilbert space. The metric inherited from the Hilbert space is cone-like hence we have all the hypotheses to apply the lemma. $\square$

3.3 Examples of $K_*C^*X_0$

In this section our aim is just to compute some examples. Note that the results of the previous section already allow us to compute the K-theory for the Roe algebras associated to a large class of spaces.

The results for products of the form $X \times \mathbb{R}_+$ with X compact also lead us to results for spaces of the form $X_1 = X \times \mathbb{R}$. We will make useof this below.

Lemma 3.18. *Let X be a compact metric space. The spaces $X_1 = X \times \mathbb{R}$ and $X_2 = (X \times \{0,1\}) \times \mathbb{R}_+$ are C_0 coarsely equivalent. Specifically the map X_1 to X_2 given by $\alpha\colon (x,t) \mapsto (x, \chi_{(-\infty,0)}(t), |t|)$, and the map X_2 to X_1 given by $\beta\colon (x,i,t) \mapsto (x, (-1)^i t)$ are coarse equivalences.*

Proof. Certainly the maps are proper. If $d(\alpha(x), \alpha(x')) > d(x,x')$, then x, x' must lie in the compact set $X \times [-1,1]$, hence α is coarse. Conversely any C_0 controlled subset of $X_2 \times X_2$ is of the form $A = B \cup K$, with K bounded and $B \subseteq (X \times \{0\} \times \mathbb{R}_+)^2 \cup (X \times \{1\} \times \mathbb{R}_+)^2$. For a pair $(x,y) \in B$ we have $d(\beta(x), \beta(y)) = d(x,y)$, while the image under $\beta \times \beta$ of K will again be bounded. Hence β is also C_0 coarse. (Note however that it is not coarse for the standard structure as it can increase distances arbitrarily.) The composition $\beta \circ \alpha$ is the identity, while $\alpha \circ \beta$ differs from the identity only on a compact set — specifically it maps $X \times \{1\} \times \{0\}$ to $X \times \{0\} \times \{0\}$ — so both compositions are close to the identity. $\square$

Example 3.19. $\tilde{\mathcal{K}}_i S^n = \begin{cases} \mathbb{Z} & i \equiv n \mod 2 \\ 0 & i \equiv n+1 \mod 2 \end{cases}$. For $n = 0$ we deduce this from the Mayer-Vietoris sequence for the decomposition $\mathbb{R} = (-\infty, 0] \cup [0, \infty)$, where we identify $S^0 \times \mathbb{R}_+$ with $\mathbb{R}$ via the coarse equivalence described above. This decomposition is coarsely excisive, indeed choosing $x_0 = 0$ as basepoint we have $(-\infty, 0]_{R,x_0} \cap [0, \infty)_{R,x_0} = \{0\}_{R,x_0}$ for all functions R. As the K-theory for rays is trivial, from the Mayer-Vietoris sequence the K-theory for $\mathbb{R}$ is obtained from that of $\{0\}$ by a dimension shift, and by example 2.29 we have $K_i C^*(\{0\}) = \mathbb{Z}$ for $i = 0$ and vanishes for $i = 1$. The result for $n > 0$ is given by induction using the usual suspension isomorphism for a reduced homology.

The above example is closely related to the question of calculating the groups $K_* C^* \mathbb{R}^n_0$. We obtain the case of $\mathbb{R}^1$ immediately from the above, however while there is a coarse map $\mathbb{R}^n_0 \to S^{n-1} \times \mathbb{R}^+$ for all n, it is a coarse equivalence only when $n = 1$. For $n > 1$ we may take a decomposition $\mathbb{R}^n = \mathbb{R}^{n-1} \times (-\infty, 0] \cup \mathbb{R}^{n-1} \times [0, \infty)$, however we have not yet developed the theory required to show that $K_* C^* (\mathbb{R}^{n-1} \times \mathbb{R}^+)_0$ is trivial.

We will now move on to slightly more complicated spaces. let us consider spaces which are 'periodic' along a ray, rather than 'constant'. Examples of these would include disjoint unions $\bigsqcup_{i=0}^{\infty} X$, with X compact metrizable. The C_0 disjoint union structure in this case is the same as that of $(X \times \mathbb{N})_0$. By lemmas 2.18 and 3.14, we again need only consider the space X up to homeomorphism, and we have good excision properties.

Lemma 3.20. *$K_0 C^*\mathbb{N}_0 = (\prod_{\mathbb{N}} \mathbb{Z})/G$ and $K_1 C^*\mathbb{N}_0 = 0$, where $\prod \mathbb{Z}$ denotes the infinite direct product, that is all sequences of integers, and G is the subgroup of finitely supported sequences whose sum is zero.*

Proof. Let $\mathfrak{H}'$ be a separable infinite dimensional Hilbert space, and represent $C_0\mathbb{N}$ on $\mathfrak{H} = l^2(\mathbb{N}) \otimes \mathfrak{H}'$, acting by pointwise multiplication on the first factor. An operator T on $\mathfrak{H}$ is given by an infinite matrix T_{ij} of operators on $\mathfrak{H}'$. Local compactness is the condition that each T_{ij} is a compact operator on $\mathfrak{H}'$. As $\mathbb{N}$ is uniformly discrete, for an operator to have C_0-propagation, it must have no propagation outside some compact set. In other words, there are only finitely many pairs (i, j) with $i \neq j$ and $T_{ij} \neq 0$.

Let $A_n = \mathfrak{K}(l^2\{0, \ldots, n\} \otimes \mathfrak{H}') \oplus l^\infty\{n+1, n+2, \ldots\} \otimes \mathfrak{K}(\mathfrak{H}')$. The above observations amount to the statement that each locally compact C_0-propagation operator lies in some A_n, hence $C^*\mathbb{N}_0 = \varinjlim_n A_n$. To compute the K-theory of A_n, note that each A_n just consists of sequences of compact operators. A projection over $\tilde{A_n}$ is therefore just a sequence of projections over $\tilde{\mathfrak{K}}$, and the restriction to each factor gives a map to the K-theory of $\mathfrak{K}$ (given by the rank in the case where the K-theory element is defined by a projection in $\mathfrak{K}$). This produces a homomorphism $K_0 A_n \to \prod_{N \geq n} \mathbb{Z}$. We will show that this is an isomorphism.

Note that $M_k(\mathfrak{K}) \cong \mathfrak{K}$. Thus it is not hard to show that every element of $K_0(\mathfrak{K})$ can be represented by a projection in $\tilde{\mathfrak{K}}$; we do not need to pass to the matrix algebra. Moreover if $[p] = [p']$ in K-theory, then there is a projection $q \in \tilde{\mathfrak{K}}$ such that $p \oplus q$ is Murray von-Neumann equivalent to $p' \oplus q$ in the algebra $M_2(\tilde{\mathfrak{K}})$. Indeed if B is any C^*-algebra then the same is true for $K_0(\mathfrak{K} \otimes B)$. We will use this for $\mathfrak{K}$, and for $A_n \cong \mathfrak{K} \otimes l^\infty$, to show that the homomorphism $K_0 A_n \to \prod_{N \geq n} \mathbb{Z}$ is injective. Given two sequences

$(p_1, p_2, \dots)$ and $(p_1', p_2', \dots)$ of projections in $\tilde{\mathfrak{K}}$, if the images in $\prod_{N \geq n} \mathbb{Z}$ are equal then for each i there is a projection q_i such that $p_i \oplus q_i$ is Murray von-Neumann equivalent to $p_i' \oplus q_i$, and the sequence of equivalences gives a Murray von-Neumann equivalence between $(p_1 \oplus q_1, p_2 \oplus q_1, \dots)$ and $(p_1' \oplus q_1, p_2' \oplus q_1, \dots)$. Hence two elements with the same image in $\prod_{N \geq n} \mathbb{Z}$ must be equal. Surjectivity is easy as we may just choose appropriate sequences of finite rank projections, and hence $K_0 A_n = \prod_{N \geq n} \mathbb{Z}$.

The map $A_{n_0} \to A_{n_1}$, for $n_0 < n_1$ is given by including all terms up to n_1 into a single larger algebra of compact operators. Hence the map on K-theory is given by $(r_{n_0}, \dots, r_{n_1}, r_{n_1+1}, \dots) \mapsto (r_{n_0} + \dots + r_{n_1}, r_{n_1+1}, \dots)$. Each of these maps is surjective, so $K_0 C^* \mathbb{N}_0 = \varinjlim_n K_0 A_n$ is a quotient of $K_0 A_0 = \prod \mathbb{Z}$. Let G be the kernel of this, and note that $(r_0, r_1, \dots) \in G$ if and only if $(r_0 + r_1 + \dots + r_n, r_{n+1}, \dots) = 0$ for n sufficiently large. Hence G is as claimed.

To establish that the K_1 group vanishes it suffices to show this for each algebra A_n. This is straightforward if we identify $K_1(A_n)$ with $K_0(A_n \otimes C_0(\mathbb{R}))$. Again an element is represented by a sequence $(p_1, p_2, \dots)$ of projections, this time in $\widetilde{\mathfrak{K} \otimes C_0}(\mathbb{R})$. As $K_0(\mathfrak{K} \otimes C_0(\mathbb{R})) = K_1(\mathfrak{K})$ vanishes, for each i there is a projection q_i with $p_i \oplus q_i$ Murray von-Neumann equivalent to q_i. Hence $(p_1 \oplus q_1, p_2 \oplus q_2, \dots)$ is equal to $(q_1, q_2, \dots)$ at the level of K-theory, so $(p_1, p_2, \dots)$ is zero, and the group vanishes. $\square$

For the remainder of this section, G will denote the subgroup of $\prod \mathbb{Z}$ consisting of finitely supported sequences whose sum is zero.

Given the calculation for a sequence of points, we may now extend to other spaces of the form $X \times \mathbb{N}$ using the Mayer-Vietoris sequence. Note however that a priori we do not have homotopy invariance for the functor $X \mapsto K_*C^*(X \times \mathbb{N})_0$, however the following example will lead to homotopy invariance under certain conditions.

Example 3.21. Let $I = [0,1]$. Then the map $K_*(C^*\mathbb{N}_0) \to K_*(C^*(I^n \times \mathbb{N})_0)$ is an isomorphism. Certainly the map is injective, as the retraction $I^n \times \mathbb{N} \to \mathbb{N}$ will induce a left inverse. To establish the isomorphism we will show that this retraction also induces an injection.

Inductively we assume that $K_*(C^*\mathbb{N}_0) \to K_*(C^*(I^n \times \mathbb{N})_0)$ is an isomorphism and we will show that the same is true for $I^{n+1} \times \mathbb{N}$. Embed $I^{n+1} \times \mathbb{N}$ coarsely into $\mathbb{R}^{n+1}$ as $X = I^n \times \bigcup_{i=0}^{\infty}[2i, 2i+1]$, and let $Y = I^n \times \bigcup_{i=0}^{\infty}[2i+1, 2i+2]$. The retraction that we wish to prove is injective is then the map from X to $2\mathbb{N}$. Note that the union $X \cup Y = I^n \times \mathbb{R}_+$ is C_0 coarsely excisive. From 3.6 we know that $K_*C^*(X \cup Y)_0$ is trivial. Hence the Mayer-Vietoris sequence gives $K_*C^*(X \cap Y)_0 \cong K_*C^*X_0 \oplus K_*C^*Y_0$. The intersection $Y \cap Z$ is $I^n \times \mathbb{N}$ so inductively $K_1C^*(X \cap Y)_0 = 0$, and the group $K_1C^*X_0$ vanishes.

For K_0, we observe that there is a surjection $K_0C^*X_0 \oplus K_0C^*Y_0$ to $K_0C^*2\mathbb{N}_0 \oplus K_0C^*(2\mathbb{N}+1)_0$. If this is an injection then $K_0C^*X_0 \cong K_0C^*2\mathbb{N}_0$ as claimed. Inductively $K_0C^*\mathbb{N}_0 \xrightarrow{\cong} K_0C^*(I^n \times \mathbb{N})_0 = K_0C^*(X \cap Y)_0$ and from the Mayer-Vietoris sequence this is also isomorphic to $K_0C^*X_0 \oplus K_0C^*Y_0$. We therefore identify $K_0C^*X_0 \oplus K_0C^*Y_0$ with $K_0C^*\mathbb{N}_0$, and with this identification we will show that the surjection $K_0C^*\mathbb{N}_0 \to K_0C^*2\mathbb{N}_0 \oplus K_0C^*(2\mathbb{N}+1)_0$ is an isomorphism.

We may explicitly compute this map; in terms of sequences (modulo G), we get $(r_0, r_1, \dots) \mapsto (r_0, r_1 + r_2, r_3 + r_4, \dots) \oplus (r_0 + r_1, r_2 + r_3, \dots)$. Now suppose this gives a pair of sequnces in $G \oplus G$. Then for all n sufficiently large, $r_0 + r_1 + \cdots + r_{2n} = 0$ and

$r_0 + r_1 + \cdots + r_{2n+1} = 0$. Thus the sequence $r_0, r_1, \ldots$ has finite support and its sum is zero, i.e. $(r_0, r_1, \ldots) \in G$. Hence the map $\prod \mathbb{Z}/G \to \prod \mathbb{Z}/G \oplus \prod \mathbb{Z}/G$ is injective as required.

Let us now use this to establish a homotopy invariance result for $K_*C^*(X \times \mathbb{N})_0$. It will be convenient to use a reduced version of this; we introduce the notation $\tilde{\mathcal{K}}_*^{\mathbb{N}}(X) = \mathrm{Ker}(K_*C^*(X \times \mathbb{N})_0 \to K_*C^*\mathbb{N}_0)$.

Lemma 3.22. *The functor $\tilde{\mathcal{K}}_*^{\mathbb{N}}$ is homotopy invariant on the category of finite simplicial complexes.*

Proof. We will apply lemma 3.16. Let $\mathcal{C}$ be the category of compact metrizable spaces admitting a finite triangulation, with continuous maps as morphisms. Note that if X admits a triangulation, then so do $\mathrm{Cone}\, X$ and $\mathrm{S}X$. We have a Mayer-Vietoris sequence for a decomposition of X into two subcomplexes Y, Z and hence all that remains is to show that $\tilde{\mathcal{K}}_*^{\mathbb{N}}(\mathrm{Cone}\, X)$ vanishes for all X. This is a simple induction over the set of simplicial complexes, partially ordered by inclusion. If X is a single simplex, then so is $\mathrm{Cone}\, X$. Thus $\mathrm{Cone}\, X \times \mathbb{N}$ is C_0-coarsely equivalent to $I^n \times \mathbb{N}$, so by the above example $\tilde{\mathcal{K}}_*^{\mathbb{N}}\, \mathrm{Cone}\, X = 0$. Otherwise $X = Y \cup Z$ with Y, Z proper subcomplexes. Note that $\mathrm{Cone}\, Y \cap \mathrm{Cone}\, Z = \mathrm{Cone}(Y \cap Z)$ where by convention we take the cone on the empty set to be a point. Hence by induction, $\tilde{\mathcal{K}}_*^{\mathbb{N}}\, \mathrm{Cone}\, Y = \tilde{\mathcal{K}}_*^{\mathbb{N}}\, \mathrm{Cone}\, Z = \tilde{\mathcal{K}}_*^{\mathbb{N}}(\mathrm{Cone}\, Y \cap \mathrm{Cone}\, Z) = 0$. The Mayer Vietoris sequence then shows that $\tilde{\mathcal{K}}_*^{\mathbb{N}}(\mathrm{Cone}\, X)$ vanishes. ∎

Example 3.23. $\tilde{\mathcal{K}}_i^{\mathbb{N}} S^n = \begin{cases} \prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_{\mathbb{N}} \mathbb{Z} & i \equiv n \mod 2 \\ 0 & i \equiv n+1 \mod 2 \end{cases}$. In the case $n = 0$ this is canonically identified with a subgroup of $K_* C^* \mathbb{N}_0$ allowing us to identify 'generators.' For $n > 0$ we identify elements via a suspension isomorphism.

Homotopy invariance and the tautological vanishing for a point are all that is needed for the suspension isomorphism, so we will just make the calculation for $n = 0$, as in 3.19. Certainly the K_1 group vanishes. For K_0, regard the map $S^0 \times \mathbb{N} \to \mathbb{N}$ as the map $\mathbb{N} \to \mathbb{N}$ given by $n \mapsto \lfloor n/2 \rfloor$. Then the induced map on the K_0 group $\prod \mathbb{Z}/G$ is given by $(r_0, r_1, r_2, \dots) \mapsto (r_0 + r_1, r_2 + r_3, \dots)$, which has kernel whose elements are represented by sequences $(r_0, r_1, r_2, \dots)$ with $r_{2n} = -r_{2n+1}$ for all $n > n_0$ sufficiently large and $r_0 + \cdots + r_{2n_0+1} = 0$. Such elements may in fact be represented by sequences with $r_{2n} = -r_{2n+1}$ for all n, so $\tilde{\mathcal{K}}_0^{\mathbb{N}}(S^0)$ is the image of the map $\prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z}/G$, given by $(r_0, r_1, r_2, \dots) \mapsto (r_0, -r_0, r_1, -r_1, \dots) + G$. The kernel of this map is $\bigoplus_N \mathbb{Z}$, so its image is isomorphic to $\prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_N \mathbb{Z}$. Hence $\tilde{\mathcal{K}}_0^{\mathbb{N}}(S^0)$ is the isomorphic image of $\prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_N \mathbb{Z}$ in $\prod_N \mathbb{Z}/G$.

As a final example we will make use of the above to calculate the groups for a periodic space which is not a product.

Example 3.24. Let X be the 'infinite ladder' given by taking a surface with four boundary components and no handles, that is a tubular H, and attaching infinitely many copies of this (all of the same size) together along pairs of boundary components, see figure 3.1. We will take the construction to be infinite only in one direction. Alternatively we may think of X as given by removing infinitely many pairs of discs from $S^1 \times \mathbb{R}_+$, and attaching a cylinder $S^1 \times I$ to each the pair of boundary circles.

Fig. 3.1. Decomposition of the ladder space

First we will compute $\tilde{\mathcal{K}}_*^{\mathbb{N}}(H)$ for H the surface with four boundary components. From a cylinder (which of course is homotopy equivalent to a circle), we can obtain a surface with three boundary components, which we shall call T, by glueing an interval in the boundary, to an interval in the boundary of another cylinder. Note that in the corresponding Mayer-Vietoris sequence, all but two of the groups vanish, and we obtain $\tilde{\mathcal{K}}_1^{\mathbb{N}}(T) \cong \tilde{\mathcal{K}}_1^{\mathbb{N}} S^1 \oplus \tilde{\mathcal{K}}_1^{\mathbb{N}} S^1 \cong \prod \mathbb{Z}/\bigoplus \mathbb{Z} \oplus \prod \mathbb{Z}/\bigoplus \mathbb{Z}$, while $\tilde{\mathcal{K}}_0^{\mathbb{N}}(T) = 0$. The two summands correspond to the inclusion of two of the three boundary components, and by homotopy invariance the inclusion of the third boundary component is given by $(r_0, r_1, \dots) \mapsto \pm(r_0, r_1, \dots) \oplus \pm(r_0, r_1, \dots)$ modulo $\bigoplus \mathbb{Z}$, where the signs depend on the choice of orientations. Repeating the construction to add another boundary component gives another copy of $\prod \mathbb{Z}/\bigoplus \mathbb{Z}$ in the $\tilde{\mathcal{K}}_1^{\mathbb{N}}(H)$ group, and will give $\tilde{\mathcal{K}}_0^{\mathbb{N}}(H) = 0$. It will be convenient to take the three 'generators' to be given by the inclusion of a circle dividing the surface into two copies of T, and the inclusion of one of the other boundary components of each T. The inclusion of the other two circles will then be given by $(r_0, r_1, \dots) \mapsto (r_0, r_1, \dots) \oplus (-r_0, -r_1, \dots) \oplus (0, 0, \dots)$, and by $(r_0, r_1, \dots) \mapsto (r_0, r_1, \dots) \oplus (0, 0, \dots) \oplus (-r_0, -r_1, \dots)$.

Now let Y be the subset of X given by taking every second copy of H from which X was built, so Y is an infinite disjoint union of copies of H. Let Z be the disjoint union of all the other pieces making up X. This gives a coarsely excisive decomposition of X, and we understand the K-theory for the pieces Y and Z. For $Y \cap Z$ we will need to compute $\tilde{\mathcal{K}}_*^{\mathbb{N}}(S^1 \sqcup S^1)$. Take the decomposition of $S^1 \sqcup S^1$ as the usion of S^1 with $S^1 \sqcup \{*\}$ with a point. As $\tilde{\mathcal{K}}_0^{\mathbb{N}}(S^1) = 0$ this decompoition gives $\tilde{\mathcal{K}}_1^{\mathbb{N}}(S^1 \sqcup S^1) = \tilde{\mathcal{K}}_1^{\mathbb{N}}(S^1) \oplus \tilde{\mathcal{K}}_1^{\mathbb{N}}(S^1)$ and $\tilde{\mathcal{K}}_0^{\mathbb{N}}(S^1 \sqcup S^1) = \tilde{\mathcal{K}}_0^{\mathbb{N}}(S^1 \sqcup \{*\})$. Decomposing $S^1 \sqcup \{*\}$ as $S^1 \cup S^0$ gives $\tilde{\mathcal{K}}_0^{\mathbb{N}}(S^1 \sqcup S^1) = \tilde{\mathcal{K}}_0^{\mathbb{N}}(S^0)$.

Note that there is an inclusion of the ray into X and a retraction from X to the ray. This takes Y, Z to disjoint unions of intervals, and with $Y \cap Z$ mapping to $\mathbb{N} + 1$. Hence we may reduce the exact sequence for $X = Y \cup Z$, by the exact sequence for this decomposition of the ray. That is, we take the kernels of the maps induced by the retraction, and the original sequence is the direct sum of the reduced sequence with the exact sequence for the decomposition of the ray. Applying example 3.21 we see that this reduction replaces $K_*C^*(Y \cap Z)_0$ with $\tilde{\mathcal{K}}_*^{\mathbb{N}}(S^1 \sqcup S^1)$ and replaces $K_*C^*Y_0 \oplus K_*C^*Z_0$ with

$\tilde{\mathcal{K}}_*^{\mathbb{N}}(H) \oplus \tilde{\mathcal{K}}_*^{\mathbb{N}}(H)$. The other terms in the sequence remain unaffected, as the K-theory for the ray vanishes. We obtain the following exact sequence:

$$\begin{array}{ccccc} \tilde{\mathcal{K}}_0^{\mathbb{N}}(S^1 \sqcup S^1) & \longrightarrow & \tilde{\mathcal{K}}_0^{\mathbb{N}}(H) \oplus \tilde{\mathcal{K}}_0^{\mathbb{N}}(H) & \longrightarrow & K_0 C^* X_0 \\ \uparrow & & & & \downarrow \\ K_1 C^* X_0 & \longleftarrow & \tilde{\mathcal{K}}_1^{\mathbb{N}}(H) \oplus \tilde{\mathcal{K}}_1^{\mathbb{N}}(H) & \longleftarrow & \tilde{\mathcal{K}}_1^{\mathbb{N}}(S^1 \sqcup S^1) \end{array}$$

Consider the map $\tilde{\mathcal{K}}_1^{\mathbb{N}}(S^1 \sqcup S^1) \to \tilde{\mathcal{K}}_1^{\mathbb{N}}(H) \oplus \tilde{\mathcal{K}}_1^{\mathbb{N}}(H)$. Elements of the latter group are given by two triples of sequences, modulo $\bigoplus \mathbb{Z}$. Corresponding to the fact that in constructing X from Y, Z we are interleaving the components of one with the other, let us instead represent elements of the group by a single triple of sequences, given by interleaving each sequence of the first triple with the corresponding sequence of the second. Then the map is given by

$$\theta \colon (r_0, r_1, \dots) \oplus (s_0, s_1, \dots) \mapsto$$
$$(s_0, s_0 + s_1, s_1 + s_2, \dots) \oplus (0, r_0 - s_0, r_1 - s_1, \dots) \oplus (r_0 - s_0, r_1 - s_1, \dots)$$

modulo $\bigoplus \mathbb{Z}$. An element of the kernel may be represented by a pair of the form $(r, -r, r, -r, \dots) \oplus (-r, r, -r, r, \dots)$, hence as $\tilde{\mathcal{K}}_0^{\mathbb{N}}(H) = 0$, we get $K_* C^* X_0 \cong \mathbb{Z}$. We may think of the generator as being analogous to that for $S^1 \times \mathbb{R}_+$, indeed we may consider it to be an 'orientation class' for the manifold. Elements of the image are all those triples with the third sequence given by a left shift of the second. The image of the map $\tilde{\mathcal{K}}_1^{\mathbb{N}}(H) \oplus \tilde{\mathcal{K}}_1^{\mathbb{N}}(H)$ to $K_1 C^* X_0$ is isomprhic to the quotient of $\tilde{\mathcal{K}}_1^{\mathbb{N}}(H) \oplus \tilde{\mathcal{K}}_1^{\mathbb{N}}(H)$ by the image of θ, and we see that this is $\prod \mathbb{Z} / \bigoplus \mathbb{Z}$.

Now consider the inclusion of $S^1 \times \mathbb{N}$ into X, as loops around the holes of the ladder. The decomposition of this space given by intersection with the pieces Y, Z, is the decomposition into pieces of the form $I \times \mathbb{N}$. The boundary map for the reduced sequence for this decomposition is the suspension isomorphism. On the other hand as the inclusion $S^0 \hookrightarrow S^1 \sqcup S^1$ induces an isomorphism on $\tilde{\mathcal{K}}_0^{\mathbb{N}}$, we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{K}}_1^{\mathbb{N}}(S^1) & \xrightarrow{\cong} & \tilde{\mathcal{K}}_0^{\mathbb{N}}(S^0) \\ \downarrow\cong & & \downarrow\cong \\ K_1C^*X_0 & \longrightarrow & \tilde{\mathcal{K}}_0^{\mathbb{N}}(S^1 \sqcup S^1) \end{array}$$

This provides a splitting of the short exact sequence

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_{\mathbb{N}} \mathbb{Z} \to K_1C^*X_0 \to \tilde{\mathcal{K}}_0^{\mathbb{N}}(S^1 \sqcup S^1) \to 0,$$

and hence we get $K_1C^*X_0 \cong \prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_{\mathbb{N}} \mathbb{Z} \oplus \prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_{\mathbb{N}} \mathbb{Z}$, where the two sequences of 'generators' are given by the inclusions of the loops through the holes, and the inclusions of loops around the holes of the ladder.

Chapter 4

The C_0 Index

In this chapter we will associate an index in the K-theory of C^*X_0 to certain differential operators on bundles over a Riemannian manifold X. The main result of this chapter is a vanishing theorem giving certain conditions under which this index will be zero. In particular we will establish that there is an index associated to the real spinor Dirac operator over a complete Riemannian manifold, and that this vanishes if the manifold has positive scalar curvature tending to infinity.

Throughout the chapter we will assume X to be a separable smooth manifold, equipped with a complete Riemannian metric. The metric will be denoted $(\cdot,\cdot)$, while $\langle\cdot,\cdot\rangle$ will denote the inner product given by integration against the volume form. Completeness ensures that the metric is proper, and hence the standard and C_0 coarse structures on X derived from the Riemannian metric will also be proper.

4.1 Recap of the bounded coarse index

For comparison with the C_0 construction, we will begin with a brief discussion of the coarse index in C^*X where X is given the standard metric coarse structure. This result appears in [19]. Some of the technical aspects of this discussion will be avoided for now, but will appear later in the chapter as needed for the development of the C_0 theory.

Let S be a bundle over a complete Riemannian manifold X and let D be an operator of Dirac type. Then using the functional calculus we produce a bounded operator $\chi(D)$ where χ is a continuous function on $\mathbb{R}$ tending to ± 1 at $\pm\infty$. It may be shown that this operator lies in D^*X. For any other such function χ', the difference $\chi - \chi'$ lies in $C_0(\mathbb{R})$ and consequently $\chi(D) - \chi'(D) = (\chi - \chi')(D)$ lies in C^*X. Likewise $\chi(D)^2 - 1$ also lies in C^*X. Hence for $q\colon D^*X \to D^*X/C^*X$ the quotient map, $q(\chi(D))$ is an involution in D^*X/C^*X, and this is independent of the choice of χ.

Associated to the short exact sequence

$$0 \to C^*X \to D^*X \to D^*X/C^*X \to 0,$$

is a long exact sequence in K-theory:

$$\cdots \to K_{m+1}(D^*X) \xrightarrow{q_*} K_{m+1}(D^*X/C^*X) \xrightarrow{\partial} K_m(C^*X) \to \ldots$$

The projection $(q(\chi(D)) + 1)/2$ gives an element in $K_0(D^*X/C^*X)$, and we define the index to be $\mathrm{Index}(D) = \partial[(q(\chi(D)) + 1)/2]$ in $K_1(C^*X)$. If the bundle has a $\mathbb{Z}_2$-grading with respect to which D is odd, then we will require χ to be an odd function. Then $q(\chi(D))$ is an odd involution, and so defines an element of $K_1(D^*X/C^*X)$. In this case we obtain an index in the even K-theory, again by applying the boundary map.

The square of the Dirac operator is given by the Weitzenbock formula, as the connection Laplacian plus a curvature term. In the case of the spinor Dirac operator, this term is multiplication by $\kappa/4$, where κ is the scalar curvature. Hence as the Laplacian is a positive operator, it follows that for a manifold with uniformly positive scalar curvature D^2 is strictly positive. Thus there is an interval about zero not meeting the spectrum of D.

Given these conditions we may choose χ to be identically 1 on the positive spectrum of D, and identically -1 on the negative spectrum. In which case $\chi(D)$ is itself an involution, and so defines an element in the K-theory of D^*X. The index is then given by applying $\partial \circ q_*$ to this. By exactness the index vanishes, and we therefore conclude that the index provides an obstruction to uniformly positive scalar curvature.

4.2 Operators of Dirac type

In this section we will give an overview of the theory of Dirac operators, cf. [12],[20]. In particular we will be interested in applying the functional calculus to them.

Definition 4.1. The *Clifford algebra* of a real vector space V, denoted $\mathrm{Cl}(V)$, is the quotient of the tensor algebra $\mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$ of V, by the relations $v \otimes v = -\|v\|^2$, for each v in V. We make the Clifford algebra into a *-algebra by defining $v^* = -v$ for v in V, and equip it with a grading by defining vectors v to be odd.

Note that the relations respect the involution and grading. When V is n-dimensional, the algebra has dimension 2^n; if $\{e_1, \dots, e_n\}$ is a basis for V, then $\{e_1^{i_1} e_2^{i_2} \dots e_n^{i_n} \mid i_1, \dots i_n \in \{0, 1\}\}$ is a basis for $\mathrm{Cl}(V)$. To a vector bundle S there is associated a bundle $\mathrm{Cl}(S)$ of Clifford algebras with fibre $\mathrm{Cl}(S_x)$ at x, and a local trivialisation of S gives rise to a local trivialisation of $\mathrm{Cl}(S)$.

Definition 4.2. A *Clifford bundle* over X is a Hermitian vector bundle S over X equipped with a bundle morphism from $\mathrm{Cl}(T^*X)$ to the bundle $\mathrm{End}(S)$ of endomorphisms of S, which is a *-algebra homomorphism on each fibre. A *graded Clifford bundle* is a Clifford bundle S equipped with a $\mathbb{Z}_2$-grading, and such that the given morphism also preserves the grading on the fibres. For ξ in T_x^*X, the map on S_x which this induces will be written $s \mapsto \xi s$ and will be referred to as *Clifford multiplication.*

Definition 4.3. A *Dirac connection* on a Clifford bundle S, is an affine connection ∇ on S such that for V a vector field on X and s_1, s_2, s sections of S

1. $V(s_1, s_2) = (\nabla_V s_1, s_2) + (s_1, \nabla_V s_2)$ and
2. $\nabla_V(\xi s) = \xi \nabla_V s + \nabla_V(\xi) s$ where the last connection term is the Levi-Civita connection on T^*X.

If S is graded, we further require that ∇ is compatible with this, that is ∇ splits to give connections separately on the positive and negative parts of S.

We may now define operators of Dirac type.

Definition 4.4. Let S be a Clifford bundle equipped with a Dirac connection. Then the *Dirac operator* D on S is the operator given locally by the formula

$$Ds = \sum_i \xi_i \nabla_{V_i} s$$

where $\{V_i\}$ is a local orthonormal frame for TX and $\{\xi_i\}$ is the dual frame.

It is straightforward to establish that this local definition does not depend on the choice of local orthonormal frame, and hence D is well defined. We note immediately that for S a graded Clifford bundle, the operator D is odd, as ∇_{V_i} is even and multiplication by ξ_i is odd.

Lemma 4.5. *Dirac operators are formally self-adjoint.*

Proof. Let S, ∇, and D be as above. Let s_1 and s_2 be sections of S, and let ν denote the volume form on X. We need to check that $\int((Ds_1, s_2) - (s_1, Ds_2))\nu = 0$. Let $\{V_i\}$ be a local orthonormal frame for TX and let $\{\xi_i\}$ be the dual frame. Given a point x in X we may choose these such that at x we have $\nabla_{V_i}\xi_i = 0$. Then

$$\begin{aligned}((Ds_1, s_2) - (s_1, Ds_2))_x &= \sum((\xi_i \nabla_{V_i} s_1, s_2) - (s_1, \xi_i \nabla_{V_i} s_2))_x \\ &= \sum((\nabla_{V_i}\xi_i s_1, s_2) + (\xi_i s_1, \nabla_{V_i} s_2))_x \\ &= \sum V_i(\xi_i s_1, s_2)_x\end{aligned}$$

Define a 1-form ω by $\omega(W) = (\zeta s_1, s_2)$, where $\zeta = (W, .)$. In coordinates $\omega(V_i) = (\xi_i s_1, s_2)$. Applying the Hodge star, we have locally

$$*\omega = \sum_i (\xi_i s_1, s_2)(-1)^{i-1}\xi_1 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_n,$$

where X has dimension n. Then $d(*\omega) = \sum V_i(\xi_i s_1, s_2)\xi_1 \wedge \cdots \wedge \xi_n = \sum V_i(\xi_i s_1, s_2)\nu$. Thus at the point x we have $(Ds_1, s_2) - (s_1, Ds_2)\nu = d(*\omega)$. As this equation does not depend on the choice of coordinates, it follows that this holds for all x. Thus the integral is 0 by Stokes' theorem. $\square$

We will now state some standard results about differential operators in general, and Dirac operators in particular. The following three results, which we will state without proof, appear in [9]. Other versions of these (in the context of compact manifolds) appear in detail in [20].

Proposition 4.6. *Let D be any formally self-adjoint first order linear differential operator on a bundle S over X. Let s be a compactly supported section in $L^2(X, S)$. Then s is in the minimal domain of D if and only if it is in the maximal domain. In particular if X is compact then the minimal and maximal domain are the same, so D is essentially self-adjoint.* □

Certainly any section may be approximated by compactly supported sections, and we may achieve this via multiplication by a sequence of smooth and compactly supported functions f_i on X. The difficulty that may in general arise, is with the commutators $[D, f_i]$. However for a Dirac operator, completeness of X guarantees the existence of such a sequence with the further property that $[D, f_i]$ tends to 0 in norm. From this and the previous two results, the following proposition may be derived.

Proposition 4.7. *Let S be a Clifford bundle over a complete Riemannian manifold X, and let D be a Dirac operator on S. Then D is essentially self-adjoint.* □

Given the essential self-adjointness of a Dirac operator D we will generally not distinguish between D and its unique closure. Essential self-adjointness also allows us to make use of the functional calculus to define bounded operators $\psi(D)$, for ψ a continuous bounded function on $\mathbb{R}$, or even a bounded Borel function.

One further result will be required; a result about ellipticity. Rather than define ellipticity here, we will just note that in particular all Dirac operators are elliptic. The domain of the closure of D may be equipped with a Hilbert space structure, and is called a Sobolev space. Essential self-adjointness implies D has real spectrum so $(D+i)^{-1}$ is a bounded operator on $L^2(X,S)$. Ellipticity implies moreover that this is bounded as a map from $L^2(X,S)$ to the Sobolev space. The inclusion of the Sobolev space into $L^2(X,S)$ is locally compact, and hence so is $(D+i)^{-1}$ as an operator on $L^2(X,S)$. The functional calculus generalises this to other functions of D. The essential part of this for our purposes is packaged in the following theorem.

Theorem 4.8. *Let D be a Dirac operator on a Clifford bundle S. Then for ψ in $C_0(\mathbb{R})$, the operator $\psi(D)$ on $L^2(X,S)$ is locally compact.* □

Another result arising from ellipticity is that whenever Ds is smooth, in fact the section s must itself be smooth. This is called elliptic regularity, and hints of this will arise later.

For the remainder of this section, S will be a Clifford bundle, equipped with a Dirac connection, and D will be the associated Dirac operator. The following two results demonstrates a relation between Dirac operators are coarse geometry, see [19].

Proposition 4.9. *The wave operator e^{itD} has propagation at most $|t|$.*

Proof. It suffices to show for K, K' closed subsets of X, with $d(K,K') > |t|$, and for s a section supported in K, that the support of $e^{itD}s$ does not meet K'. We will suppose that $t > 0$; replacing D with $-D$ will then give the full result.

Choose T with $t < T < d(K,K')$. There exists a smooth function f on X with $f(K) = \{1\}$ and $f(K') = \{0\}$, and with gradient everywhere less than $1/T$. Note that the commutator $Df - fD$ is given by $s \mapsto \sum V_i(f)\xi_i s$ for V_i, ξ_i an orthonormal frame and its dual. As the Clifford multiplication operators ξ_i have norm one, it follows from the bound on the gradient that $\|Df - fD\| < 1/T$.

Let ϕ be a smooth and non-decreasing function on $\mathbb{R}$ which is identically one on $\mathbb{R}_+$, and otherwise strictly less than one. Let $f_\tau(x) = \phi(f(x) + \tau/T - 1)$. Our aim is to show that $\frac{\partial}{\partial\tau}\langle f_\tau e^{i\tau D}s, e^{i\tau D}s\rangle \geq 0$ for $0 \leq \tau \leq t$. This inequality implies that $\langle f_\tau e^{i\tau D}s, e^{i\tau D}s\rangle$ is non-decreasing, while on the other hand it is at most $\langle e^{i\tau D}s, e^{i\tau D}s\rangle$ which is constant. Hence as these are the same for $\tau = 0$, we have $\langle (1 - f_\tau)e^{i\tau D}s, e^{i\tau D}s\rangle = 0$ for all $\tau \leq t$. In particular setting $\tau = t$, for x in the support of $e^{itD}s$, we must have $f(x) + t/T \geq 1$ and hence $f(x) > 0$. Thus as required, the support of $e^{itD}s$ does not meet K'.

Now we shall examine the inequality.

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\langle f_\tau e^{i\tau D}s, e^{i\tau D}s\rangle \\
&\qquad = \langle (\frac{\partial}{\partial\tau}f_\tau + f_\tau iD)e^{i\tau D}s, e^{i\tau D}s\rangle + \langle f_\tau e^{i\tau D}s, iDe^{i\tau D}s\rangle \\
&\qquad = \langle (\frac{\partial}{\partial\tau}f_\tau - i(Df_\tau - f_\tau D))e^{i\tau D}s, e^{i\tau D}s\rangle
\end{aligned}$$

The commutator calculation above shows

$$Df_\tau - f_\tau D = \phi'(f(x) + \tau/T - 1)(Df - fD) = T(\frac{\partial}{\partial\tau}f_\tau)(Df - fD).$$

As ϕ is increasing, $\frac{\partial}{\partial\tau}f_\tau$ is non-negative. The operator $iT(Df - fD)$ has norm at most one, and is self-adjoint, thus $1 - iT(Df - fD)$ is a positive operator. This commutes with the positive operator $\frac{\partial}{\partial\tau}f_\tau$, and hence their product is positive. This provides the necessary inequality. ☐

Making use of Fourier analysis we may now examine the propagation of operators produced from D via the functional calculus.

Proposition 4.10. *Let ψ be a continuous bounded function on $\mathbb{R}$ with compactly supported distributional Fourier transform $\hat{\psi}$. If $\hat{\psi}$ is supported in $[-R, R]$, then $\psi(D)$ has propagation at most R.*

Proof. Note that as $\hat{\psi}$ is compactly supported we may integrate it against any smooth function, not just Schwartz-class functions. We may define an operator T by $\langle s_1, Ts_2\rangle = \frac{1}{2\pi}\int \hat{\psi}(t)\langle s_1, e^{itD}s_2\rangle dt$. Let p_j be the polynomial partial sums for the exponential function. Then for fixed z and for t in a compact set, $p_n(itz) \to e^{itz}$ uniformly, and likewise for all derivatives. Let $\psi_j(z) = \frac{1}{2\pi}\int \hat{\psi}(t)p_j(itz)dt$. Then $\psi_j \to \psi$ pointwise, and hence $\psi_j(D) \to \psi(D)$ strongly.

On the other hand, we certainly have

$$\langle s_1, \psi_j(D)s_2\rangle = \frac{1}{2\pi}\int \hat{\psi}(t)\langle s_1, p_j(itD)s_2\rangle dt.$$

Consider sections s_2 in the image of $\chi_{[-N,N]}(D)$, for some N. Such sections are dense, and D is bounded on each of these spaces. For such s_2, the functions $\langle s_1, p_j(itD)s_2\rangle$ converge uniformly on compact sets, and likewise for all derivatives. Thus we conclude that $\langle s_1, \psi_j(D)s_2\rangle \to \langle s_1, Ts_2\rangle$ for all s_1 and for a dense space of sections s_2, as j tends to infinity. But this also converges to $\langle s_1, \psi(D)s_2\rangle$, and hence we must have $T = \psi(D)$.

It is now immediate from proposition 4.9, that $\psi(D)$ has the claimed propagation. If f, g are functions on X with supports of distance greater than R apart, then for $|t| \leq R$, we have $\langle fs_1, e^{itD} gs_2\rangle = 0$. Thus $\langle s_1, f\psi(D)gs_2\rangle = \langle fs_1, Tgs_2\rangle = \int \hat{\psi}(t) \cdot 0 dt = 0$, for all s_1, s_2, and hence $f\psi(D)g = 0$. □

Corollary 4.11. *For ψ in $C_0(\mathbb{R})$, the operator $\psi(D)$ is a norm limit of finite propagation operators.*

Proof. If ψ is C_0 then it is a uniform limit of C_0 functions with compactly supported Fourier transform.[1] Thus $\psi(D)$ is a norm limit of finite propagation operators. □

Combining this last result with elliptic regularity we deduce that for ψ in C_0, and for X equipped with the standard coarse structure, $\psi(D) \in C^*X$. The following result gives conditions under which $\psi(D)$ need not be locally compact, but is at least pseudolocal.

Proposition 4.12. *Let ψ be a function on $\mathbb{R}$ extending continuously to $[-\infty, \infty]$. Suppose that ψ is a C_0 perturbation of a function with compactly supported distributional Fourier transform. Then for X with the standard coarse structure, $\psi(D) \in D^*X$.*

Proof. Given our previous observations about C_0 functions, we may assume that ψ itself has compactly supported Fourier transform, say with support in $[-R, R]$. This ensures that $\psi(D)$ has finite propagation. Define $\psi_\varepsilon(z) = \psi(\varepsilon z/R)$. Then we have $\hat{\psi}_\varepsilon$ supported in $[-\varepsilon, \varepsilon]$.

[1] Schwartz-class functions are dense in C_0, and these in turn may be approximated by inverse Fourier transforms of C_c^∞ functions.

Given any compactly supported functions f and g on X with disjoint supports, we have $d(\mathrm{Supp}(f), \mathrm{Supp}(g)) > \varepsilon$ for some $\varepsilon > 0$. Then as $\psi_\varepsilon(D)$ has propagation at most ε, we have $f\psi_\varepsilon(D)g = 0$. But as ψ tends to a limit at $\pm\infty$, it follows that $\psi - \psi_\varepsilon$ in C_0, so $(\psi - \psi_\varepsilon)(D)$ is locally compact. Thus $f\psi(D)g = f(\psi - \psi_\varepsilon)(D)g$ which is compact. As this holds for all disjointly supported f, g, we have $\psi(D)$ pseudolocal as required. □

We conclude this section with a simple result about gradings.

Proposition 4.13. *Let S be a graded Clifford bundle, and D a Dirac operator on S. For ψ a continuous bounded function on $\mathbb{R}$, if ψ is even then $\psi(D)$ is even, and likewise if ψ is odd then $\psi(D)$ is odd.*

Proof. If ψ is even then $\psi(z) = \phi(z^2)$ for some bounded Borel function ϕ. As D is odd, D^2 is even, and so we may apply the functional calculus separately to D^2 on the positive and negative parts of S. Hence we obtain $\psi(D) = \phi(D^2)$ operating separately on $L^2(X, S^+)$ and $L^2(X, S^-)$. Thus $\psi(D)$ is even.

If ψ is odd then $\psi(z) = z\phi(z)$ for some function ϕ. Provided that ψ is differentiable at 0, it follows that ϕ is continuous. Then as ϕ is even, so is $\phi(D)$, and consequently $\psi(D) = D\phi(D)$ is odd. To be precise, it acts as an odd operator on a dense subspace, but is bounded and hence extends by continuity. The general result follows by approximating continuous functions by functions differentiable at 0. □

4.3 Definition of the C_0 coarse index

We can now proceed to construct the index. We will suppose throughout the section that S is a Clifford bundle over a complete Riemannian manifold X, and D is an operator of Dirac type on sections of the bundle. The bundle S may be graded or ungraded, and in the former case D will be an odd operator.

For constructing C^*X_0 and D^*X_0, we will use an ample representation on the Hilbert space of sections $L^2(X,S)$ in the ungraded case. In the graded case we will use an ample representation on the space $L^2(X,S^+)$ of positively graded sections. In each case the representation of $C_0(X)$ is given by pointwise multiplication.

Definition 4.14. A *chopping function* is a continuous function χ from $\mathbb{R}$ to $[-1,1]$, such that $\chi(x) \to \pm 1$ as $x \to \pm\infty$. We will also require this to be odd unless otherwise stated.

We note that there exist chopping functions with compactly supported distributional Fourier transform, and every other chopping function is of course a C_0 perturbation of such. For example, consider the function $2/it$ interpreted as a distribution via principal value of the integrals. This is the Fourier transform of the sign function. Truncation of the distribution, smooths its inverse transform; for $\hat{\chi}(t) = 2/it$ on $[-1,1]$, and $\hat{\chi}(t) = 0$ elsewhere, we obtain for χ a real-valued odd function, tending to 0 at 0, and tending to ± 1 at $\pm\infty$.

Lemma 4.15. *Let S be a Clifford bundle over a complete Riemannian manifold X, and D an operator of Dirac type on S. Let $\{f_i\}$ be a locally finite partition of unity by compactly supported functions, and to each f_i associate a chopping function χ_i. Then* *the sum,*

$$T = \sum_i f_i^{1/2} \chi_i(D) f_i^{1/2},$$

converges strongly, and is a locally compact perturbation of $\chi(D)$ *for any chopping function* χ.

Proof. For convergence it suffices to show this separately for the two sums obtained by replacing each χ_i with respectively its positive and negative part. Thus we may suppose instead that each χ_i has image in $[0,1]$. In this case $0 \leq f_i^{1/2} \chi_i(D) f_i^{1/2} \leq 1$ for each i, so the sum will converge strongly provided that the partial sums $T_n = \sum_{i \leq n} f_i^{1/2} \chi_i(D) f_i^{1/2}$ are bounded in norm.

As T_n is positive, its norm is given by

$$\begin{aligned}
\|T_n\| &= \sup\{\langle \sum_{i \leq n} f_i^{1/2} \chi_i(D) f_i^{1/2} v, v\rangle \mid \|v\| \leq 1\} \\
&= \sup\{\sum_{i \leq n} \langle \chi_i(D) f_i^{1/2} v, f_i^{1/2} v\rangle \mid \|v\| \leq 1\} \\
&\leq \sup\{\sum_{i \leq n} \|\chi_i(D)\| \|f_i^{1/2} v\|^2 \mid \|v\| \leq 1\} \\
&\leq \sup\{\sum_{i \leq n} \langle f_i v, v\rangle \mid \|v\| \leq 1\} \\
&\leq 1
\end{aligned}$$

and hence we have the required bound.

The second part of the statement asserts that for any g in $C_0(X)$, and for χ any chopping function, $(T - \chi(D))g$ is compact. It suffices to check this for g in the dense subalgebra $C_c(X)$. The support of such g will meet the support of only finitely many

f_i in the partition of unity. Thus $(T - \chi(D))g = (T_n - \chi(D))g$, for n sufficiently large. Pseudolocality of $\chi(D)$, proposition 4.12, implies that

$$(T_n - \chi(D))g = \sum_{i \leq n} f_i^{1/2}(\chi_i(D) - \chi(D))f_i^{1/2} g,$$

modulo compacts. For each i, as $\chi_i - \chi$ tends to 0 at $\pm\infty$ it follows that $(\chi_i - \chi)(D)$ is locally compact (4.8). Hence each term is compact, so $(T - \chi(D))g$ is compact as required. □

As the above operator T is a locally compact perturbation of $\chi(D)$ it follows that it shares many of the properties of this, for example pseudolocality. We will show that though $\chi(D)$ need not lie in D^*X_0, for appropriate choices of f_i, χ_i, the operator T will have C_0-propagation, and so in particular it is in D^*X_0.

Proposition 4.16. *Let S be a Clifford bundle over a complete Riemannian manifold X, and D an operator of Dirac type on S.*

- *For any sequence of chopping functions χ_i, there exists a locally finite partition of unity f_i with compact supports, and such that $\sum_i f_i^{1/2}\chi_i(D)f_i^{1/2}$ has C_0 propagation.*

- *Conversely for any such partition of unity f_i, there is a sequence of chopping functions χ_i, such that the sum again has C_0 propagation. Specifically we may take $\chi_i(t)$ to be $\chi(t/t_i)$ for any chopping function χ with compactly supported fourier transform and any sequence t_i with $y_i \to \infty$.*

- *For any such partition of unity f_i, and for $\chi_i(t) = \chi(t/t_i)$ where χ is a chopping function with no support restriction and t_i is a sequence with $t_i \to \infty$, the operator $\sum_i f_i^{1/2} \chi_i(D) f_i^{1/2}$ iss a norm limit of C_0 propagation operators.*

Proof. First, let χ_i be any sequence of chopping functions. We observed in chapter 1 remark 2.7, that for a proper metrizable coarse space, coarse and topological separability agree. Hence X has a C_0-uniformly bounded countable open cover. Less abstractly, there is a sequence of relatively compact open sets U_i, with diameters tending to zero, and such that $\bigcup_i U_i = X$. By paracompactness, for each such cover, there exists a locally finite partition of unity $\{f_i\}$ subordinate to it.

For a strongly convergent sequence of operators T_n, the support of the limit lies in the closure of $\mathrm{LimSup}_\mathrm{n} \, \mathrm{Supp}(T_n)$, by lemma 2.23. Thus for any sequence of chopping functions, and for a partition of unity subordinate to $\{U_i\}$, the operator $\sum_i f_i^{1/2} \chi_i(D) f_i^{1/2}$ is supported in $\overline{\bigcup_i U_i \times U_i}$. As the cover is C_0-uniformly bounded, by definition $A = \bigcup_i U_i \times U_i$ is C_0-controlled. As A contains an open neighbourhood of the diagonal the composition $A \circ A$ contains $\overline{A}$. Hence $\overline{A}$ is also C_0-controlled and the operator has C_0 propagation.

For the second statement suppose $\{f_i\}$ is any locally finite partition of unity with compact supports. Let χ be a chopping function with compactly supported distributional Fourier transform, and let t_i be a sequence of positive real numbers tending to infinity. We define $\chi_i(t) = \chi(t/t_i)$. If $\hat{\chi}$ is supported in $[-N, N]$, then $\hat{\chi}_i$ is supported in $[-N/t_i, N/t_i]$. Hence by proposition 4.10, $\chi_i(D)$ has propagation at most N/t_i.

Let $r_{ij} = \max\{N/t_k \mid \mathrm{Supp}(f_k) \text{ meets } \mathrm{Supp}(f_i) \text{ and } \mathrm{Supp}(f_j)\}$ and let $r(x, y) = \sum_{i,j} r_{ij} f_i(x) f_j(y)$. For any $\varepsilon > 0$, there will exist only finitely many pairs (i, j) such that $r_{ij} \geq \varepsilon$. This follows from the assumption that the partition of unity is locally finite and each f_i is compactly supported. Hence r is C_0. For (x, y) in the support of $\sum_i f_i^{1/2} \chi_i(D) f_i^{1/2}$, we must have (x, y) in the support of $f_k^{1/2} \chi_k(D) f_k^{1/2}$ for some k. Thus $d(x, y) \leq N/t_k \leq r_{ij}$ for all (i, j) with $f_i(x) f_j(y) \neq 0$. Hence $d(x, y) \leq r(x, y)$ for all (x, y) in the support of $\sum_i f_i^{1/2} \chi_i(D) f_i^{1/2}$, and so this has C_0 propagation as required.

For the final statement, let χ be any chopping function, and let χ' be a chopping function with compactly supported fourier transform. Let t_i be a sequence tending to infinity, let f_i be a locally finite partition of unity, and let $\chi_i(t) = \chi(t/t_i), \chi_i'(t) = \chi'(t/t_i)$. We know that $\sum_i f_i^{1/2} \chi_i'(D) f_i^{1/2}$ has C_0 controlled propagation, thus it suffices to establish that for $\psi_i(t) = \chi_i(t) - \chi_i'(t) = (\chi - \chi')(t/t_i)$, the operator $\sum_i f_i^{1/2} \psi_i(D) f_i^{1/2}$ is a norm limit of C_0 propagation operators. But $\psi = \chi - \chi'$ is C_0, and so is a uniform limit of C_0 functions with compactly supported Fourier transform. By the norm estimates in the proof of lemma 4.15, this uniform convergence will give norm convergence of operators, so $\sum_i f_i^{1/2} \psi_i(D) f_i^{1/2}$ is a norm limit of C_0 propagation operators as claimed. □

To define the index, we shall now make a few further observations about the operators that we have constructed, especially in the graded case.

Proposition 4.17. *Let χ_i be chopping functions, and $\{f_i\}$ a partition of unity, such that the operator $T = \sum_i f_i^{1/2} \chi_i(D) f_i^{1/2}$ lies in $D^* X_0$. Let q denote the quotient map $q \colon D^* X_0 \to D^* X_0 / C^* X_0$. Then $q(T)$ is an involution which is independent of the choice of chopping functions and of partition of unity, and which is odd if a grading is provided.*

In the graded case there exists a unitary $V\colon L^2(X,S^+)\to L^2(X,S^-)$ *covering the identity on* X*, and we may write* $T=\begin{pmatrix}0 & (VU)^*\\ VU & 0\end{pmatrix}$*, with* U *an operator on* $L^2(X,S^+)$*. Then* $q(U)$ *is unitary, and the class* $[q(U)]$ *in* K*-theory is independent of the choice of* V*.*

Proof. As χ^2-1 is in $C_0(\mathbb{R})$, modulo locally compact operators $\chi(D)^2$ is the identity, and certainly $\chi(D)$ is self adjoint, thus $\chi(D)$ is an involution modulo locally compact operators. In the graded case we noted in the previous section that $\chi(D)$ is odd. Both these properties hold for any operator T constructed as above: we again have the former as T is a locally compact perturbation of $\chi(D)$ (lemma 4.15), while the latter follows as the multiplication operators $f_i^{1/2}$ are even. Any two such constructions must be locally compact perturbations of one another, and so $q(T)$ is independent of the choices made.

All that remains is to check the additional assertions about the graded case. As the operator D is a Dirac operator, it must be the case that S^+ and S^- have the same fibre dimension. Hence as vector bundles, S^+ and S^- are locally isomorphic. For a Borel subset B of X, over which both S^+ and S^- are trivialised, the isomorphism of vector bundles gives rise to a unitary isomorphism $V_B\colon L^2(B,S^+|_B)\to L^2(B,S^-|_B)$. We may regard these two spaces as included in $L^2(M,S^\pm)$, and we note that they are invariant subspaces for the multiplication representations. Thus V_B exactly intertwines the multiplication representations i.e. $V_Bf=fV_B$ for all f in $C_0(X)$. By separability we may pick a countable partition of X into Borel sets over which S^+,S^- are trivialized. Thus we obtain direct sum decompositions of $L^2(M,S^\pm)$, with the summands paired by unitaries exactly intertwining the representations. The direct sum $V=\sum_B V_B$ is a unitary from $L^2(X,S^+)$ to $L^2(X,S^-)$ which intertwines the representations, and so in particular V covers the identity.

Let V be any unitary from $L^2(X,S^+)$ to $L^2(X,S^-)$ covering the identity. As T is odd and self-adjoint it has an off-diagonal matrix form and we may express the matrix entries as VU and $(VU)^*$. Modulo locally compact operators T^2 is the identity, while on $L^2(X,S^+)$ it is U^*U, and on $L^2(X,S^-)$ it is VUU^*V^*. Hence $q(U^*U)=1$, and as V covers the identity we also have $q(UU^*)=1$, i.e. $q(U)$ is a unitary as required. To see that $[q(U)]$ is independent of the choice of V all we need is to observe that different choices of V correspond to conjugation by a unitary on $L^2(X,S^+)$ which covers the identity. From chapter 1 (remark 2.35) this induces the identity on K-theory. □

We have now made all the observations necessary to define the C_0 index, and to see that it is well defined.

Definition 4.18. The C_0 *index* of a Dirac type operator D is defined as follows. Let $q\colon D^*X_0 \to D^*X_0/C^*X_0$ in the short exact sequence

$$0 \to C^*X \to D^*X \to D^*X/C^*X \to 0,$$

and let $\partial_m\colon K_m(D^*X/C^*X) \to K_{m-1}(C^*X)$ denote the boundary maps in the associated K-theory long exact sequence. Let $T=\sum_i f_i^{1/2}\chi_i(D)f_i^{1/2}$.

In the ungraded case the C_0 index of D is

$$\mathrm{Index}(D)=\partial_0[q((T+1)/2)] \in K_1(C^*X_0).$$

In the graded case the C_0 index of D is

$$\mathrm{Index}(D) = \partial_1[q(U)] \in K_0(C^*X_0),$$

where $T = \begin{pmatrix} 0 & U^*V^* \\ VU & 0 \end{pmatrix}$ for some unitary V from $L^2(M, S^+)$ to $L^2(M, S^-)$ covering the identity on X.

4.4 Spectral obstructions and spin

We will be particularly interested in Clifford bundles arising from spin structures. We will set up the machinery of spin structures, and define the spinor Dirac operator. We will then establish the Weitzenbock formula, relating the spinor Dirac operator to the scalar curvature. We will use this formula to establish a spectral obstruction to properly positive scalar curvature, defined below.

Definition 4.19. The group $\mathrm{Spin}(n)$ is the even part of the subgroup of the multiplicative group of $\mathrm{Cl}(\mathbb{R}^n)$ generated by vectors v with $\|v\| = 1$.

The group $\mathrm{Spin}(n)$ is a two-fold cover of $\mathrm{SO}(n)$, the covering being given by the map taking a in $\mathrm{Spin}(n)$ to the orthogonal map $v \mapsto \mathrm{Ad}_a(v)$.

Definition 4.20. A *spin structure* on an oriented n-dimensional manifold X is a principal $\mathrm{Spin}(n)$ bundle equipped with a morphism to the principal $\mathrm{SO}(n)$ bundle of positively oriented orthonormal frames of T^*X, with the underlying group homomorphism equal to the canonical covering map.[2] A manifold equipped with a spin structure is called a *spin manifold.*

Given a spin structure $P_{\mathrm{Spin}(n)} \xrightarrow{\eta} P_{\mathrm{SO}(n)}$ we may construct the balanced product $\hat{S} = P_{\mathrm{Spin}(n)} \times_{\mathrm{Spin}(n)} \mathrm{Cl}(\mathbb{R}^n)$, where $\mathrm{Spin}(n)$ acts on $\mathrm{Cl}(\mathbb{R}^n)$ by left multiplication. Applying a similar construction we have $T^*X = P_{\mathrm{SO}(n)} \times_{\mathrm{SO}(n)} \mathbb{R}^n$. The bundle $\hat{S}$ is a Clifford bundle as follows. Given smooth sections s of $\hat{S}$, and ξ of T^*X, we represent s as (p, a) and ξ as $(\eta(p), v)$, where p is a section of $P_{\mathrm{Spin}(n)}$, and v, a are functions on X with values in $\mathbb{R}^n, \mathrm{Cl}(\mathbb{R}^n)$ respectively. Then we may define ξs to be (p, va) modulo the group action. It is easy to show that this does not depend on the choice of representative and hence from a spin structure we obtain a well defined Clifford bundle.

Note that in addition to a left multiplication by sections of T^*X, the bundle $\hat{S}$ also has a right multiplication by $\mathrm{Cl}(\mathbb{R}^n)$. Let $\varepsilon_1, \dots, \varepsilon_n$ denote the standard basis vectors in $\mathbb{R}_n$. On the complexification of $\hat{S}$, right multiplication by $i\varepsilon_{n-1}\varepsilon_n$ defines an involution. The operator $(i\varepsilon_{n-1}\varepsilon_n + 1)/2$ gives the projection onto the bundle of $+1$ eigenspaces of $i\varepsilon_{n-1}\varepsilon_n$. By associativity the left multiplications by sections of T^*X commute with the right multiplication by $\mathrm{Cl}(\mathbb{R}^n)$, and hence with this projection. Each vector $\varepsilon_1, \dots, \varepsilon_{n-2}$ will also commute with the product $i\varepsilon_{n-1}\varepsilon_n$ and hence with the projection. Thus the $+1$ eigenspace of $i\varepsilon_{n-1}\varepsilon_n$ is an invariant subbundle for the left multiplications by sections of T^*X and the right multiplication by $\mathrm{Cl}(\mathbb{R}^{n-2})$, where $\mathbb{R}^{n-2}$ is included in R^n as the span of $\varepsilon_1, \dots, \varepsilon_{n-2}$.

[2] Strictly speaking a spin structure should be an equivalence class of these, and it must be shown that the construction of indices of the associated Dirac operators respects the equivalence relation. For our purposes however we may suppose a single representative has been chosen.

Restricting to the invariant subbundle of $+1$ eigenspaces halves the fibre dimension as $\varepsilon_{n-1}, \varepsilon_n$ give isomormorphisms between the positive and negative eigenspaces. This isomorphism also means that the full bundle can be reconstructed from the subbundle. If $n = 2k$ then we may repeat this restriction process k times. If $n = 2k + 1$ then once we have repeated this k times, we may carry out a further reduction using the grading of the bundle, as follows. Right multiplication by $i\varepsilon_1$ defines an isomorphism between the positively and negatively graded parts of the bundle. We take the positive part of the bundle and define a new left multiplication agreeing with the existing one for even elements of $\mathrm{Cl}(T^*M)$, and given for odd elements by composing the existing left multiplication with right multiplication by $i\varepsilon_1$.

Definition 4.21. The *spinor bundle* associated to a spin structure is the reduced Clifford bundle constructed above.

For an even-dimensional manifold of dimension $2k$, the spinor bundle has dimension 2^k and is graded. For an odd-dimensional manifold of dimension $2k + 1$, it has dimension 2^k and is ungraded.

We will now consider connections on the spinor bundle S. From a connection on $\hat{S}$ we obtain one on the complexification. This will restrict to a connection on S provided that the connection commutes with the right multiplication, that is $\nabla_V(s\varepsilon_j) = (\nabla_V s)\varepsilon_j$ for all j. In fact given a spin structure there exists a unique Dirac connection with this property. Suppose a positively oriented orthonormal frame $\xi_1, \dots, \xi_n$ is defined near a point x. Then this gives a local trivialisation of T^*X; in terms of principal bundles we obtain a section q of $P_{\mathrm{SO}(n)}$ near x, such that ξ_j is given by (q, ε_j) modulo the group action. As $P_{\text{Spin}(n)}$ is locally trivial, and as η induces a covering map on groups, in some possibly smaller neighbourhood U of x there exists a section p of $P_{\text{Spin}(n)}$ lifting q. This provides a trivialisation of $\hat{S}$, which is compatible with the trivialisation of T^*X in the sense that the Clifford multiplication on $\hat{S}|_U$ is just given by the Clifford multiplication of the trivialisation $U \times \text{Cl}(\mathbb{R}^n)$. For these trivialisations there is a formula for the Dirac connection.

Lemma 4.22. *For $P_{\text{Spin}(n)} \to P_{\text{SO}(n)}$ a spin structure, there exists a unique Dirac connection on $\hat{S} = P_{\text{Spin}(n)} \times_{\text{Spin}(n)} \text{Cl}(\mathbb{R}^n)$ that commutes with the right multiplication. Suppose T^*X and $\hat{S}$ are compatibly trivialised over some open subset U of X. Let s_1 denote the section of $\hat{S}$ with constant value $1 \in \text{Cl}(\mathbb{R}^n)$ for this trivialisation, and denote the frame for T^*X corresponding to the trivialisation by $\xi_1, \dots, \xi_n$. Then*

$$\nabla_V(s_1) = {}^1\!/_4 \sum_j (\xi_j \nabla_V \xi_j) s_1.$$

Proof. Write $\nabla_V s_1 = a s_1$. The inner product (s_1, s_1) is constant and hence $(\nabla_V s_1, s_1) = 0$, so a has no scalar term. For each j we may write $a = \xi_j a_j + b_j$, where a_j is odd, b_j is even (as ∇ preserves the grading), and ξ_j graded commutes with a_j, b_j. Then $\nabla_V(\xi_j s_1) = (\xi_j(\xi_j a_j + b_j) + \nabla_V \xi_j) s_1 = (-a_j + \xi_j b_j + \nabla_V \xi_j) s_1$, but on the other hand, $\nabla_V(\xi_j s_1) = \nabla_V s_1 \varepsilon_j = (\xi_j a_j + b_j)\xi_j s_1 = (a_j + \xi_j b_j) s_1$. Thus $2a_j = \nabla_V \xi_j$. As a_j is a vector for each j, there are no terms in a which are a product of more than two linearly independent vectors, so in fact all terms are given by precisely two linearly independent vectors. Summing $\xi_j a_j$ will thus count every term twice, hence $\nabla_V s_1 = {}^1\!/_2 \sum (\xi_j a_j) s_1 = {}^1\!/_4 \sum (\xi_j \nabla_V \xi_j) s_1$.

Given this formula, uniqueness follows as we may write every section as a sum of sections of the form $s = f\xi_{j_1}\dots\xi_{j_m}s_1$, and then $\nabla_V s = V(f)\xi_{j_1}\dots\xi_{j_m}s_1 + f\nabla_V s_1\varepsilon_{j_1}\dots\varepsilon_{j_m}$. For existence it is clear that this formula for $\nabla_V s$, along with the above formula for $\nabla_V s_1$ locally defines a connection which commutes with the right multiplication. Uniqueness will imply compatibility of the local definitions, so it suffices to check that locally this is a *Dirac* connection. From the definition $(\nabla_V s_1, s_1) = 0$, which implies the metric compatibility. We need to check that $\nabla_V(\xi_j s) = \nabla_V(\xi_j)s + \xi_j\nabla_V s$. It suffices to check this for $s = s_1$, and then the condition amounts to checking that $as_1\varepsilon_j = (\nabla_V\xi_j + \xi_j a)s_1$. The calculation $(\nabla_V\xi_j, \xi_j) = 0$ shows that ξ_j anticommutes with $\nabla_V\xi_j$, while the definition of the Clifford algebra gives $\nabla_V(\xi_i)\xi_j + \xi_j\nabla_V\xi_i = -2(\nabla_V\xi_i, \xi_j)$. Note that $(\nabla_V(\xi_i), \xi_j) = -(\xi_i, \nabla_V\xi_j)$, so we have

$$\begin{aligned}
[a, \xi_j] &= {}^1\!/_4\xi_j(\nabla_V(\xi_j)\xi_j - \xi_j\nabla_V\xi_j) + {}^1\!/_4\sum_{i\neq j}\xi_i(\nabla_V(\xi_i)\xi_j + \xi_j\nabla_V\xi_i) \\
&= {}^1\!/_2\nabla_V\xi_j + {}^1\!/_2\sum_{i\neq j}\xi_i(\xi_i, \nabla_V\xi_j) \\
&= \nabla_V\xi_j.
\end{aligned}$$

This shows that $a\xi_j s_1 = (\xi_j a + \nabla_V\xi_j)s_1$ as required. □

We will now establish the Weitzenbock formula. In general this says that the square of a Dirac operator is given by the Laplacian $\nabla^*\nabla$ of the Dirac connection, plus a curvature term. In the case of the unique compatible connection on a spinor bundle, the latter is a scalar; it is a multiple of the scalar curvature of the manifold. We will denote the Riemann curvature tensor by R. By definition $R(V, W) =$

$\nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}$. For V_j an orthonormal frame we express this in coordinates as $R(V_i, V_j)V_k = \sum_l R_{lkij} V_l$.[3] It will be convenient to apply $R(V_i, V_j)$ to covectors as well as vectors. For coordinates with $\nabla_{V_i} \xi_j = 0$ at a point x, we note that $(\nabla_{V_i} \nabla_{V_j} \xi_k)(V_l) = V_i((\nabla_{V_j} \xi_k)(V_l)) = V_i(-\xi_k(\nabla_{V_j} V_l)) = -\xi_k(\nabla_{V_i} \nabla_{V_j} V_l)$. Hence $R(V_i, V_j)\xi_k = -\sum_l R_{klij} \xi_l$ at x and as this is a tensor equation it holds for any choice of (orthonormal) frame, and for all x. The scalar curvature will be denoted by κ. This is a contraction of R; in orthonormal coordinates $\kappa = \sum_{i,j} R_{ijij}$.

Proposition 4.23. *Let S be a Clifford bundle over X and D a Dirac operator on S. Let $\{\xi_j\}$ be a local orthonormal frame for T^*X with dual frame $\{V_j\}$. Then $D^2 = \nabla^*\nabla + \sum_{i<j} \xi_i \xi_j R(V_i, V_j)$. If X is a spin manifold, and S the spinor bundle then $D^2 = \nabla^*\nabla + {}^1\!/_4 \kappa$.*

Proof. We will prove the general formula first. Note that as R is a tensor, the claimed formula for D^2 is independent of the choice of frames. To simplify calculations, we will choose a frame such that at a point x, we have $\nabla_{V_i} \xi_j = 0$ for all i, j. It follows that at that point $[V_i, V_j]$ vanishes simplifying the curvature term. From the assumptions we get,

$$\begin{aligned} D^2 s &= \sum_{i,j} \xi_i \nabla_{V_i} (\xi_j \nabla_{V_j} s) \\ &= \sum_{i,j} \xi_i \xi_j \nabla_{V_i} \nabla_{V_j} s \\ &= \sum_j -\nabla^2_{V_j} s + \sum_{i<j} \xi_i \xi_j (\nabla_{V_i} \nabla_{V_j} s - \nabla_{V_j} \nabla_{V_i} s). \end{aligned}$$

[3] All coordinates considered will be orthonormal, so we need not use raised/lowered indices R^l_{kij}.

We just need to check that the first term is $\nabla^*\nabla s$. We have $\nabla\colon C^\infty(S)\to C^\infty(T^*X\otimes S)$ given by $\nabla s=\sum \xi_j\otimes\nabla_{V_j}s$, and we need to compute the formal adjoint $\nabla^*\colon C^\infty(T^*X\otimes S)\to C^\infty(S)$. This is given in orthonormal coordinates by $\nabla^*(\xi_j\otimes s)=-\nabla_{V_j}s$. This may be checked in a similar way to the proof (lemma 4.5) that D is self-adjoint. A straightforward calculation shows that for ν the volume form, $(\nabla s,\xi_j\otimes s')\nu+(s,\nabla_{V_j}s')\nu=d(*\omega)$, where ω is defined by $\omega(V_i)=\delta_{ij}(s,s')$. Thus $\langle\nabla s,\xi_j\otimes s'\rangle=\langle s,-\nabla_{V_j}s'\rangle$, which establishes the formula for the formal adjoint. Thus $\nabla^*\nabla s=\sum_j\nabla^*(\xi_j\otimes\nabla_{V_j}s)=-\sum_j\nabla^2_{V_j}s$ as required.

For the second assertion, we may consider the square of the Dirac operator on $\hat{S}$, as the operator D^2 on the spinor bundle is a restriction of this. We will make the same convenient choice of coordinates, and will make use of the above lemma. Let K denote the curvature term. Every section may be written locally as a sum of sections of the form $s=fs_1\varepsilon_{j_1}\dots\varepsilon_{j_m}$ and so it suffices to check that $Ks={}^1\!/_4\kappa s$ for such sections. As ε_j commutes with the connection for all j, we need only consider fs_1. The only asymmetric term in $\nabla_{V_i}\nabla_{V_j}fs_1=V_iV_j(f)s_1+V_i(f)\nabla_{V_j}s_1+V_j(f)\nabla_{V_i}s_1+f\nabla_{V_i}\nabla_{V_j}s_1$ is the last one, so we have $Kfs_1=fK(s_1)$, and it suffices to check that $K(s_1)={}^1\!/_4\kappa s_1$. From the lemma we have $\nabla_{V_j}s_1={}^1\!/_4\sum_k(\xi_k\nabla_{V_j}\xi_k)s_1$, so $\nabla_{V_i}\nabla_{V_j}s_1={}^1\!/_4\sum_k(\nabla_{V_i}(\xi_k)\nabla_{V_j}\xi_k+\xi_k\nabla_{V_i}\nabla_{V_j}\xi_k)s_1$. The former term vanishes by the assumptions on the coordinates, so we have $4Ks_1=\sum_{i<j}\sum_k\xi_i\xi_j\xi_k(\nabla_{V_i}\nabla_{V_j}\xi_k-\nabla_{V_j}\nabla_{V_i}\xi_k)s_1$.

It will be convenient to sum over all i,j, which given the symmetries will count all terms twice. In coordinates we have $4Ks_1=-{}^1\!/_2\sum_{i,j,k,l}\xi_i\xi_j\xi_k\xi_l R_{klij}$. We now use the symmetries of the curvature tensor. From the first Bianchi identity, terms with three of more distinct indices must cancel, as $\xi_i\xi_j\xi_k\xi_l$ is then invariant under 3-cycles of these indices, while the sum of R_{klij} cycling three indices gives zero. Antisymmetry of R_{klij} under transposition of k, l or of i, j, means that the only other non-zero terms are for $i = k \neq j = l$, and for $i = l \neq j = k$. The sums for both of these sets of indices are the same so we have $4Ks_1 = -\sum_{i,j} \xi_i\xi_j\xi_i\xi_j R_{ijij} = \sum_{i,j} R_{ijij} = \kappa$. □

Definition 4.24. A Riemannian manifold X has *properly positive scalar curvature* if the scalar curvature $\kappa\colon X \to \mathbb{R}$ is a proper function and has range in an interval $[\kappa_0, \infty)$ for some $\kappa_0 \in \mathbb{R}$.

We make the following definition for a not necessarily bounded operator, which agrees with the usual definition in the bounded case.

Definition 4.25. The *essential spectrum* of an essentially self-adjoint operator D is the set of points $\lambda \in \mathbb{C}$, for which either λ is a limit point of the spectrum of D, or λ is in the spectrum of D and the λ-eigenspace is infinite dimensional.

The following theorem provides a spectral obstruction to properly positivescalar curvature. In particular, any essential spectrum provides an obstruction to this.

Theorem 4.26. *Let X be a complete Riemannian spin manifold, S the spinor bundle over X, and D the Dirac operator on S. Suppose that X has properly positive scalar curvature. Then D has no essential spectrum, i.e. the spectrum of D is discrete, and the corresponding eigenspaces are finite dimensional.*

Proof. We will assume that X has scalar curvature tending to $+\infty$. If D has no essential spectrum then for any function ψ in $C_0(\mathbb{R})$, the operator $\psi(D)$ will be compact. The converse statement follows from the spectral theorem for compact operators, so we will prove compactness of $\psi(D)$ for $\psi \in C_0(\mathbb{R})$. For this it in turn suffices to prove the result for compactly supported functions ψ. We suppose that ψ is supported in $[-N, N]$, and define a projection $P = \chi_{[-N,N]}(D)$.

Let $\kappa\colon X \to \mathbb{R}$ denote the scalar curvature function. Note κ is bounded from below, so there exists $t \in \mathbb{R}$, with $\kappa + t > 0$. Let $k = (\kappa + t)^{1/2}$. The multiplication operator given by k is unbounded, but has bounded inverse. We will show that the operator kP is bounded, or equivalently that $k\colon PL^2(X, S) \to L^2(X, S)$ is bounded.

Given this, it is easy to establish the compactness of $\psi(D)$. Note that k^{-1} is in $C_0(X)$. Thus elliptic regularity (theorem 4.8) implies that $\psi(D)k^{-1}$ is compact. But $\psi(D) = \psi(D)P = \psi(D)k^{-1}kP$, where kP is bounded. Thus $\psi(D)$ is compact.

We will now show that kP is bounded. We have seen above the Weitzenbock formula, $D^2 = \nabla^*\nabla + \kappa/4$. So for any smooth section s of S we have, $4\langle Ds, Ds\rangle = 4\langle \nabla s, \nabla s\rangle + \langle \kappa s, s\rangle$. Using positivity of the connection term, and the definition of k we get an inequality, $\langle ks, ks\rangle \leq 4\langle Ds, Ds\rangle + t\langle s, s\rangle$.

Sections in $PL^2(X, S)$ are in the maximal domain of D. This is immediate from the construction of the Borel calculus. Thus such sections in fact lie in the minimal domain, by essential self-adjointness. So for $s \in PL^2(X, S)$, there is a sequence of smooth sections $s_j \to s$, and with $Ds_j \to \bar{D}s$. The closure of D is bounded on $PL^2(X, S)$, indeed it has norm at most N. Thus $\limsup_j \|ks_j\| \leq C\|s\|$, where $C^2 = 4N^2 + t$. For any compact subset K of X, the restrictions to K of ks_j converge in $L^2(K, S|_K)$ to the restriction of ks. As the norm of ks is the supremum of the norms of such restrictions, and likewise for ks_j, we conclude that $\|ks\| \leq C\|s\|$. Thus multiplication by k has norm at most C on $PL^2(X, S)$, so in particular we have established the claim that kP is a bounded operator. □

4.5 The index obstruction to properly positive scalar curvature

In this section we will prove a vanishing theorem for the C_0 coarse index. This is analogous to the fact that the standard coarse index vanishes given a spectral gap at 0. The results of the previous section provide geometric conditions under which the spectral conditions of the vanishing theorem will hold.

As in the construction of the index, we will suppose throughout that S is a Clifford bundle over X, with Dirac operator D, and that $C_0(X)$ is represented either on $L^2(X, S)$ or on $L^2(X, S^+)$, depending on whether S is graded.

The following lemma provides the main technical aspect of the vanishing theorem.

Lemma 4.27. *Suppose that the Dirac operator D has no essential spectrum. Then for any chopping function χ, the operator $\chi(D)$ is a compact perturbation of a C_0 propagation operator. In particular $\chi(D) \in D^*(X_0)$.*

Given that $\chi(D)$ is a compact perturbation of a C_0 propagation operator, to see that $\chi(D) \in D^*(X_0)$, write $\chi(D) = T + k$ with T of C_0 propagation, and k compact. By proposition 4.12, the operator $\chi(D)$ is pseudolocal, and certainly so is k. Hence T is pseudolocal and it has C_0 propagation by assumption, thus $T \in D^*(X_0)$. On the other hand by lemma 2.30 all compact operators lie in $C^*(X_0)$, so a foritori $k \in D^*(X_0)$.

We will postpone the proof of the lemma, and show first that it implies the vanishing theorem.

Suppose then that D has no essential spectrum. Let T be an operator defining the index of D. By lemma 4.15 if χ is a chopping function then $\chi(D)$ is a locally compact perturbation of T. On the other hand, by the above lemma $\chi(D)$ is in D^*X_0, so $q(T) = q(\chi(D))$ in D^*X_0/C^*X_0 where q is the quotient map $D^*X_0 \to D^*X_0/C^*X_0$. Hence $q(\chi(D))$ determines a K-theory element, say a, in either $K_0(D^*X_0/C^*X_0)$ or $K_1(D^*X_0/C^*X_0)$ depending on the grading, and $\partial(a) = \operatorname{Index} D$. In fact the spectral hypothesis also implies that $\chi(D)^2 = 1$ modulo compact operators and hence $\chi(D)$ determines a K-theory element, say a', in $K_*(D^*X_0/\mathfrak{K})$, and the map on K-theory induced from the quotient $D^*X_0/\mathfrak{K} \to D^*X_0/C^*X_0$ takes a' to a. We thus obtain a commutative diagram:

$$\begin{array}{ccc} a' \in K_{*+1}(D^*X_0/\mathfrak{K}) & \xrightarrow{\partial} & K_*(\mathfrak{K}) \\ \downarrow & & \downarrow \\ a \in K_{*+1}(D^*X_0/C^*X_0) & \xrightarrow{\partial} & K_*(C^*X_0) \ni \operatorname{Index}(D) \end{array}$$

Hence when D has no essential spectrum the index factors through $K_*(\mathfrak{K})$. We thus obtain the following theorem.

Theorem 4.28. *If S is an ungraded Clifford bundle over X, and D is a Dirac type operator with no essential spectrum, then* $\operatorname{Index}(D) = 0$ *in $K_1(C^*X_0)$. If S is graded and D has no essential spectrum then* $\operatorname{Index}(D)$ *lies in the image of i_* in $K_0(C^*X_0)$ where i denotes the inclusion of the compact operators into C^*X_0. More precisely identifying $K_0(\mathfrak{K})$ with $\mathbb{Z}$, the index is given by* $\operatorname{Index}(D) = i_*(n)$ *where n is the graded Fredholm index of D.*

Corollary 4.29. *Let X be an odd dimensional complete Riemannian spin manifold and let D be the spinor Dirac operator. If X has properly positive scalar curvature, then* $\mathrm{Index}(D) = 0$ *in* $K_1(C^*X_0)$.

Proof. From theorem 4.26 the hypothesis implies that D has discrete spectrum. The result is now immediate from the previous theorem. □

Corollary 4.30. *Let X be an even dimensional complete Riemannian spin manifold and let D be the spinor Dirac operator. If X has properly and uniformly positive scalar curvature, then* $\mathrm{Index}(D) = 0$ *in* $K_0(C^*X_0)$.

Proof. From the Weitzenbock formula, under the conditions of uniform positivity, D^2 is strictly positive. Thus D^2 has no kernel, and so D also has no kernel so the graded Fredholm index of D is zero. Theorem 4.26 ensures that D has discrete spectrum, hence the theorem applies and the index is $i_*(0)$. □

Let ι be the inclusion of a point into X. The algebra $C^*(pt)$ is given by the compact operators on some Hilbert space $\mathfrak{H}$. For K any compact subset of X with non-empty interior, a unitary from $\mathfrak{H}$ to $L^2(K, S^+|_K)$ will cover ι. The inclusion of the compact operators on $L^2(K, S^+|_K)$ into the compact operators on $L^2(X, S^+)$ induces an isomorphism on K-theory, so the image of i_* is the same as the image of ι_*.

Lemma 4.31. *Let X be a complete Riemannian manifold, and suppose that X has an unbounded component. Then there exists a coarse map from the ray $(\mathbb{R}_+)_0$ to X_0.*

Proof. We will inductively pick a sequence of points in an unbounded component, and paths between them. Let X_1 be an unbounded component of X, and pick a point x_1 in

X_1. Inductively, having defined x_j and $X_j \subseteq X_1$ we will pick an unbounded component X_{j+1} of $X_j \setminus B(x_1, j)$. We must therefore show that $X_j \setminus B(x_1, j)$ has an unbounded component. For each component C of $X_j \setminus B(x_1, j)$ not contained in $B(x_1, j+2)$ let $U_C = \{x \in C | j+1 < d(x, x_1) \leq j+3\}$, and let U_0 be the open ball about x_1 of radius $j+2$. Clearly the sets U_C along with U_0 cover the closed ball of radius $j+3$, but there is no proper subcover. Hence by compactness of the ball, there are only finitely many sets in the cover, and hence only finitely many components C of $X_j \setminus B(x_1, j)$ that are not contained in $B(x_1, j+2)$. As X_j is unbounded, so also is one of these components. Thus we take X_{j+1} to be some unbounded component of $X_j \setminus B(x_1, j)$, and pick x_{j+1} in X_{j+1}.

Choose curves from x_j to x_{j+1} in X_j; this is possible as X_j is connected by definition. We will take these curves to be parametrized by path length. Define γ from $\mathbb{R}_+$ to X to be the join of all these curves, preserving the parametrization. Then the map does not increase distances, as for all $t < t'$ in $\mathbb{R}_+$, the restriction of γ to $[t, t']$ gives a path from $\gamma(t)$ to $\gamma(t')$ of length t. To check that it is coarse, all we additionally need is that it is proper. Given any compact subset K of X, there exists a j_0 such that K lies inside $B(x_1, j_0)$. Thus for all $j \geq j_0$, the set X_j does not meet K. Let t_0 be the minimum of the interval in $\mathbb{R}_+$ corresponding to the curve from x_{j_0} to x_{j_0+1}. Then the restriction of γ to $[t_0, \infty)$ lies within X_{j_0}, hence does not meet K. The preimage of K is thus compact, and hence γ is proper. $\square$

This gives rise to a final corollary.

Corollary 4.32. *Let X be an even dimensional spin manifold with scalar curvature tending to $+\infty$ at infinity, and with an unbounded component. Then for D the spinor Dirac operator,* $\mathrm{Index}(D) = 0$.

Proof. Let γ be a coarse map from $(\mathbb{R}_+)_0$ to X_0. Let ι be the inclusion of $\gamma(0)$ into X. Then i factors through γ, thus the image of ι_* is contained in the image of γ_*. But from proposition 3.4 the K-theory for $C^*(\mathbb{R}_+)_0$ vanishes, thus this image is zero. Scalar curvature tending to infinity implies that the index lies in the image of ι_*, and hence the index vanishes. □

We will conclude with the proof the main lemma.

Proof of Lemma 4.27. We are given a function χ, and we must establish that $\chi(D)$ is a compact perturbation of a C_0-controlled operator. Define boundedly controlled neighbourhoods B_j of the diagonal by $B_j = \{(x,y) \in M \times M \mid d(x,y) \leq 2^{-j}\}$.

Claim: for $j = 1, 2, \ldots$ there exist operators T_j over M, and closed subsets A_j of $M \times M$ and C_j of M, such that

- $T_j - \chi(D)$ is compact;
- $\|T_j - T_{j-1}\| \leq 2^{-j}$ for $j > 1$;
- $\mathrm{Support}(T_j)$ is contained in A_j;
- $A_j \subseteq A_{j-1}$ for $j > 1$;
- $B_j \subseteq A_j \subseteq B_j \cup C_j \times C_j$;
- C_j is compact.

Fig. 4.1. The induction step from A_j, C_j to A_{j+1}, C_{j+1}

Once we have established the claim, we proceed as follows. Note that T_j is a Cauchy sequence and hence converges to an operator T. For all j, the difference $T_j - \chi(D)$ is compact, and hence the limit $T - \chi(D)$ is also compact. We also want T to be C_0-controlled, and we observe that $\mathrm{Support}(T) \subseteq \bigcap_j A_j$ by lemma 2.23. For any $\varepsilon > 0$ there exists j with $2^{-j} < \varepsilon$. For $(x, y) \in \mathrm{Support}(T) \setminus (C_j \times C_j)$ we must have $(x, y) \in B_j$, and hence $d(x, y) < \varepsilon$. As C_j is compact, the operator T is C_0-controlled as required.

We will now prove the claim. Let χ_j be a sequence of normalizing functions, with distributional fourier transform $\hat{\chi}_j$ supported in $[-2^{-j}, 2^{-j}]$. Then $\chi_j(D)$ has propagation at most 2^{-j} for each j.

We prove the claim by induction on j. Let $T_1 = \chi_1(D)$, let $A_1 = B_1$, and let $C_1 = \varnothing$. The difference $T_1 - \chi(D)$ is given by $(\chi_1 - \chi)(D)$ which is compact as D has no essential spectrum. As $\chi_j(D)$ has propagation at most 2^{-1} we have $\mathrm{Support}(T_1) \subseteq A_1$, and the rest of the conditions are immediate.

We now carry out the induction step. We assume that T_j, A_j, and C_j have been constructed, and we will construct T_{j+1}, A_{j+1}, and C_{j+1}. Let $K_j = \chi_{j+1}(D) - T_j$. This is compact as both terms are compact perturbations of $\chi(D)$. Here we again using the fact that D has no essential spectrum. For a sequence of compactly supported real-valued functions on M converging to 1 uniformly on compact sets, the corresponding multiplication operators converge strongly to 1. Thus multiplying by a compact operator will give a norm convergent sequence. In particular, given K_j we may find a compactly supported function $g_j \colon X \to [0, 1]$, such that $\|K_j - g_j K_j g_j\| \leq 2^{-(j+1)}$.

Let $T_{j+1} = T_j + K_j - g_j K_j g_j$, let $C_{j+1} = \text{Support}(g_j)$, and let $A_{j+1} = B_{j+1} \cup ((C_{j+1} \times C_{j+1}) \cap A_j)$. See figure 4.1. Certainly T_{j+1} is a compact perturbation of T_j, hence it is also a compact perturbation of $\chi(D)$. As $B_{j+1} \subseteq B_j \subseteq A_j$, it follows that $A_{j+1} \subseteq A_j$. We need to check that T_{j+1} is supported in A_{j+1}; the remaining conditions are immediate. Note that $T_{j+1} = \chi_{j+1}(D) - g_j K_j g_j$, and $\chi_{j+1}(D)$ is supported in B_{j+1}. It therefore suffices to show that $g_j K_j g_j$ is supported in $(C_{j+1} \times C_{j+1}) \cap A_j$. Certainly it is supported in $C_{j+1} \times C_{j+1}$, and it will in turn suffice to show that K_j is supported in A_j. But $K_j = \chi_{j+1}(D) - T_j$, and we already noted that the first term is supported in A_j, while the second is also supported in A_j by the induction hypothesis. This completes the proof of the claim. □

4.6 Non-vanishing examples of the index

In this section we will give an example illustrating the distinction between the bounded and C_0 coarse structures. This in will be an example where the C_0 higher index is non-vanishing, but the bounded higher index vanishes. We will also use the non-vanishing of certain C_0 higher indices to prove a theorem about perturbations of metrics on compact spin manifolds.

Our example will be the manifold illustrated in figure 4.2(a). This has uniformly positive scalar curvature and hence the bounded coarse index vanishes, however it does not have *properly* positive scalar curvature and indeed the C_0 coarse index does not vanish, so no C_0 perturbation of this metric has properly positive scalar curvature. The calculation of the index is simplified by the fact that this manifold is C_0-contractible as defined below.

Fig. 4.2. The product metric (a), and the cusp metric (b), on $S^2 \times \mathbb{R}^+$ capped by a hemisphere

Definition 4.33. A proper separable metric space X_0 equipped with the C_0-coarse structure is C_0*-contractible* if for every controlled set A in $X \times X$ there is a controlled set B in $X \times X$ such that for every subset U of X with $U \times U \subseteq A$ there is a subset V of X containing U with $V \times V \subseteq B$ and with the inclusion of U into V null homotopic.

Remark 4.34. The calculation of the index is an application of the coarse Baum-Connes conjecture. For C_0-contractible spaces, the C_0 coarse Baum-Connes conjecture claims that $K_*(C^*(X_0))$ is isomorphic to the K-homology group $K_*(X)$, and in this case the conjecture holds.

Theorem 4.35. *Let X denote the manifold illustrated in figure 4.2(a). Let X_0 denote X equipped with the C_0 coarse structure, and let D be the spinor Dirac operator on X. Then $K_1(C^*X_0) \cong \mathbb{Z}$ with generator* $\operatorname{Index} D$*. In particular this C_0 higher index is non-zero.*

Proof. The Dirac operator on X gives the generator for the K-homology group $K_1(X) \cong K_1(\mathbb{R}^3) \cong \mathbb{Z}$. This group may be computed by repeated suspension isomorphisms, starting from the K-homology of a point, $K_0(\mathrm{pt})$. We may regard X as being embedded in $\mathbb{R}^4$, and then splitting $\mathbb{R}^4$ into two half spaces, will split X into two spaces with trivial K-homology, see figure 4.3. The Mayer-Vietoris sequence then gives a suspension isomorphism from X to $X \cap \mathbb{R}^3$ which is a union of a hemisphere of S^2 with the product $S^1 \times \mathbb{R}^+$. Repeating the process two more times, by splitting $\mathbb{R}^3$ and then $\mathbb{R}^2$ into half-spaces, will reduce X to $X \cap \mathbb{R}^1$ which is a point. In each step, the boundary map takes the Dirac element in K-homology to the Dirac element for the corresponding spin structure on the intersection. Note that the Dirac operator for a point amounts to the zero operator on a 1-dimensional space graded as $\mathbb{C} \oplus 0$. This has graded index 1, and gives the generator for $K_0(\mathrm{pt})$. We then conclude that the K-homology is infinite cyclic with generator $[D]$ as claimed.

Fig. 4.3. The decomposition of $S^2 \times \mathbb{R}^+$ capped by a hemisphere

Note that these decompositions are C_0-excisive. However the final decomposition is *not* boundedly excisive, which is why the result fails for bounded coarse geometry. As the decompositions are C_0-excisive, there are corresponding Mayer-Vietoris sequences for $K_*C^*(X_0)$. The calculation 3.6 shows that the intersection of X with either of the half-spaces of $\mathbb{R}^4$ gives a space for which the corresponding C^*-algebra has trivial K-theory. It follows that this again induces a suspension isomorphism and the same is true for the decompositions if $\mathbb{R}^3$ and $\mathbb{R}^2$. Hence we may repeat the above computation of the K-homology to show that $K_*C^*(X_0)$ is obtained from $K_*C^*(\mathrm{pt})$ by a dimension shift. As $K_*(\mathrm{pt}) \cong K_*C^*(\mathrm{pt})$, combining the suspension isomorphisms we conclude $K_*(X) \cong K_*(C^*X_0)$, i.e. the coarse Baum-Connes conjecture holds for the space X with the C_0-structure. The isomorphism identifies $\operatorname{Index} D$ with the generator $[D]$ in $K_1(X)$ which completes the result. □

This contrasts with the following theorem, which is a specific case of the bounded version of the vanishing theorem from [19].

Theorem 4.36. *Let X denote the manifold illustrated in figure 4.2(a). Let X_b denote X equipped with the* bounded *coarse structure, and let D be the spinor Dirac operator on X. Then* $\operatorname{Index} D = 0$. □

We conclude by sketching an application of the C_0 vanishing theorem in the case where the components of X are compact.

Definition 4.37. Let d_1, d_2 be Riemannian metrics on a compact manifold C. The metrics are *ε-close* if $d_1(x,y) - \varepsilon \leq d_2(x,y) \leq d_1(x,y) + \varepsilon$ for all $x, y \in C$.

Definition 4.38. Let C be a compact Riemannian manifold, and let κ denote the scalar curvature of C. The Riemannian manifold C has *R-uniformly positive scalar curvature* if κ is bounded below by R.

Theorem 4.39. *Let C be any compact Riemannian spin manifold. Then there exists $R > 0, \varepsilon > 0$, such that no metric on C which is ε-close to the given metric, has R-uniformly positive scalar curvature.*

Note that it follows from the results of [13] that given C there exists $R > 0, \varepsilon > 0$ such that there is no R-uniformly positive metric on C for which the metric tensor differs from the given one by at most ε. Our result strengthens this, however the bounds we obtain are not explicit.

The corresponding statement for uniformly negative scalar curvature would be false. The existence of metrics of negative curvature approximating any given metric on a manifold of dimension at least three is discussed in [14].

Proof of 4.39. Let $X = C \times \mathbb{N}$ with the product metric, and let X_0 denote X with the C_0 structure. The statement of the theorem is equivalent to the statement that X admits no metric of properly positive scalar curvature, which is C_0 coarsely equivalent to the given one. Let us suppose that such a metric exists. On the complement of a finite set of components of X the metric will be uniformly positive, and by a further C_0 perturbation of the metric we may assume that the metric has both properly positive and uniformly positive scalar curvature. This further condition implies that the Fredholm index of the spinor Dirac operator is zero, and hence the vanishing theorem implies that $\text{Index}(D_X)$ vanishes in $K_*C^*(X_0)$.

We will now show that the index cannot vanish in this case. First let us suppsoe for simplicity that $C = S^m$. By example 3.23 we have $\mathrm{Ker}(K_m(C^*(S^m \times \mathbb{N})_0) \to K_m(C^*\mathbb{N}_0)) = \prod_{\mathbb{N}} \mathbb{Z}/\bigoplus_{\mathbb{N}} \mathbb{Z}$. As in 4.35, the boundary of the Dirac element in K-homology gives the Dirac element of the boundary, so the suspension isomorphism arising from the decomposition of S^m into two hemispheres takes $\mathrm{Index}(D_{S^m \times \mathbb{N}})$ to $\mathrm{Index}(D_{S^{m-1} \times \mathbb{N}})$. Hence inductively it suffices to show that the C_0 index of the Dirac operator on $S^0 \times \mathbb{N}$ is non-zero. But the Dirac operator on S^0 arising from the boundary map is positively oriented on one point, and negatively oriented on the other. Hence the Dirac operator for S^0 amounts to the zero operator on the 1-dimensional bundle over S^0 graded as $\mathbb{C} \oplus 0$ on one point, and $0 \oplus \mathbb{C}$ on the other. Thus $\mathrm{Index}(D_{S^0 \times \mathbb{N}}) = (1, -1, 1, -1, \dots) \in K_0(C^*(S^0 \times \mathbb{N})_0) \cong \prod_{S^0 \times \mathbb{N}} \mathbb{Z}/G$ where G denotes the subgroup of finitely supported sequences of sum zero. This in fact lies in the reduced theory $\mathrm{Ker}(K_m(C^*(S^m \times \mathbb{N})_0) \to K_m(C^*\mathbb{N}_0)) = \prod_{\mathbb{N}} \mathbb{Z}/\bigoplus_{\mathbb{N}} \mathbb{Z}$, where it is identified with $(1, 1, \dots)$ modulo $\bigoplus_{\mathbb{N}} \mathbb{Z}$. Hence we have shown that in the case where $C = S^m$, the index is non-zero in $K_*C^*(X_0)$ giving a contradiction.

For the general case let U be an open ball in C, let A be the closure of U, and let $B = C \backslash U$. Let $Y = A \times \mathbb{N}, Z = B \times \mathbb{N}$, and consider the decomposition $X = Y \cup Z$. This is excisive by lemma 3.14. The boundary map in the exact sequence will take $\mathrm{Index}(D_X)$ to $\mathrm{Index}(D_{S^{m-1} \times \mathbb{N}})$ where $m = \dim X$, as the boundary of the Dirac element in K-homology is the Dirac element of the boundary. However we have already shown that this is non-zero, so $\mathrm{Index}(D_X)$ is non-zero and this contradiction completes the proof. $\square$

Chapter 5

The Coarse Baum-Connes Conjecture

In this chapter we will give a proof of the coarse Baum-Connes conjecture for spaces of finite asymptotic dimension. The conjecture is:

Conjecture 5.1 (The Coarse Baum-Connes Conjecture). *If X is a bounded geometry metric space equipped with the bounded coarse structure then the assembly map $\mu : KX_*(X) \to K_*(C^*X)$ is an isomorphism.*

We will begin by defining the left hand side of the conjecture, the coarse K-homology of X denoted $KX_*(X)$, and then we will show that the index map from the previous chapter $D \mapsto \operatorname{Index} D \in K_*(C^*M)$ where M is a smooth manifold, generalizes to an assembly map $\mu : KX_*(X) \to K_*(C^*X)$. We will then make use of the C_0 coarse structure to reformulate the conjecture in such a way that the two sides are more easily comparable. We will see that the left hand side may be formulated as the K-theory of an ideal of C_0-controlled operators on a certain space, while the right hand side is the K-theory of a corresponding ideal of finite propagation operators. With this reformulation the assembly map is induced from the inclusion of the C_0 ideal into the bounded one.

In [25] Yu proved the following:

Theorem 5.2. *If X is a metric space with finite asymptotic dimension, then the coarse Baum-Connes conjecture holds for X.*

This chapter provides an alternative proof of Yu's theorem. The main idea of the proof is the interplay of the C_0 structure, the bounded structure, and a 'hybrid' structure which has certain properties in common with each of the other two.

5.1 Spherical metrics on simplicial complexes

Many of the constructions in this chapter will involve metric simplicial complexes. We will begin by defining a good class of metrics on a locally finite simplicial complex, and developing some of their properties.

Definition 5.3. The *spherical m-simplex* is the intersection of the m-sphere in $\mathbb{R}^{m+1}$ with the positive cone, equipped with the spherical path metric. Barycentric coordinates on this are defined by taking convex combinations of the vertices $(1,0,\ldots,0),\ldots,(0,\ldots,0,1)$, and then projecting radially onto the sphere.

The following lemma relates the spherical metric to the flat metric on a simplex.

Lemma 5.4. *Let σ_f be the m-simplex in $\mathbb{R}^{m+1}$ which is the convex hull of the points $(1,0,\ldots,0),\ldots,(0,\ldots,0,1)$, equipped with the inherited Euclidean metric. Let σ_s be the spherical m-simplex. Then the radial projection from σ_s to σ_f is contractive, while the radial projection σ_f to σ_s expands distances by a factor of at most ${}^1\!/_2\pi(m+1)^{1/2}$.*

Proof. Given two points x,y in σ_f let s be the distance from x to y, and let θ be the spherical distance between the projections of x,y onto σ_s, that is the angle between the rays from the origin through x and y. It is clear that $s\leq\theta$ which gives the first assertion.

Let a,b be the distances from the origin to x,y respectively. Note that $a,b\geq (m+1)^{-1/2}$, as the distance from the origin to the simplex σ_f is $(m+1)^{-1/2}$. The triangle defined by x, y and the origin has area given by ${}^1\!/_2 s(m+1)^{-1/2} = {}^1\!/_2 ab\sin\theta \geq {}^1\!/_2(m+1)^{-1}\sin\theta$. As $0 \leq \theta \leq {}^1\!/_2\pi$, we have $\theta \leq {}^1\!/_2\pi\sin\theta \leq {}^1\!/_2\pi(m+1)^{1/2}s$. This completes the calculation. □

Definition 5.5. A *uniform spherical* metric on a locally finite simplicial complex is a metric with the following properties:

- each simplex is isometric to the spherical m-simplex;
- the restriction of the metric to each component is a path metric;
- for all $R > 0$ there is a finite subcomplex K such that if (x, y) lies in the complement of $K \times K$, and if $d(x, y) < R$ then x and y are in the same component of the complex.

It is not difficult to see that such a metric must always exist, and indeed it is unique up to coarse equivalence (for the bounded coarse structure).

Lemma 5.6. *Let X be a locally finite simplicial complex with a uniform spherical metric. For any vertex v_0 and simplex σ in the same component of X, there exists a sequence of adjacent vertices $v_0, v_1, \ldots, v_k$ with $v_k \in \sigma$ and $d(v_0, \sigma) = k\pi/2$.*

Proof. Let k be the smallest number such that there exists a sequence of adjacent vertices $v_0, v_1, \ldots, v_k$ with $v_k \in \sigma$. It is clear that $d(v_0, \sigma) \leq k\pi/2$. We will prove by induction on k that we have equality. Certainly if $k = 0$ the result holds, for then $v_0 \in \sigma$ and $d(v_0, \sigma) = 0$.

For $k > 0$ let K be the union of all simplices spanned by vertices v connected to v_0 by a sequence of at most $k - 1$ adjacent edges in X. Let $x \in \sigma$ be a point minimizing $d(v_0, x)$, and let γ be a geodesic from v_0 to x. Now let y be the last point on γ lying in K. Let τ be the simplex of K containing y in its interior. Then inductively we know that $d(v_0, \tau) = (k-1)\pi/2$, so as $d(v_0, \sigma) \leq k\pi/2$ and γ is a geodesic, it follows that $d(x, y) \leq \pi/2$, and in particular $d(\sigma, \tau) \leq \pi/2$. To complete the result, it suffices to prove the following claim.

Claim. *If σ, τ are simplices in a simplicial complex with uniform spherical metric, and if $d(\sigma, \tau) \leq \pi/2$ then either $d(\sigma, \tau) = \pi/2$ or σ meets τ.*

To prove the claim let δ be a geodesic between closest points of τ, σ. It is clear that δ is piecewise linear and we will break it into linear segments. We will prove the result by induction on the number of segments. The result is clear for a single segment. Now suppose σ and τ do not meet, and that $d(\sigma, \tau) < \pi/2$. Let us remove the first segment of δ. Let z denote the end of this segment, and let σ' be the simplex containing z in its interior. Then by induction, as $d(\sigma', \tau) < \pi/2$ we must have $\tau \cap \sigma'$ non-empty. On the other hand certainly $\sigma \cap \sigma'$ is non-empty. Hence the shortest paths from z to τ and to σ lie in σ'. As $\sigma \cap \sigma'$ and $\tau \cap \sigma'$ are non-intersecting faces of σ' they are distance $\pi/2$ apart, and hence the length of the path δ is at least $\pi/2$. Thus in fact $d(\sigma, \tau) = \pi/2$ which completes the proof. □

Definition 5.7. Let (X, d) be a proper metric space. The *associated path-length metric* is

$$d_l(x, x') = \inf\{l(\gamma) : \gamma \text{ a path from } x \text{ to } x',\ \gamma\colon [0,1] \to X\}, \text{ where}$$

$$l(\gamma) = \sup\{\sum_i d(\gamma(t_i), \gamma(t_{i+1})) : 0 = t_0 < t_1 < \cdots < t_n = 1 \text{ a partition}\}.$$

Note that the path length $l(\gamma)$ is at least $d(\gamma(0), \gamma(1))$, and hence $d_l(x, x') \geq d(x, x')$.

Definition 5.8. Let (X, d) be a path metric space, and let Y be a closed subset of X. Let d_l be the path metric on Y associated to the metric inherited from X. The *distortion* of Y in X is

$$\sup \left\{ \frac{d_l(y, y')}{d(y, y')} : y, y' \in Y \right\}.$$

The *distortion* of a path $\gamma\colon [0,1] \to Y$ is $l(\gamma)/d(\gamma(0), \gamma(1))$.

We will need the following technical lemma relating different metrics.

Lemma 5.9. *Let X be a connected finite dimensional simplicial complex equipped with the uniform spherical metric. Let Y be a connected subcomplex of $X^{(n)}$, the n^{th} barycentric subdivision of X, and let $Y_\sigma = Y \cap \sigma$ for σ a simplex of X. Suppose that*

1. *for each simplex σ of X, the subcomplex Y_σ is connected, and*

2. *the 1-skeleton of X is contained in Y, or*

2'. *for all pairs of simplices σ, σ' in X, such that $Y_\sigma, Y_{\sigma'}$ are nonempty, the simplex σ meets σ' and the intersection $Y_\sigma \cap Y_{\sigma'} = Y_{\sigma \cap \sigma'}$ is non-empty.*

Then there is a finite bound on the distortion of Y in X, depending only on n and the dimension of X. That is, the inherited metric on Y, and the associated path-length metric on Y are bi-Lipschitz equivalent. Moreover, the uniform spherical metric on Y is also bi-Lipschitz equivalent to the other two metrics, and all the Lipschitz constants depend only on n and the dimension of X.

Proof. Let d_X denote the spherical metric on X, let d_Y be the inherited metric on Y, let d_l be the associated path metric on Y, and d_s the spherical metric on Y.

First we will observe that the distortion introduced by restricting to simplicial paths is bounded. Let $X^{(n)}$ denote the n^{th} barycentric subdivision of X, and let Z be the 1-skeletion of $X^{(n)}$. For x, x' in the interiors of simplices τ, τ' of $X^{(n)}$, let

$$d'_X(x, x') = \inf\{l(\gamma) : \gamma \text{ a path from } x \text{ to } x' \text{ in } \tau \cup Z \cup \tau'\},$$

where l denotes the path-length with respect to the metric d_X[1]. Note that for a single simplex τ of $X^{(n)}$ the boundary $\partial\tau$ has finite distortion. As there are only finitely many τ up to isometry, the distortion is bounded over all simplices τ. Now suppose that the $j+1$-skeleton of $X^{(n)}$ has finite distortion in $X^{(n)}$, which is true for j sufficiently large as X is finite dimensional. Then breaking a geodesic in the $j+1$-skeleton of $X^{(n)}$ into segments each contained in a single simplex τ we conclude that the j-skeleton of $X^{(n)}$ in the $j+1$ skeleton has finite distortion. Hence inductively the j-skeleton of $X^{(n)}$ has finite distortion in $X^{(n)}$.

Let c be a bound for all j and $j' \geq j$ on the distortion of the j-skeleton in the j' skeleton. Now for an arbitrary pair of points x, x' in X let τ, τ' be the simplices of $X^{(n)}$ containing x, x' in their interior, and let j be $\max\{\dim\tau, \dim\tau'\}$. Take a geodesic γ_j from x to x' through the j skeleton of $X^{(n)}$, and let z, z' be respectively the first and last points on the path which lie in the $j-1$-skeleton of $X^{(n)}$. From what we've already

[1]The function d'_X need not be a metric. We cannot compose such paths so the triangle inequality may fail. Note however that the restriction of d'_X to Z is a metric.

observed the distortion of γ in $X^{(n)}$ is at most c. On the other hand we also know there is a path γ_{j-1} from z to z' in the $j-1$ skeleton of $X^{(n)}$ with distortion at most c. Inductively we find that there is a constant c' such that for all x, x' and corresponding τ, τ' there is a path γ from x to x' in $\tau \cup Z \cup \tau'$ with distortion at most c'. Hence d_X and d'_X are bi-Lipschitz equivalent.

Now we will carry out a similar construction for d_l on Y. For y, y' in the interiors of simplices τ, τ' of Y, let $d'_l(y, y') = \inf\{l(\gamma) : \gamma \text{ a path from } x \text{ to } x' \text{ in } \tau \cup (Y \cap Z) \cup \tau'\}$, where Z is the 1-skeleton of $X^{(n)}$ as above. Let d'_Y denote the restriction of d'_X to Y. We will show that there is a path γ from y to y' in $\tau \cup (Y \cap Z) \cup \tau'$ and a bound on the 'distortion of γ relative to d'_Y' which is independent of y, y'. More precisely we will show there is a a constant c'' with $d'_l \leq c'' d'_Y$. Take a geodesic from y to y' in $\tau \cup Z \cup \tau'$ where $y \in \tau$ and $y' \in \tau'$. Now replace the segment of the path contained in the 1-skeleton of $X^{(n)}$ by a path contained in the 1-skeleton of Y. Hypotheses 2 or $2'$ along with 1 ensure that we can do this one simplex of X at a time, that is each segment lying in a single simplex σ of X can be replaced by a segment in $Z \cap \sigma$. There are no difficulties joining up over the boundaries, as we can connect via the 1-skeleton of X using hypothesis 2, or given $2'$, for y, y' in simplices σ, σ' of X we know that σ meets σ' and we can connect up via $Y_\sigma \cap Y_{\sigma'} = Y_{\sigma \cap \sigma'}$.

Let γ be the path so constructed. There are only finitely many simplicial isomorphism classes for pairs $(\sigma^{(n)}, Y_\sigma)$ where σ is a simplex of X, and $\sigma^{(n)}$ is its subdivision in $X^{(n)}$, hence there are only finitely many types of replacement occurring within the simplices of X. Thus there is a bound on the distortion of γ relative to d'_Y, depending only on the finite set of path-lengths for simplicial paths in a subdivided simplex $\sigma^{(n)}$

of X. Combining this with the distortion bound for d'_X relative to d_X we conclude that there a bound of the distortion of γ in X. As $d_l \leq d'_l$ this is moreover a bonud on the distortion of Y in X.

Finally we note that on a single simplex τ the metrics d_l, d_s are bi-Lipschitz equivalent, and again as there are only a finite number of cases, the Lipschitz constants may be chosen independent of τ. A geodesic for d_l may be divided up into segments each contained in a single simplex of Y. Each segment has d_l-length bounded by a constant multiple of its d_s-length, which gives one of the required Lipschitz inequalities. Interchanging the roles of d_s and d_l will give the other inequality, and hence we conclude that d_l and d_s are bi-Lipschitz equivalent. □

5.2 Coarse K-homology and assembly

In this section we will define the coarse K-homology $KX_*(X)$ and the assembly map $\mu : KX_*(X) \to K_*(C^*X)$.

We will now proceed to define the coarse K-homology of a space. This is a geometric process; the space will be replaced by a sequence of 'coarser' spaces, each of which is coarsely equivalent to the original, though perhaps topologically different. We will construct coarsenings of a space using open covers.

Definition 5.10. Let $\mathcal{U}$ be an open cover of a metric space X.

- The *degree* of $\mathcal{U}$ is the supremum over points $x \in X$ of the number of elements U of $\mathcal{U}$ containing x.

- The *diameter* of $\mathcal{U}$ is the supremum of the diameters of the sets U in $\mathcal{U}$.

- The *Lebesgue number* of $\mathcal{U}$, denoted Lebesgue($\mathcal{U}$) is the largest number R such that for every open subset V of X of diameter at most R, there exists $U \in \mathcal{U}$ with $V \subseteq U$.

Note that the diameter of $\mathcal{U}$ is finite if and only if $\mathcal{U}$ is uniformly bounded for the standard metric coarse structure.

Definition 5.11. An *anti-Čech sequence* for a metric space X is a sequence $\mathcal{U}_i$ of open covers of X with the properties that Lebesgue($\mathcal{U}_i$) tends to infinity, and $\mathrm{Diam}(\mathcal{U}_i) \leq$ Lebesgue($\mathcal{U}_{i+1}$).

Definition 5.12. The *nerve* $N_{\mathcal{U}}$ of a cover $\mathcal{U}$ is the simplicial complex defined abstractly to have the members of $\mathcal{U}$ as vertices, and with $(U_1, U_2, \dots, U_k)$ a simplex iff $U_1 \cap U_2 \cap \dots \cap U_k$ is non-empty.

If the degree of $\mathcal{U}$ is finite then so is the dimension of $N_{\mathcal{U}}$ and $\mathrm{Degree}(\mathcal{U}) = \mathrm{Dim}(N_{\mathcal{U}}) + 1$.

The anti-Čech property implies that for each $V \in \mathcal{U}_i$ there exists $U \in \mathcal{U}_{i+1}$ with $V \subseteq U$. Correspondingly there are simplicial *connecting* maps $\phi_i \colon N_{\mathcal{U}_i} \to N_{\mathcal{U}_{i+1}}$, given by mapping a vertex $[V]$ of $N_{\mathcal{U}_i}$ to a vertex $[U]$ of $N_{\mathcal{U}_{i+1}}$ with $V \subseteq U$. We make the convention that an anti-Čech sequence comes equipped with particular choices for these connecting maps.

Note that the anti-Čech property additionally implies that for each finite subcomplex K of $N_{\mathcal{U}_i}$ and for j sufficiently large depending on K, there is a vertex $[V]$ of$N_{\mathcal{U}_{i+j+1}}$ such that the complex $\phi_{i+j} \circ \dots \circ \phi_i(K)$ in $N_{\mathcal{U}_{i+j+1}}$ lies in the star of $[V]$. Indeed we can do better than this in the following sense.

Lemma 5.13. *For $\mathcal{U}_i$ an anti-Čech sequence of covers of a countable discrete metric space there exists a subsequence $\mathcal{U}_{i_k}$ and a sequence of connecting maps $\phi_{i_k}\colon N_{\mathcal{U}_{i_k}} \to N_{\mathcal{U}_{i_{k+1}}}$ such that for each k, and for each finite subcomplex K of $N_{\mathcal{U}_{i_k}}$, there exists $j \in \mathbb{N}$ such that $\phi_{i_{k+j}} \circ \cdots \circ \phi_{i_k}(K)$ is a single vertex.*

It will be convenient to assume that the maps have this property, and we will pass to an appropriate subsequence where necessary.

Proof. We can choose such a subsequence as follows. Let $K_1, K_2, \ldots$ be a list of all finite subcomplexes appearing in any of the nerves $N_{\mathcal{U}_i}$. Define $\mathcal{U}_{i_1}$ to be the cover whose nerve contains K_1 and let U_1 be the union of all sets V with $[V]$ a vertex of K_1. The set U_1 has finite diameter and hence is contained in some $U_1' \in \mathcal{U}_{i_2}$ for i_2 sufficiently large. We may now define ϕ_{i_1} to be a simplicial connecting map between the nerves of $\mathcal{U}_{i_1}$ and $\mathcal{U}_{i_2}$ with the property that each vertex of K_1 maps to $[U_1']$, i.e. K_1 is collapsed to a vertex.

Now inductively suppose that for some j, k, covers $\mathcal{U}_{i_1}, \ldots, \mathcal{U}_{i_k}$ have been chosen, and that each subcomplex $K_1, \ldots, K_j$ appearing in the nerve of one of these covers is collapsed to a vertex of the nerve of $\mathcal{U}_{i_k}$ by the composition of connecting maps so far constructed. If K_{j+1} lies in the nerve of one of the covers $\mathcal{U}_i$ which we have already skipped, that is $i < i_k$ but $i \neq i_{k'}$ for any $k' \leq k$, then move on to $j+2$. If K_{j+1} lies in some $\mathcal{U}_{i_{k'}}$ then let K_{j+1}' be the image of K_{j+1} in $N_{\mathcal{U}_{i_k}}$, and note that this is still a finite subcomplex. Otherwise let $\mathcal{U}_{i_{k+1}}$ be the cover whose nerve contains K_{j+1}, let ϕ_{i_k} be any simplical connecting map between the nerves of $\mathcal{U}_{i_k}$ and $\mathcal{U}_{i_{k+1}}$, and let $K_{j+1}' = K_{j+1}$. Now as for K_1 we may extend the sequence by one further cover and connecting map such that K'_{j+1} is collapsed to a vertex. The construction ensures that each K_j appearing in each $N_{\mathcal{U}_{i_k}}$ is ultimately collapsed to a vertex. $\square$

Definition 5.14. A metric space X is *uniformly discrete* if there exists $\varepsilon > 0$ such that $d(x_1, x_2) \geq \varepsilon$ for any two points $x_1 \neq x_2$ of X.

Definition 5.15. A uniformly discrete metric space X has *bounded geometry* if for all $R \in \mathbb{R}$, the number of points in a set $U \subseteq X$ of diameter at most R, is bounded independent of U.

Note that if X is a uniformly discrete space with bounded geometry and $\mathcal{U}$ is a cover of X with $\mathrm{Diam}(\mathcal{U}) < \infty$, then $N_{\mathcal{U}}$ is finite dimensional and locally finite, with a uniform bound on the number of simplices meeting at a point.

We can now define the coarse K-homology of a metric space X.

Definition 5.16. Let X be a bounded geometry metric space. The *coarse K-homology groups of X* are the groups $KX_*(X) = \varinjlim_i K_*(N_{\mathcal{U}_i})$[2] where $\mathcal{U}_*$ is any anti-Čech sequence, and the maps on K-homology groups are induced from the connecting maps $\phi_i \colon N_{\mathcal{U}_i} \to N_{\mathcal{U}_{i+1}}$.

Note that this does not depend on the choice of anti-Čech sequence, indeed the direct limit could be taken over the directed system of all open covers of finite diameter, equipped with the relation $\mathcal{U} \preceq \mathcal{U}'$ iff $\mathrm{Diam}(\mathcal{U}) \leq \mathrm{Lebesgue}(\mathcal{U}')$ or $\mathcal{U} = \mathcal{U}'$.

[2] When we refer to K-homology, we mean K-homology groups with locally finite supports. Using a compactly supported homology we would just get the homology of a point from this construction. The specific model for K-homology that we will use is the definition of K-homology groups given by the K-theory of the dual algebra for a space, see [9].

There is a similar definition of coarse K-homology $KX_*(X_0)$ for the C_0 coarse structure on a metric space X, or indeed for any coarse structure. The major difference is that the C_0 structure is not generated by a countable collection of its controlled sets, except in some trivial cases such as the case where X is compact. As the structure is not countably generated we cannot restrict our attention to sequences of covers but must instead use a directed system.

The collection of all covers of a space X is partially ordered by 'coarsening':

$$\mathcal{U}_1 \preceq \mathcal{U}_2 \text{ if for each } U_1 \in \mathcal{U}_1 \text{ there exists } U_2 \in \mathcal{U}_2 \text{ with } U_1 \subseteq U_2.$$

We will need to restrict this to locally finite open covers of a space X, and the following lemma allows us to do so.

Lemma 5.17. *Let X be a proper separable coarse space. Then for any uniformly bounded open cover $\mathcal{U}_1$ of X there exists a locally finite uniformly bounded open cover $\mathcal{U}_2$ with $\mathcal{U}_1 \preceq \mathcal{U}_2$.*

Proof. The axioms for a proper separable coarse space ensure the existence of an open cover $\{U_1, U_2, \dots\}$ such that $\bigcup_i U_i \times U_i$ is controlled, and hence each U_i is relatively compact. Thus the sequence $V_i = U_1 \cup \cdots \cup U_i$ is an exhaustion of X by relatively compact open sets. This shows that X is σ-compact, so in particular it is paracompact.

Thus there exists a locally finite refinement $\mathcal{U}$ of $\{U_1, U_2, \dots\}$, and we note that as $\{U_1, U_2, \dots\}$ is uniformly bounded, so is the refinement $\mathcal{U}$. Now given any controlled cover $\mathcal{U}_1$, we will show that there is a locally finite uniformly bounded open cover $\mathcal{U}_2$ with $\mathcal{U}_1 \preceq \mathcal{U}_2$. Let $C_1 = \bigcup_{U \in \mathcal{U}_1} U \times U$. This is controlled by assumption, and we will define $\mathcal{U}_2$ to be the C_1-thickening of $\mathcal{U}$, that is

$$\mathcal{U}_2 = \{C_1 \circ U : U \in \mathcal{U}\}$$

$$\text{where } C_1 \circ U = \{x : \text{there exists } y \in U \text{ with } (x, y) \in C_1\}.$$

We must show:

- $\mathcal{U}_2$ is uniformly bounded. This will follow from the uniform boundedness of $\mathcal{U}$,$\mathcal{U}_1$.
- $\mathcal{U}_2$ is locally finite. This will follow from local finiteness of the cover $\mathcal{U}$.
- $\mathcal{U}_1 \preceq \mathcal{U}_2$.

First let $C_2 = \bigcup_{U \in \mathcal{U}_2} U \times \mathcal{U}$ and let $C = \bigcup_{U \in \mathcal{U}} U \times \mathcal{U}$. We will show that $\mathcal{U}_2$ is uniformly bounded, that is, C_2 is controlled. But C, C_1 are controlled as $\mathcal{U}$,$\mathcal{U}_1$ are uniformly bounded, and $C_2 = C_1 \circ C \circ C_1$, hence this is also controlled.

Now we will show that $\mathcal{U}_2$ is locally finite. Let K be a compact subset of X and let K' be the C_1-thickening of K, that is $K' = C_1 \circ K$. If $C_1 \circ U$ meets K then there exists $(x, y) \in C_1$ with $x \in K$ and $y \in U$. Thus U meets K'. But K' which is relatively compact, so by local finiteness of $\mathcal{U}$ there are only finitely many $U \in \mathcal{U}$ for which U meets K'. Correspondingly there are only finitely many $C_1 \circ U \in \mathcal{U}_2$ which meet K, hence $\mathcal{U}_2$ is locally finite.

Finally we show that $\mathcal{U}_1 \preceq \mathcal{U}_2$. That is, for all $V_1 \in \mathcal{U}_1$ there exists $V_2 \in \mathcal{U}_2$ with $V_1 \subseteq V_2$. Given $V_1 \in \mathcal{U}_1$, pick a point $x \in V_1$. As $\mathcal{U}$ covers X, there is an open set $U \in \mathcal{U}$ with $x \in \mathcal{U}$. Then for all $y \in V_1$, we have $(y, x) \in V_1 \times V_1 \subseteq C_1$, and $x \in U$. Thus $y \in C_1 \circ U$, so for $V_2 = C_1 = circU$, we have $V_2 \in \mathcal{U}_2$ and $V_1 \subseteq V_2$. $\square$

Definition 5.18. Let $\mathcal{C}_0(X)$ denote the collection of all locally finite C_0-uniformly bounded open covers of X, directed by the above relation $\mathcal{U} \prec \mathcal{U}'$. The C_0 *coarse K-homology* of X is $KX_*(X_0) = \varinjlim_{\mathcal{U}\in\mathcal{C}_0(X)} K_*(N_{\mathcal{U}})$, where the maps on K-homology groups are induced from connecting maps $\phi\colon \mathcal{U} \to \mathcal{U}'$ for $\mathcal{U} \prec \mathcal{U}'$.

Note that this direct limit is well behaved with respect to spherical metrics on the nerves: Given two covers $\mathcal{U}$,$\mathcal{U}'$ of a space X with $\mathcal{U} \prec \mathcal{U}'$, and a simplicial connecting map $\phi\colon N_{\mathcal{U}} \to N_{\mathcal{U}'}$, there exist uniform spherical metrics on $N_{\mathcal{U}}, N_{\mathcal{U}'}$ for which ϕ is a contraction.

Definition 5.19. Let X be a locally compact topological space. Then the *K-homology of X at infinity* is $K_*^{\infty}(X) = \varinjlim_{C\subset X \text{ compact}} K_*(X/C)$, where the directed system is given by inclusions, and for $C \subseteq C'$ the map $K_*(X/C) \to K_*(X/C')$ is induced from the quotient map.

We will see below that in a wide range of cases the C_0 coarse K-homology groups are just given by the K-homology at infinity.

Proposition 5.20. *Let X be a proper, separable coarse space, and let $\mathcal{U}$ be a locally finite uniformly bounded open cover. Equip $N_{\mathcal{U}}$ with a uniform spherical metric, and the corresponding bounded coarse structure. Then there exist maps $\eta\colon N_{\mathcal{U}} \to X$ such that if $y \in \mathrm{Star}[V]$, then $\eta(y) \in V$. Any map η with this property is coarse and any two such maps are close.*

Proof. It is straightforward to see that η exists. Suppose y lies in the interior of the simplex spanned by $[V_1], \ldots, [V_k]$. Then $y \in \mathrm{Star}[V_i]$ for $i = 1, \ldots, k$, and there are no other vertices $[V]$ for which $y \in \mathrm{Star}[V]$. As $[V_1], \ldots, [V_k]$ span a simplex $V_1 \cup \cdots \cup V_k$ is non-empty, and we can pick any $\eta(y) \in V_1 \cup \cdots \cup V_k$.

That two such maps η, η' are close is immediate from the fact that $C = \bigcup_{U \in \mathcal{U}} U \times U$ is controlled; if $y \in \mathrm{Star}[V]$ then $\eta(y), \eta'(y) \in V$, so as $V \in \mathcal{U}$, we have $(\eta(y), \eta'(y)) \in C$.

For a bounded subset of X, properness of X and the condition that $\mathcal{U}$ is locally finite together imply that the preimage under η is a finite subcomplex of $N_{\mathcal{U}}$ and hence η is coarsely proper. It remains to show that for all $R > 0$ there is a controlled subset A of $X \times X$ such that if $d(y, y') < R$ then $(\eta(y), \eta(y')) \in A$. For any R the set of pairs (y, y') with y, y' in different components of $N_{\mathcal{U}}$ and such that $d(y, y') < R$ is bounded. Hence the set of images $(\eta(y), \eta(y'))$ of such pairs is relatively compact, and hence controlled in $X \times X$. Thus it remains to show that the set

$$\{(\eta(y), \eta(y')) : d(y, y') < R, \text{ and } y, y' \text{ in the same component of } N_{\mathcal{U}}\}$$

is controlled. If y, y' lie in the same component of $N_{\mathcal{U}}$ and $d(y, y') < R$ then by lemma 5.6 there is a sequence $[V_0], \ldots, [V_k]$ of adjacent vertices of $N_{\mathcal{U}}$ with $y \in V_0, y' \in V_k$, and $d([V_0], [V_k]) = \pi k/2 < R + \pi$. Thus (y, y') is contained in the $k+1$-fold composition of the controlled set $\bigcup_{U \in \mathcal{U}} U \times U$ with itself. As k is bounded there is a single controlled set containing all such pairs as required. $\square$

We will need the assembly map in the contexts of the bounded and C_0 coarse structures. To encompass both definitions we will define the assembly map for an abstract coarse structure, and will consequently make use of directed systems of open covers, however for the bounded coarse structure the reader may assume that all such systems are anti-Čech sequences.

The assembly map will be defined as follows. First we will construct natural[3] homomorphisms $K_*(X) \to K_*(C^*X)$ for proper coarse spaces X. Then for a locally finite open cover $\mathcal{U}$ of X we obtain maps $K_*(N_\mathcal{U}) \to K_*(C^*(N_\mathcal{U}))$ where $N_\mathcal{U}$ is equipped with the bounded coarse structure for a uniform spherical metric. As any connecting map $\phi\colon N_\mathcal{U} \to N_{\mathcal{U}'}$ is both continuous and coarse, naturality implies there is a map $\varinjlim K_*(N_\mathcal{U}) \to \varinjlim K_*(C^*(N_\mathcal{U}))$ where the direct limit is over an anti-Čech sequence if X was equipped with the bounded coarse structure, or in general over the directed system of locally finite controlled covers for any abstract coarse structure. Finally from 5.20 there are coarse maps $\eta N_\mathcal{U} \to X$ defined up to closeness, and we note that if $\phi\colon \mathcal{U} \to \mathcal{U}'$ is a connecting map and $\eta N_\mathcal{U} \to X, \eta' N_{\mathcal{U}'} \to X$ then $\eta = \eta' \circ \phi$ up to closeness. Hence we will obtain the assembly map as the composition

$$KX_*(X) \to \varinjlim_\mathcal{U} K_*(C^*(N_\mathcal{U})) \to K_*(C^*X).$$

The latter map of the composition is in fact an isomorphism.

Theorem 5.21. *Let X be a proper separable coarse space. Then*

$$\varinjlim_\mathcal{U} K_*(C^*(N_\mathcal{U})) \to K_*(C^*X)$$

[3] Any coarse and continuous map $X \to Y$ should give rise to a commutative diagram relating the assembly maps X and Y.

is an isomorphism, where the direct limit is taken over the directed system of locally finite open covers $\mathcal{U}$ such that $\bigcup_{U\in\mathcal{U}} U \times U$ is controlled.

Proof. The maps $K_*(C^*(N_{\mathcal{U}})) \to K_*(C^*X)$ are induced from coarse maps $\eta_{\mathcal{U}} \colon N_{\mathcal{U}} \to X$. The idea of the proof is to take an 'inverse' map $\psi_{\mathcal{U}} \colon X \to N_{\mathcal{U}}$. Let $\psi_{\mathcal{U}} \colon X \to N_{\mathcal{U}}$ be any map taking points $x \in X$ to vertices $[U] \in N_{\mathcal{U}}$ with $x \in U$. Certainly the compositions $\psi_{\mathcal{U}} \circ \eta_{\mathcal{U}}$ and $\eta_{\mathcal{U}} \circ \psi_{\mathcal{U}}$ are close to the identity. The issue is that $\psi_{\mathcal{U}}$ will not in general be a coarse map.

Let C^*X be defined using a representation of $C_0(X)$ on $\mathfrak{H}$ and let $C^*(N_{\mathcal{U}})$ be defined using a representation of $C_0(N_{\mathcal{U}})$ on $\mathfrak{H}_{\mathcal{U}}$. Write $A = C^*X$ as the direct limit $A = \varinjlim_C A_C$ where the direct limit is over controlled open subsets C of $X \times X$ containing the diagonal, and A_C is the C^*-subalgebra of A generated by operators with support contained in C. Note that the axioms for a proper coarse structure imply that every controlled set is contained in some set C appearing in the direct limit, and hence $A = \varinjlim_C A_C$ as claimed.

Let $\widetilde{\mathcal{U}_C}$ be the collection of all open sets U with $U \times U \subset C$, and let $\mathcal{U}_C$ be a locally finite cover with $\bigcup_{U\in\mathcal{U}_C} U \times U$ controlled and $\widetilde{\mathcal{U}_C} \preceq \mathcal{U}_C$. Such a cover exists by lemma 5.17. Let $\psi = \psi_{\mathcal{U}_C} \colon X \to N_{\mathcal{U}_C}$ and $\eta = \eta_{\mathcal{U}_C} \colon N_{\mathcal{U}_C} \to X$. Let V_1 be a covering isometry for η and in the same spirit let V_2 be an isometry from $\mathfrak{H}$ to $\mathfrak{H}_{\mathcal{U}_C}$ with $\{(y, \psi(x)) : (y, x) \in \operatorname{Supp} V\}$ controlled. Then certainly V_1V_2 is an isometry on $\mathfrak{H}$ covering the identity on X, and V_2V_1 is an isometry on $\mathfrak{H}_{\mathcal{U}_C}$ covering the identity on $N_{\mathcal{U}_C}$.

Let B_C be the preimage of $C^*N_{\mathcal{U}_C}$ under Ad_{V_2}. As V_2V_1 covers the identity on $N_{\mathcal{U}_C}$ is follows that Ad_{V_1} maps $C^*N_{\mathcal{U}_C}$ into B_C. Indeed at the level of K-theory the composition $K_*(C^*(N_{\mathcal{U}_C})) \to K_*(B_C) \to K_*(C^*(N_{\mathcal{U}_C}))$ is the identity. On the other hand as V_1V_2 also covers the identity, the composition $K_*(B_C) \to K_*(C^*(N_{\mathcal{U}_C})) \to K_*(A)$ is the map induced by the inclusion $B_C \hookrightarrow A$. It is not difficult to see that for C contained in another controlled open set C', and corresponding isometries V_1', V_2' we get commutative diagrams at the level of K-theory relating $\mathrm{Ad}_{V_1 *}$ and $\mathrm{Ad}_{V_2 *}$ with $\mathrm{Ad}_{V_1' *}$ and $\mathrm{Ad}_{V_2' *}$ respectively. Hence we may pass to the direct limit.

We will show that $\mathrm{Ad}_{V_2}(A_C)$ is contained in $C^*N_{\mathcal{U}_C}$, that is $A_C \subseteq B_C$. Given this it follows that the direct limit of the subalgebras B_C is A. Hence we will have homomorphisms

$$K_*(C^*(N_{\mathcal{U}_C})) \to K_*(A) \to K_*(C^*(N_{\mathcal{U}_C})) \to K_*(A)$$

where the composition of each pair of consecutive maps induces the identity, and this suffices to establish the theorem. To prove the assertion, it suffices to show that for $T \in A$ supported in C, the operator $T' = \mathrm{Ad}_{V_2}(T)$ has finite propagation (it is not difficult to see that it must be locally compact). But the support of T' is contained in

$$\{(y, y') : \exists (x, x') \in C \text{ such that } (y, x), (y', x') \in \mathrm{Supp}(V_2)\}.$$

We thus know that there is a constant $R > 0$ such that if $(y, y') \in \mathrm{Supp}\, T'$ then there exists $(x, x') \in C$ with $d(y, \psi(x)), d(y', \psi(x')) < R$. But as $\widetilde{\mathcal{U}_C} \preceq \mathcal{U}_C$ it follows that there exists $U \in \mathcal{U}_C$ with $x, x' \in U$. Hence $d(\psi(x), [U]), d(\psi(x'), [U]) \leq \pi/2$ and so $d(y, y') < 2R + \pi$ for all (y, y') in the support of T'. $\square$

In preparation for defining the assembly map we will recall the definition of K-homology in terms of dual algebras.

Definition 5.22. Let A be a C^*-algebra, and let ρ be a representation of A. Then the *dual algebra* $\mathfrak{D}_\rho(A)$ is defined to be the algebra of operators T which commute with the representation modulo compact operators. For J an ideal in A, the *relative dual algebra* $\mathfrak{D}_\rho(A//J)$ is defined to be the subalgebra of $\mathfrak{D}_\rho(A)$ consisting of operators T for which $\rho(j)T$ and $T\rho(j)$ are compact for $j \in J$. For a locally compact topological space X and an open subset U of X we will denote $\mathfrak{D}_\rho(C_0(X))$ by $\mathfrak{D}_\rho(X)$, and $\mathfrak{D}_\rho(C_0(X)//C_0(U))$ by $\mathfrak{D}_\rho(X//U)$.

Definition 5.23. Let A be a C^*-algebra, and let ρ be a faithful representation of A whose range contains no non-zero compact operators. Then $K^*(A) = K_{*+1}(\mathfrak{D}_\rho(A)/\mathfrak{D}_\rho(A//A))$. For a locally compact topological space X we will denote $K^*(C_0(X))$ by $K_*(X)$.

The K-homology of a space or algebra is well defined i.e. it independent of ρ by a theorem in the spirit of 2.34.

The dual algebra definition of K-homology is related to the coarse algebras by the following theorem.

Theorem 5.24. *For X a proper separable coarse space the inclusion $D^*X \hookrightarrow \mathfrak{D}_\rho(X)$ induces an isomorphism of the quotient algebras D^*X/C^*X and $\mathfrak{D}_\rho(X)/\mathfrak{D}_\rho(X//X)$.*

Proof. We will begin by showing that $\mathfrak{D}(X) = D^*(X) + \mathfrak{D}(X//X)$. Let $\mathcal{U}$ be a locally finite, uniformly bounded open cover of X, and let $\{f_i\}$ be a partition of unity subordinate to this. Let $S \in D^*(X)$ be self-adjoint. Then the operator $T = \sum_i f_i^{1/2} S f_i^{1/2}$ converges, is a locally compact pertubation of S, and has norm at most $2\|S\|$, cf. 4.15.

The operator T is controlled by lemma 2.23, hence it lies in $D^*(X)$. We find that any self-adjoint operator in $\mathfrak{D}(X)$ is a locally compact perturbation of a self-adjoint operator in $D^*(X)$. Noting that $\mathfrak{D}(X//X)$ is the algebra of all locally compact operators it follows that $\mathfrak{D}(X) = D^*(X) + \mathfrak{D}(X//X)$.

Hence there is an isomorphism

$$\mathfrak{D}(X)/\mathfrak{D}(X//X) \cong D^*(X)/(D^*(X) \cap \mathfrak{D}(X//X)).$$

For the first part of the statement it therefore suffices to check that $D^*(X)\cap\mathfrak{D}(X//X) = C^*(X)$, and it is immediate from the definitions that the former contains the latter.

For the converse inclusion we will repeat the construction of T. Suppose $S \in D^*(X) \cap \mathfrak{D}(X//X)$ is self-adjoint, and let T be constructed as above. Then T is controlled, it is locally compact as S is, and hence $T \in C^*(X)$. Now for S_j a sequence of controlled operators in $\mathfrak{D}(X)$ converging to S, construct operators T_j using the same partition of unity. Each operator $S_j - T_j$ is also controlled and locally compact, and moreover $S_j - T_j$ converges to $S - T$ as $\|T - T_j\| \leq 2\|S - S_j\|$. Hence $S - T$ also lies in $C^*(X)$, and splitting an arbitrary operator into self-adjoint and skew-adjoint parts, we conclude that $D^*(X) \cap \mathfrak{D}(X//X) \subseteq C^*(X)$. □

We now obtain the required homomorphism from the boundary map of the K-theory long exact sequence associated to the short exact sequence of C*-algebras

$$0 \to C^*X \to D^*X \to D^*X/C^*X \to 0.$$

The homomorphism is the composition

$$K_*(X) \cong K_{*+1}(D^*X/C^*X) \xrightarrow{\partial} K_*(C^*X)$$

and the assembly map then given by the composition

$$\mu\colon KX_*(X) = \varinjlim K_*(N_{\mathcal{U}}) \xrightarrow{\partial} \varinjlim K_*(C^*(N_{\mathcal{U}})) \to K_*(C^*X).$$

5.3 Properties of KX_* and homology uniqueness

In chapter 2 we noted that $K_*(C^*X)$ has various homological properties. These homological properties are also true for the coarse K-homology (they are quite straightforward from the homological properties of K-homology), and hence we may think of the coarse Baum-Connes conjecture as being a uniqueness result for 'coarse homologies'. This view of the coarse Baum-Connes conjecture is explored in [15].

In the topological context, uniqueness of homology theories on finite dimensional simplicial complexes can be proved inductively. Any finite dimensional complex can be constructed as a union of two pieces, each of which is homotopy equivalent to a complex of lower dimension. The two results employed in the induction step — the Mayer-Vietoris sequence, and homotopy invariance — can both be used in the coarse context, though with certain restrictions. These restrictive results are usually sufficient for the required decompositions and homotopies.

The more fundamental difficulty is in finding a decomposition of X into pieces for which the coarse Baum-Connes conjecture can be proved. The case case of the bounded coarse structure one might hope to prove the conjecture by induction on the asymptotic dimension[4], however there is a difficulty; it is not in general possible to decompose a space of asymptotic dimension m into finitely many spaces which are coarsely homotopy equivalent to spaces of asymptotic dimension less than m. On the other hand it is certainly possible to decompose a simplicial complex of dimension m into complexes which are homotopy equivalent (either topologically, or C_0 coarsely) to to complexes of dimension less than m. Thus for C_0 coares geometry we can reduce finite dimensional simplicial complexes to 0-dimensional (i.e. uniformly discrete) complexes. For the C_0 structure, unlike the bounded case all infinite uniformly discrete spaces are coarsely equivalent, so the 0-dimensional case amounts to a single calculation. Hence for the C_0 coarse structure, this homological uniqueness in fact proves the conjecture for locally finite simplicial complexes with a uniform spherical metric.

We will compare the following three functors on proper coarse spaces: $X \mapsto K_*(C^*X)$, $X \mapsto KX_*(X)$ and $X \mapsto K_*^\infty(X)$. Note that the K-homology at infinity is a not a functor on the coarse category, but on the category of locally compact topological spaces with maps which are continuous at infinity i.e. maps whose points of discontinuity form a relatively compact set. We will find a posteriori that it may be regarded as a functor on certain C_0 coarse spaces. Specifically we will prove that if X is a finite dimensional simplicial complex with uniform spherical metric then $K_*(C^*X_0), KX_*(X)$, and $K_*^\infty(X)$ are all isomorphic.

We will begin with a Mayer-Vietoris sequence for KX_*.

[4]Fow now we will focus mainly on the C_0 coarse structure, so we will postpone the discussion of asymptotic dimension to the final section.

Theorem 5.25. *For a coarsely excisive decomposition $X = Y \cup Z$, there is a cyclic Mayer Vietoris exact sequence:*

$$\cdots \to KX_*(Y \cap Z) \xrightarrow{i_{1*} \oplus i_{2*}} KX_*(Y) \oplus KX_*(Z) \xrightarrow{j_{1*} - j_{2*}} KX_*(X)$$
$$\xrightarrow{\partial} KX_{*-1}(Y \cap Z) \to \dots$$

where i_1, i_2 are respectively the inclusions of $Y \cap Z$ into Y, Z, and j_1, j_2 are respectively the inclusions of Y, Z into X. There is also a corresponding exact sequence for K-homology at infinity[5].

The assembly maps from this Mayer-Vietoris sequence to the sequence of 3.13 give rise to a commutative ladder.

Proof. From an open cover $\mathcal{U}$ of X, we obtain covers $\mathcal{U}_Y, \mathcal{U}_Z$ of Y, Z, naturally giving a decomposition $N_{\mathcal{U}} = N_{\mathcal{U}_Y} \cup N_{\mathcal{U}_Z}$. We also obtain a cover $\mathcal{U}_{Y \cap Z}$ of $Y \cap Z$ with $N_{\mathcal{U}_{Y \cap Z}} \subseteq N_{\mathcal{U}_Y} \cap N_{\mathcal{U}_Z}$. The condition of coarse excisiveness ensures that for some cover $\mathcal{U}'$ with $\mathcal{U} \prec \mathcal{U}'$, the image of $N_{\mathcal{U}_Y} \cap N_{\mathcal{U}_Z}$ in $N_{\mathcal{U}'}$ is contained in $N_{\mathcal{U}'_{Y \cap Z}}$. Taking the direct limit, the result for KX_* then follows from the K-homology exact sequence. ☐

Corollary 5.26. *If $X = Y \cup Z$ is a coarsely excisive decomposition and if the coarse Baum-Connes conjecture holds for the subsets Y, Z and $Y \cap Z$ of X, then the coarse Baum-Connes conjecture holds for X. Likewise if for Y, Z and $Y \cap Z$ the coarse K-homology agrees with the K-homology at infinity, then the same is true for $X = Y \cup Z$.*

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[5] *The Mayer-Vietoris sequence for the K-homology at infinity is exact for any decomposition of X into two closed sets.*

Now we will give a homotopy invariance result. As this is more general than the homotopy invariance results already proved for $K_*(C^*X)$ in chapter 2, we will prove homotopy invariance both for this functor and for $KX_*(X)$.

Definition 5.27. Let X, Y be proper separable coarse spaces, and suppose $\phi, \psi\colon X \to Y$ are coarse. Then ϕ, ψ are *directly coarsely homotopic* if there is a map $\eta\colon X \times \mathbb{R}^+ \to Y$ which we will denote by $\eta(x,t) = \eta_t(x)$, such that

- $\eta(x,0) = \phi(x)$, and for any bounded subset K of X $\eta_t|_K \equiv \psi|_K$ for t sufficiently large;
- for $\mathbb{R}^+$ equipped with the bounded coarse structure, and for $X \times \mathbb{R}^+, Y \times \mathbb{R}^+$ with the product coarse structures, the map $\tilde{\eta}\colon X \times \mathbb{R}^+ \to Y \times \mathbb{R}^+$ given by $(x,t) \mapsto (\eta(x,t), t)$ is coarse.

Maps ψ, ϕ are *coarsely homotopic* if there is a sequence of maps $\psi = \psi_0, \psi_1, \ldots, \psi_k = \phi$, with ψ_j, ψ_{j+1} directly coarsely homotopic or ψ_{j+1}, ψ_j directly coarsely homotopic for each j. In other words coarse homotopy is the equivalence relation generated by direct coarse homotopy.

Remark 5.28. For a map $\eta(x,t) = \eta_t(x)$ satisfying the first condition of the above definition, the condition that $\tilde{\eta}$ is coarse can be stated explicitly as follows:

1. for any controlled set A in $X \times X$, the sets

$$B_t = \{(\eta(x,t), \eta(x',t)) \mid (x,x') \in A\}$$

are controlled uniformly in t, that is $\bigcup_t B_t$ is controlled;

2. about each $t \in \mathbb{R}^+$ there is an interval U such that η_t is close to $\eta_{t'}$ for each $t' \in U$ uniformly in t', that is

$$\{(\eta_t(x), \eta_{t'}(x)) : x \in X, t' \in U\}$$

is controlled;

3. for any bounded set K in Y, the projection of $\eta^{-1}(K)$ onto X is bounded.

Definition 5.29. Proper coarse spaces X, Y are *coarsely homotopy equivalent* if there exist maps $\phi\colon X \to Y$, and $\psi\colon Y \to X$ with $\psi \circ \phi$ and $\phi \circ \psi$ coarsely homotopic to the identity on X, Y. The maps ϕ, ψ are called *coarse homotopy equivalences.*

Theorem 5.30. *Let X, Y be proper separable coarse spaces, and suppose that $\phi, \psi\colon X \to Y$ are coarse maps which are coarsely homotopic. Then the induced maps ϕ_* and ψ_* on the K-theory of C^*X and on the coarse K-homology of X are equal.*

It follows from this theorem that coarse homotopy equivalences induce isomorphisms on coarse K-homology.

Corollary 5.31. *If X, Y are coarsely homotopy equivalent and if the coarse Baum-Connes conjecture holds for X, then the coarse Baum-Connes conjecture holds for Y. Suppose moreover that the coarse homotopy equivalences are continuous at infinity, and there is a homotopy between them which is continuous at infinity. If the coarse K-homology, and K-homology at infinity agree for X then the same is true for Y.*

Proof of 5.30. The general homotopy invariance result can be deduced from lemma 3.2. We will sketch the argument: Suppose η is a direct coarse homotopy from α to β. Write

$X = K_1 \cup K_2 \cup \ldots$ with $\{K_i\}$ a locally finite cover by closed bounded sets. Then by hypothesis there are numbers t_i such that $\eta(x,t) = \beta(x)$ for $(x,t) \in Y = \bigcup_i K_i \times [t_i, \infty)$. Let $Z = \bigcup_i K_i \times [0, t_i]$. By 3.2 applied to positive translations of $\mathbb{R}^+$ the C^*-algebra of $X \times \mathbb{R}^+$ has trivial K-theory and likewise for Y, while negative translations show that the same holds for $X \times (-\infty, 0]$ and $X \times (-\infty, 0] \cup Z$.

The Mayer-Vietoris sequence for the union of $X \times (-\infty, 0]$ and Z thus shows that the inclusion of X into Z induces an isomorphism on K-theory with inverse given by the projection of Z onto X. Similarly the decomposition of $X \times \mathbb{R}^+$ into Y and Z gives an isomorphism at the level of K-theory between C^*Z and the ideal J of operators supported near both Y and Z[6]. We conclude that there are isomorphisms $K_*(J) \to K_*(C^*Z) \to K_*(C^*X)$. Note that on any of the generating subsets $Y_A \cap Z_A$ for J the map η is close to $\beta \circ \pi_X$ where π_X is the projection onto X. Thus the first of the isomorphisms tells us that $\eta_* = \beta_* \circ \pi_{X*}$ on Z while the latter gives us that $\eta_* = \eta_* \circ \pi_{X*} = \alpha_* \circ \pi_{X*}$. Hence as π_{X*} is an isomorphism $\alpha_* = \beta_*$. □

In the spirit of uniqueness of homologies, we now need a result corresponding to the homologies agreeing on 0-dimensional complexes i.e. on discrete spaces. We will prove this in the context of C_0 coarse geometry.

Theorem 5.32. *Let X be an infinite uniformly discrete proper metric space. Then the coarse Baum-Connes conjecture holds for X equipped with the C_0 coarse structure.*

[6] We cannot be sure that this decomposition is excisive, so we must use the more general formulation of 5.25

Moreover any such space is coarsely equivalent to $\mathbb{N}_0$ *and*

$$K_*^\infty(\mathbb{N}) \cong KX_*(\mathbb{N}_0) \cong K_*(C^*\mathbb{N}_0).$$

Proof. We have already shown (lemma 3.20) that $\prod \mathbb{Z}$ maps to $K_0(C^*\mathbb{N}_0)$ surjectively with kernel consisting of finitely supported sequences of sum zero, while $K_1(C^*\mathbb{N}_0) = 0$. The map $\prod \mathbb{Z} \to K_0(C^*\mathbb{N}_0)$ was specifically identified as taking a sequence of positive integers to a sequence of projections having the given sequence of ranks.

It is easy to see that X_0 is coarse equivalent to $\mathbb{N}_0$; the coarse structure on X is such that a C_0 controlled operator is one for which the support contains only finitely many points off the diagonal of $X \times X$, and a C_0 cover is a cover in which all but finitely many of the sets are singletons. For a C_0 cover let C be the union of all non-singletons (which is finite). Then the cover can be coarsened to a cover consisting of all subsets of C along with all singletons of $\mathbb{N} \setminus C$. For such a cover the nerve is the union of a simplex with $\mathbb{N} \setminus C$, and the canonical map from this nerve to $\mathbb{N}/C$ is a homotopy equivalence. It follows therefore that the C_0 coarse K-homology of $\mathbb{N}$ is isomorphic to $K_*^\infty(\mathbb{N}) = \varinjlim_{C \subseteq \mathbb{N} \text{ compact}} K_*(\mathbb{N}/C)$.

We will now explicitly compute this group in order zero. For each C, we have $K_0(\mathbb{N}/C) = \prod_{x \in \mathbb{N}/C} \mathbb{Z}$, and for $C \subset C'$ the map $K_0(\mathbb{N}/C) \to K_0(\mathbb{N}/C')$ is surjective and its kernel consists of those elements of $K_0(C'/C) \subset K_0(\mathbb{N}/C)$ which have sum zero. Hence the homomorphism $K_0(\mathbb{N}) \to K_0^\infty(\mathbb{N}) = KX_0(\mathbb{N}_0)$ is surjective, and its kernel consists of the finitely supported elements of $K_0(\mathbb{N}) = \prod_{x \in \mathbb{N}} \mathbb{Z}$ with sum zero. It is easy to see that $K_1(\mathbb{N}/C) = 0$ for all C and hence $K_1^\infty(\mathbb{N}) = 0$.

Now we will describe the assembly map for this space. Let us represent $C_0(X)$ on $l^2(X) \otimes l^2$. The K-homology is given by the K-theory of $\mathfrak{D}(X)/\mathfrak{D}(X//X)$. But in this case the algebra $\mathfrak{D}(X)$ of pseudolocal operators consists of operators which are a sum of a locally compact operator, that is one in $\mathfrak{D}(X//X)$, with a diagonally supported operator. Hence $\mathfrak{D}(X)/\mathfrak{D}(X//X) \cong \prod_{\mathbb{N}} \mathcal{B}(l^2)/\mathfrak{K}(l^2)$. The boundary map $K_0(\mathbb{N}) = K_1(\prod_{\mathbb{N}} \mathcal{B}(l^2)/\mathfrak{K}(l^2)) \to K_*(\prod_{\mathbb{N}} \mathfrak{K}(l^2))$ is an isomorphism and gives the identification $K_0(\mathbb{N}) = \prod_{\mathbb{N}} \mathbb{Z}$, while the composition of this map with the inclusion of $\prod_{\mathbb{N}} \mathfrak{K}(l^2)$ into $C^* N_0$ is the assembly map.

Explicitly we have found that the assembly map $\prod \mathbb{Z} = K_0(X) \to K_0(C^*\mathbb{N}_0)$ is given by taking a sequence of positive integers to a sequence of projections having the given sequence of ranks. But we know that this map is surjective, and that the kernels of the maps $\prod \mathbb{Z} \to KX_0(\mathbb{N}_0)$ and $\prod \mathbb{Z} \to K_0(C^*\mathbb{N}_0)$ agree, hence the assembly map is an isomorphism in degree zero. As $KX_1(X)$ and $K_1(C^*X_0)$ vanish the assembly map is also an isomorphism in degree one. □

We now get the following theorem as a uniqueness result for coarse homologies in the C_0 context.

Theorem 5.33. *If X is a finite dimensional simplicial complex equipped with a uniform spherical metric then*

$$K_*^{\infty}(X) \cong KX_*(X_0) \xrightarrow[\cong]{\mu} K_*(C^*X_0).$$

Proof. First let $X^{(2)}$ denote the second barycentric subdivision of X, and let Y_j be the union of simplicial stars in $X^{(2)}$ about the barycentres of the j-simplices of X. Then

$X = Y_0 \cup \cdots \cup Y_m$ where m is the dimension of X, and each Y_j is a disjoint union of uniformly separated stars, see fig. 5.1. To see that these are uniformly separated it suffices to consider pairs of stars $\mathrm{Star}(x), \mathrm{Star}(y)$ in the same component of X, and as we have a path metric on each component the distance will be the length of a path between the boundaries of the stars. If these meet a common simplex σ of X then the distance between the stars will be the length of a path within σ. Otherwise for some simplex σ containing x the path between the stars must connect $\partial\,\mathrm{Star}(x)$ to a face of σ not containing x. In either case we get a lower bound on the distance between the stars, which does not depend on the simplex σ as all simplices (of the same dimension) are isometric.

Fig. 5.1. The second barycentric subdivision of a simplex, decomposed in terms of $\{Y_j\}$.

Each Y_j is therefore either compact if it consists of finitely many stars, or is coarsely homotopy equivalent to the infinite uniformly discrete set consisting of the j-barycentres. The homotopy is continuous, and moreover it is contractive on each star. As the stars are uniformly separated this ensures that it is a C_0 coarse homotopy. Hence in either case the coarse Baum-Connes conjecture holds for Y_j and the coarse K-homology agrees with the K-homology at infinity.

We now consider the unions $Z_k = Y_0 \cup \cdots \cup Y_k$ for $k = 0, 1, \ldots$. We will show that for all k the conjecture holds for Z_k and the coarse K-homology of Z_k agrees with the K-homology at infinity. In particular we will conclude that this holds for $k = m$. Certainly the statement holds for $Z_0 = Y_0$. Inductively, we suppose it is true for Z_{k-1}. The statement also holds for Y_k, so by the Mayer-Vietoris theorem (5.25) it will suffice to show that the result holds for $Z_{k-1} \cap Y_k$, and that the decomposition of Z_k as $Z_{k-1} \cap Y_k$ is coarsely excisive. For $k < m$ the metric on Z_k will not in general be uniform, in particular it will not be a length metric. However by 5.9 (using conditions 1,2,3) there is a finite bound on the distortion for the components of Z_k in X. Thus the given metric on Z_k and the associated path metric are at least uniformly continuous with respect to one another, which implies that they are C_0-coarsely equivalent by lemma 2.18. It follows that the decomposition is excisive.

The intersection $Z_{k-1} \cap Y_k$ has dimension $m - 1$, and hence we would like to conclude by a further induction on the dimension m of X, that the statement must also hold for $Z_{k-1} \cap Y_k$. Again there is a slight nuance as the metric on $Z_{k-1} \cap Y_k$ is not a uniform spherical metric. However if we replace the given metric by a uniform spherical metric then 5.9 (using conditions 1,2,3$'$) implies that on each component of

$Z_{k-1} \cap Y_k$ the two metrics are bi-Lipschitz equivalent, for some Lipschitz constant which is independent of the component. Hence as for Z_k, the two metrics on $Z_{k-1} \cap Y_k$ are uniformly continuous with respect to one another, so they are C_0 coarsely equivalent, and moreover this equivalence is a homeomorphism so it is compatible with K_*^∞. Thus inductively the result holds for $Z_{k-1} \cap Y_k$, completing the proof. □

5.4 The coarsening space

In this section we will construct a 'total coarsening space' associated to the coarsenings $N_{\mathcal{U}_i}$ coming from an anti-Čech sequence $\mathcal{U}_i$. Using the C_0 version of the coarse Baum-Connes conjecture from the previous section, we will then identify the left hand side of the conjecture with the K-theory of an ideal in the C_0 Roe algebra of the total coarsening space. The right hand side can be identified with a corresponding ideal in the bounded Roe algebra, and the assembly map becomes a forgetful functor from the C_0 ideal to the bounded ideal. Moreover the bounded version of the ideal can also be regarded as an ideal in the Roe algebra associated to a 'hybrid' structure on the total coarsening space, which is coarser than the C_0 structure, but not as coarse than the bounded structure. This new structure will be exploited in the next section, to enable calculations which could not be done directly for the bounded structure.

As we will be dealing with several coarse structures on the same space, we will use the following conventions. A space equipped with the C_0 coarse structure will be denoted X_0. When we have defined the hybrid coarse structure, this will be denoted X_h. The bounded coarse structure will be denoted by an undecorated X.

Definition 5.34. Let W be a proper metric space, and let $\mathcal{U}_*$ be an anti-Čech sequence for W. The *coarsening space* of $(W,\mathcal{U}_*)$ is

$$X = X(W,\mathcal{U}_*) = N_{\mathcal{U}_1} \times [1,2] \cup_{\phi_1} N_{\mathcal{U}_2} \times [2,3] \cup_{\phi_2} \cdots$$

equipped with the following path metric. Equip each subset $\sigma \times [i,i+1]$ of $N_{\mathcal{U}_i} \times [i,i+1]$ with the product metric; σ being given the spherical metric and $[i,i+1]$ the linear metric. The image of $\sigma \times [i,i+1]$ in X is then equipped with the largest metric bounded by the product metric[7]. Denote by $\pi\colon X \to [1,\infty)$ the map on X arising from the projection maps $N_{\mathcal{U}_i} \times [i,i+1] \to [i,i+1]$.

Definition 5.35. The *partial coarsening spaces* of $W,\mathcal{U}_*$ are the spaces

$$X_i = X_i(W,\mathcal{U}_*) = N_{\mathcal{U}_1} \times [1,2] \cup_{\phi_1} \cdots \cup_{\phi_{i-1}} N_{\mathcal{U}_i}$$

equipped with the metrics they inherit as subspaces of the coarsening space. In other words $X_i = \pi^{-1}([1,i])$.

We will use various homotopy arguments which will involve collapsing partial coarsening spaces in X. We will also use these collapsing maps to show that C^*X has trivial K-theory for X equipped with various coarse structures; we may think of this as a homotopy from the identity to the 'constant map at infinity'.

[7]If $\sigma \times [i,i+1]$ injects into X then its image will just have the product metric, however if $\phi_i\colon N_{\mathcal{U}_i} \to N_{\mathcal{U}_{i+1}}$ maps σ to a simplex of lower dimension, then the metric on the image in X will be reduced.

Definition 5.36. The *collapsing map* from X to $\pi^{-1}([t,\infty))$ is the map

$$\Phi_t(x,s) = \begin{cases} (\phi_{i'-1}\circ\cdots\circ\phi_i(x),t) & \text{for } (x,s)\in N_{\mathcal{U}_i}\times[i,i+1), \\ & \text{with } s\le t, t\in[i',i'+1) \\ (x,s) & \text{for } s\ge t \end{cases}.$$

Note that these maps are contractive, and for $t' > t$ we have $\Phi_{t'}\circ\Phi_t = \Phi_{t'}$.

Theorem 5.37. *Let W be a uniformly discrete bounded geometry metric space, and let $\mathcal{U}_*$ be an anti-Čech sequence for W. Then $C^*(X(W,\mathcal{U}_*)_0)$ has trivial K-theory.*

For the purposes of the following section it would suffice to establish this when X is finite dimensional. In that case by the C_0 coarse Baum-Connes isomorphism (5.33) it suffices to show that $K_*^\infty(X) = 0$ which is straightforward. It is interesting however to note that the result can be proved in the greater generality stated here.

Proof. We will apply lemma 3.2. We must find maps α_k from X to X satisfying the hypotheses of the lemma. Let $r_k\colon \mathbb{R}^+\to\mathbb{R}^+$ be given by $r_k(t) = \log k - t$ for $0\le t\le \log t$ and $r_k(t) = 0$ for $t\ge \log k$. Pick a basepoint x_0 in X, and define $\alpha_k\colon X\to X$ by $\alpha_k(x) = \Phi_{r_k(d(x_0,x))}(x)$. We will show that this has the required properties.

A bounded subset K of X must lie in some X_i. Note that $\Phi_i(K)$ is then also bounded, and so is $K' = \Phi_i^{-1}\Phi_i(K)$. The set K' has the property that if $\Phi_t(x)\in K'$ for some t then $x\in K'$. As K' contains K, to show that the range of α_k does not meet K for k sufficiently large it therefore suffices to show that $\alpha_k(K')$ does not meet K. The set K' lies in some ball $B(x_0,R)$ in X, and then for $x\in K$ we have $d(x_0,x)\le R$ so

$r_k(d(x, x_0)) \geq \log k - R$. Thus for $\log k > R + i$ the set $\alpha_k(K')$ does not meet X_i and so in particular it does not meet K.

If A is a C_0 controlled subset of $X \times X$, then let

$$B_A = \{(\alpha_k(x), \alpha_k(x')) : k = 1, 2, \ldots, \text{ and } (x, x') \in A\}.$$

We must show that this is C_0 controlled. For each $\varepsilon > 0$ we may write $A = K_\varepsilon \cup A_\varepsilon$ where K_ε is bounded and $d < \varepsilon$ on A_ε. The maps α_k expand distances by at most a factor of 2 as

$$d(\alpha_k(x), \alpha_k(x')) \leq d(x, x') + \left| r_k(d(x_0, x)) - r_k(d(x_0, x')) \right| \leq 2d(x, x').$$

Thus for all k we have $d(\alpha_k(x), \alpha_k(x')) \leq 2\varepsilon$ for $(x, x') \in A_\varepsilon$. On the other hand K_ε is bounded, and using the fact that Φ_t ultimately collapses any finite complex to a point we find that for k sufficiently large, $(\alpha_k(x), \alpha_k(x'))$ lies on the diagonal for all $(x, x') \in K_\varepsilon$. Therefore B_A lies in the union of a bounded set and a set on which the distance function is bounded by 2ε. As ε is arbitrary it follows that B_A is controlled.

It is not hard to see that α_1 is close to the identity; indeed α_1 equals the identity outside of a bounded set. It therefore remains to show that

$$C = \{(\alpha_k(x), \alpha_{k+1}(x)) : k = 1, 2, \ldots \text{ and } x \in X\}$$

is controlled. Note that

$$\begin{aligned} d(\alpha_k(x), \alpha_{k+1}(x)) &\leq r_{k+1}(d(x, x_0)) - r_k(d(x, x_0)) \\ &\leq \log(k+1) - \log(k) < 1/k. \end{aligned}$$

Fixing k_0 we note that there is a bounded set outside of which α_k is the identity for $k \leq k_0$. Hence for $k < k_0$ the set $\{(\alpha_k(x), \alpha_{k+1}(x))\}$ lies in the union of a bounded set with the diagonal. On the other hand for $k \geq k_0$ we have $d(\alpha_k(x), \alpha_{k+1}(x)) < 1/k_0$. Thus C lies in the union of a bounded set, the diagaonal, and a set on which the distance function is bounded by $1/k_0$. As k_0 is arbitrary C is controlled, which completes the proof. □

Now we will construct the ideal whose K-theory gives the left hand side of the conjecture.

Definition 5.38. Let $I_0 = I_0(W, \mathcal{U}_*) = \varinjlim_i C^*(X_i(W, \mathcal{U}_*)_0)$.

Note that we may regard I_0 as an ideal of $C^*(X(W, \mathcal{U}_*)_0)$; each algebra $C^*(X_i(W, \mathcal{U}_*)_0)$ is naturally included in $C^*(X(W, \mathcal{U}_*)_0)$, and the closure of their union forms an ideal.

Theorem 5.39. *Let W be a uniformly discrete bounded geometry metric space, and $\mathcal{U}_*$ an anti-Čech sequence. Then $KX_*(W)$ is naturally isomorphic to $K_*(I_0(W, \mathcal{U}_*))$.*

Proof. We have

$$KX_*(W) = \varinjlim_i K_*(N_{\mathcal{U}_i}) \cong \varinjlim_i \varinjlim_{C \subseteq N_{\mathcal{U}_i} \text{ compact}} K_*(N_{\mathcal{U}_i}/C)$$

as for any compact subset C of $N_{\mathcal{U}_i}$ and for j sufficiently large, $\phi_{i+j} \circ \cdots \circ \phi_i(K)$ is a contractible subset of $N_{\mathcal{U}_{i+j+1}}$. We therefore get

$$KX_*(W) \cong \varinjlim_i KX_*((N_{\mathcal{U}_i})_0) = \varinjlim_i K_*(C^*(N_{\mathcal{U}_i})_0)$$

by the coarse Baum-Connes conjecture for C_0 coarse geometry.

Theorem 5.37 tells us that $K_*(C^*(X_0))$ is zero and for each i the group $K_*(C^*(\pi^{-1}[i,\infty))_0)$ is zero, the latter following from the fact that we use the antiČech sequence $\mathcal{U}_i, \mathcal{U}_{i+1}, \ldots$. Thus using the Mayer-Vietoris sequence of 5.25 for the decomposition $X = X|i \cup \pi^{-1}[i,\infty)$, the inclusion of $\pi^{-1}\{i\}$ into X_i induces an isomorphism at the level of K-theory.

We now observe that $\pi^{-1}\{i\}$ and $N_{\mathcal{U}_i}$ are coarsely equivalent with the C_0 structure. Given a set A which is C_0-controlled for $\pi^{-1}\{i\}$ we may write $A = B \cup C$ with $d_X(x,y) \leq 1$ for $(x,y) \in B$ and with C bounded. Then for pairs $(x,y) \in B$ the distance d_X agrees with the distance $d_{N_{\mathcal{U}_i}}$, hence B is also C_0 controlled for $N_{\mathcal{U}_i}$. The set C will also be bounded for the $N_{\mathcal{U}_i}$ metric, hence A is also C_0 controlled for $N_{\mathcal{U}_i}$. To show conversely that if A is C_0 controlled for $N_{\mathcal{U}_i}$ then it is also C_0 controlled for $\pi^{-1}\{i\}$, we use the same argument; the only difference is that if $N_{\mathcal{U}_i}$ is not connected then we must write $A = B \cup C$ with $d_{N_{\mathcal{U}_i}}(x,y) \leq \min\{1,\varepsilon\}$ for $(x,y) \in B$ where ε is less than the least distance between two components of $N_{\mathcal{U}_i}$. We conclude that $C^*(N_{\mathcal{U}_i})_0 \to C^*(X_i)_0$ indcues an isomorphsim on K-theory and hence

$$KX_*(W) \cong \varinjlim_i K_*(C^*(X_i(W,\mathcal{U}_*)_0)) \cong K_*(I_0(W,\mathcal{U}_*)).$$

Naturality follows from the naturality of the assembly map. □

Now we will formulate the right hand side in terms of an analogous ideal. The direct limit we use can be described as an ideal in the algebra $C^*(X(W,\mathcal{U}_*))$ with the bounded structure. However it will be more useful to describe it as an ideal of a slightly different algebra.

Definition 5.40. Let $X = X(W,\mathcal{U}_*)$ be a coarsening space. The *hybrid coarse structure* on X, denoted $X(W,\mathcal{U}_*)_h$ is the coarse structure for which a set $A \subseteq X \times X$ is controlled iff

- $d|_A$ is bounded, i.e. A is controlled for the bounded coarse structure, and
- $\sup\{d(x,y) : (x,y) \in A \setminus (X_i \times X_i)\} \to 0$ as i tends to infinity.

Note that the restriction of this structure to any partial coarsening space agrees with the bounded structure.

Definition 5.41. Let $I_h = I_h(W,\mathcal{U}_*) = \varinjlim_i C^*(X_i(W,\mathcal{U}_*))$.

We may regard I_h as an ideal of $C^*(X(W,\mathcal{U}_*)_h)$; as the bounded and hybrid coarse structures on $X_i(W,\mathcal{U}_*)$ agree, each algebra $C^*(X_i(W,\mathcal{U}_*))$ is naturally included in $C^*(X(W,\mathcal{U}_*)_h)$, and the closure of their union forms an ideal.

To reformulate the conjecture as a forgetful map from I_0 to I_h all that remains is to establish the following theorem.

Theorem 5.42. *There is an isomorphism* $K_*(I_h) \cong K_*(C^*W)$*, and moreover the forgetful map* $I_0 \hookrightarrow I_h$ *gives rise to the following commutative diagram:*

$$\begin{array}{ccc} KX_*(W) & \xrightarrow{\mu} & K_*(C^*W) \\ \downarrow\cong & & \downarrow\cong \\ K_*(I_0) & \longrightarrow & K_*(I_h) \end{array}$$

The coarse Baum-Connes conjecture is therefore equivalent to the statement that the forgetful map $I_0 \to I_h$ *induces an isomorphism on* K*-theory.*

Proof. Define $\zeta\colon X_i \to W$ be defined by $\zeta = \eta \circ \Phi_i$ where $\eta\colon N_{\mathcal{U}_i} \to W$ is any map such that if $x \in \mathrm{Star}[V]$ then $\eta(x)$ lies in V, as in proposition 5.20. We will show that ζ is a coarse map, indeed a coarse equivalence[8].

First let us show that ζ is coarse. If $d(x, x') \leq j$ then there is a path in $N_{\mathcal{U}_{i+j}}$ from $\Phi_{i+j}(x)$ to $\Phi_{i+j}(x')$ of length at most j. Hence by lemma 5.6 there exists a sequence of open sets $V_0, \ldots, V_k$ in $\mathcal{U}_{i+k}$ with the intersection of consecutive pairs non-empty, with $\zeta(x) \in V_0$ and $\zeta(x') \in V_k$, and with k at most $2j/\pi + 2$. Hence if $d(x, x') < j$ then $d(\zeta(x), \zeta(x')) \leq (2k/\pi + 4)\,\mathrm{Diam}\,\mathcal{U}_{i+k}$. It is not hard to see that ζ is proper, and hence it follows that ζ is coarse.

Now let $\psi\colon W \to \pi^{-1}\{i\}$ be any map taking $w \in W$ to a vertex $[V]$ of $\pi^{-1}\{i\}$ with $w \in V$. If $d(w, w') < R$ then let $j = j_R \geq 0$ be such that $\mathcal{U}_{i+j}$ has Lebesgue number at least R. Then it follows that there exists $[V] \in N_{\mathcal{U}_{i+j}}$ with $w, w' \in V$ and hence such that $\Phi_{i+j}(\psi(w))$ and $\Phi_{i+j}(\psi(w'))$ are vertices of $N_{\mathcal{U}_{i+j}}$ which are adjacent

[8] We know that η is coarse for the uniform spherical metric on $N_{\mathcal{U}_i}$, but for that metric it is not in general a coarse equivalence. However for the metric inherited from X we will see that it is in fact a coarse equivalence.

to $[V]$. Thus if $d(w,w') < R$ then $d(\psi(w),\psi(w')) < 2j_R + \pi$. Hence ψ is also coarse. It is easy to see that $\psi \circ \zeta$ and $\zeta \circ \psi$ are close to the identity. The former is at most $i + \pi$ from the identity, while the latter is at most $\mathrm{Diam}\,\mathcal{U}_i$ from the identity.

We have shown that ζ gives a coarse equivalence $\zeta\colon X_i \to W$, hence $K_*(C^*X_i) \cong K_*(C^*W)$. By continuity of K-theory under direct limits we therefore get isomorphisms $K_*(I_h) \cong \varinjlim_i K_*(C^*X_i) \cong K_*(C^*W)$.

The inclusion of $N_{\mathcal{U}_i}$ into X_i is contractive on each component of $N_{\mathcal{U}_i}$, and for any controlled subset A of $N_{\mathcal{U}_i} \times N_{\mathcal{U}_i}$ the set of pairs $(x,x') \in A$ with x and x' in different components is bounded. Hence the inclusion is coarse. Thus for each i we have maps

$$K_*(C^*(N_{\mathcal{U}_i})) \to K_*(C^*(X_i)) \underset{\cong}{\xrightarrow{\zeta_*}} K_*(C^*(W)).$$

To see that under the identifications of $KX_*(W)$ with $K_*(I_0)$, and of $K_*(C^*X)$ with $K_*(I_h)$, the assembly map μ corresponds to the inclusion of I_0 into I_h — in other words to see that the diagram in the statement of the theorem commutes — it suffices to observe that the diagram

$$\begin{array}{ccccc}
KX_*(C^*X_i(W,\mathcal{U}_*)_0) & \longleftarrow & K_*(N_{\mathcal{U}_i}) & \longrightarrow & KX_*(W) \\
\downarrow & & \downarrow & & \downarrow \\
K_*(C^*X_i(W,\mathcal{U}_*)) & \longleftarrow & K_*(C^*(N_{\mathcal{U}_i})) & \longrightarrow & K_*(C^*W)
\end{array}$$

commutes for each i. □

5.5 The coarse Baum-Connes conjecture for spaces of finite asymptotic dimension

In this section we will begin by defining asymptotic dimension. As the algebras I_0, I_h can be considered as ideals in the Roe algebras of X_0, X_h, we obtain a pair of short exact sequences, which gives rise to the following commutative ladder:

$$\begin{array}{ccccccc} \ldots K_*(I_0) & \longrightarrow & K_*(C^*(X_0)) & \longrightarrow & K_*(C^*(X_0)/I_0) & \xrightarrow{\partial} & K_{*-1}(I_0)\ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ldots K_*(I_h) & \longrightarrow & K_*(C^*(X_h)) & \longrightarrow & K_*(C^*(X_h)/I_h) & \xrightarrow{\partial} & K_{*-1}(I_h)\ldots \end{array}$$

By the 5-lemma, to prove the coarse Baum-Connes conjecture for W, it will therefore suffice to prove that the forgetful maps $C^*(X_0) \to C^*(X_h)$, and $C^*(X_0)/I_0 \to C^*(X_h)/I_h$ induce isomorphisms on K-theory.

We will prove that when W has finite asymptotic dimension, there exists an anti-Čech sequence $\mathcal{U}$ such that for $X = X(W, \mathcal{U}_*)$

- the group $K_*(C^*(X_h))$ vanishes so $K_*(C^*(X_0))$ vanishes by 5.37 the forgetful map is an isomorphism;
- the map $C^*(X_0)/I_0 \to C^*(X_h)/I_h$ induces an isomorphism on K-theory.

This will allow us to conclude that the coarse Baum-Connes conjecture holds for spaces of finite asymptotic dimension.

The former assertion will follow by another application of lemma 3.2, while that latter is suggested by the fact the two coarse structures on $\overline{X \setminus X_i}$ 'become closer together' as i tends to infinity, and hence the quotients should in some sense agree. More precisely, for certain subsets Y of X, the two coarse structures on $\overline{Y \setminus X_i}$ agree for i sufficiently large, and homological arguments will provide a reduction to such cases.

Definition 5.43. Let W be a metric space, and let $R > 0$. The *R-degree* of an open cover $\mathcal{U}$ of W is the supremum over $w \in W$ of the cardinality of $\{U \in \mathcal{U} : d(w, U) < R\}$.

Definition 5.44. A metric space W has *asymptotic dimension at most m* if for all $R > 0$ there exists an open cover $\mathcal{U}$ of W with $\mathrm{Diam}(\mathcal{U}) < \infty$ and with the R-degree of $\mathcal{U}$ at most $m+1$. The *asymptotic dimension* of W is the smallest m such that W has asymptotic dimension at most m.

The following theorem is due to Yu, see [25].

Theorem 5.45. *If W is a bounded geometry metric space of finite asymptotic dimension, then the assembly map $\mu\colon KX_*(W) \to K_*(C^*W)$ is an isomorphism.*

We will give a new proof of this using the techniques outlined above.

If W has asymptotic dimension at most m then for each R there exists a cover $\mathcal{U}$ with R-multiplicity at most $m+1$. We can thicken this cover to a cover $\mathcal{U}' = \{\{w \in W : d(w, U) < R\} : U \in \mathcal{U}\}$. This cover has Lebesgue number at least R, and has degree at most $m+1$. This allows us to construct an anti-Čech sequence with bounded degrees.

Proposition 5.46. *For a uniformly discrete metric space W of asymptotic dimension at most m, there exists an anti-Čech sequence $\mathcal{U}_*$ for W, with* $\mathrm{Degree}(\mathcal{U}_i) \leq m+1$ *for all i. Correspondingly there in an anti-Čech sequence such that each coarsening $N_{\mathcal{U}_i}$ has dimension at most m.* □

The bound on the dimensions of the complexes $N_{\mathcal{U}_i}$ is all that we will require to prove the coarse Baum-Connes conjecture.

First we use this to compute the groups $K_*(C^*(X(W,\mathcal{U}_*)_h))$. We proceed in several stages. We will need to construct a sequence of maps from $\pi^{-1}([i,\infty))$ to itself such that the restrictions to $N_{\mathcal{U}_i}$ are more and more contractive. The following lemma will be useful in dealing with homotopies that appear in the construction.

Lemma 5.47. *Let Y be a path metric space, and let Z be a uniform metric simplicial complex. Let η_0, η_1 be Lipschitz maps from Y to Z with Lipschitz constant $\lambda \geq 1$, and suppose that for all y the images $\eta_0(y), \eta_1(y)$ lie in a common simplex. Let η_t denote the linear homotopy from η_0 to η_1. Then for $Y \times [0,1]$ equipped with the metric $d((y_0,t_0),(x_1,t_1)) = d_Y(y_0,y_1) + |t_0 - t_1|$, the map $\eta\colon Y \times [0,1] \to Z$ is Lipschitz, with constant at most $\pi\lambda$.*

Proof. Given points y_0, y_1 in Y, and $\varepsilon > 0$, let γ be a path from y_0 to y_1 of length at most $d(y_0,y_1) + \varepsilon$. We know that $\eta_t \circ \gamma$ is a path of length at most $\lambda(d(x,x') + \varepsilon)$ for $t = 0, 1$. It will suffice to show that $\eta_t \circ \gamma$ is of length at most $\pi\lambda(d(y_0,y_1) + \varepsilon)$ for all t; the result will then follow by letting $\varepsilon \to 0$.

Breaking the path γ up in to segments each having images under η_0, η_1 lying within a single simplex, it will suffice to show that for such a segment γ' from y_0' to y_1', the distance $d(\eta_t(y_0'), \eta_t(y_1'))$ is at most $\pi\lambda d(y_0', y_1')$ for all t. If the simplex was equipped with a flat metric d_f this would be clear, indeed we would have $d_f(\eta_t(y_0'), \eta_t(y_1')) \leq \lambda d_f(y_0', y_1')$. We will use lemma 5.4 to deduce the result for the spherical metric from a flat inequality.

Let S denote the affine span of $\eta_0(y_0'), \eta_0(y_1'), \eta_1(y_0'), \eta_1(y_1')$ in the simplex, and note that S can be isometrically embedded into a spherical 3-simplex. Let d_f denote the metric on S induced from the flat metric on the 3-simplex and note that from lemma 5.4 we have $d_f \leq d \leq {}^1\!/_2 \pi 4^{1/2} d_f = \pi d_f$. Thus $d(\eta_t(y_0'), \eta_t(y_1')) \leq \pi d_f(\eta_t(y_0'), \eta_t(y_1')) \leq \pi\lambda d_f(y_0', y_1') \leq \pi\lambda d(y_0', y_1')$ for all t.

To complete the proof note that

$$\begin{aligned} d(\eta_{t_0}(y_0), \eta_{t_1}(y_1)) &\leq d(\eta_{t_0}(y_0), \eta_{t_0}(y_1)) + d(\eta_{t_0}(y_1), \eta_{t_1}(y_1)) \\ &\leq \pi\lambda d(y_0, y_1) + \pi|t_0 - t_1| \\ &\leq \pi\lambda d((y_0, t_0), (x_1, t_1)). \end{aligned}$$

□

We will now state a technical lemma, asserting the existence of certain 'good' partitions of unity which will give the coefficients for the contractions of $N_{\mathcal{U}_i}$.

Lemma 5.48. *Let W be a uniformly discrete bounded geometry metric space of asymptotic dimension at most m, and let $\mathcal{U}_*$ be an anti-Čech sequence for W with degrees bounded by $m+1$. For each i and each $\varepsilon > 0$ there is an $i' > i$, and a partition of unity $\{h_U\}$ of $N_{\mathcal{U}_i}$ indexed by sets U in $\mathcal{U}_i'$ such that*

- *all the maps h_U are ε-Lipschitz;*
- *for x in the interior of a simplex σ of $N_{\mathcal{U}_i}$, if $[V_1], \ldots, [V_j]$ are the vertices of σ and $U \in \mathcal{U}_{i'}$ with $h_U(x) \neq 0$ then U contains the intersection $V_1 \cap \cdots \cap V_j$.*

We will defer the proof of the lemma.

Proposition 5.49. *Let W be a uniformly discrete bounded geometry metric space of asymptotic dimension at most m, and let $\mathcal{U}_*$ be an anti-Čech sequence for W with degrees* *bounded by $m+1$. Let $X = X(W,\mathcal{U}_*)$ the total coarsening space of $(W,\mathcal{U}_*)$ and let π be the quotient map from X to $[1,\infty)$. Then for each i and each $\varepsilon > 0$ there is an $i' > i$, and a map $\beta\colon \pi^{-1}([i,\infty)) \to \pi^{-1}([i',\infty))$ such that*

- *$d(\beta(x),\beta(x')) \leq \varepsilon d(x,x')$ for $x,x' \in N_{\mathcal{U}_i}$;*
- *β is Lipschitz with constant 3π;*
- *$\beta(x) = x$ for $x \in X$ with $\pi(x) > i'$;*
- *if $x \in X$ with $i \leq \pi(x) \leq i'$ then $\beta(x) \in N_{\mathcal{U}_{i'}}$ and there is a simplex σ of $N_{\mathcal{U}_{i'}}$ containing both $\Phi_{i'}(x)$ and $\beta(x)$, hence $\Phi_{i'}$ is linearly homotopic to β, as a map from $\pi^{-1}([i,\infty)) \to \pi^{-1}([i',\infty))$.*

Proof. Apply the lemma for the given value of i, and for contraction factor $\varepsilon/(m+1)$. For i' as provided by the lemma we will begin by defining $\beta\colon N_{\mathcal{U}_i} \to N_{\mathcal{U}_{i'}}$ by

$$\beta(x) = \sum_{U \in \mathcal{U}_{i'}} h_U(x)[U].$$

This is well defined as whenever $h_{U_1}(x) \neq 0, \ldots, h_{U_j}(x) \neq 0$, the sets $U_1, \ldots, U_j$ have non-empty intersection (containing the intersection of the sets V for which $[V]$ is a vertex of the simplex containing x), and hence $[U_1], \ldots, [U_j]$ span a simplex. This moreover implies that for any given x at most $m+1$ of the terms are non-zero. A path γ from x and x' can be approximated by a path composed of segments $\gamma_1, \ldots, \gamma_k$ such that each $\beta \circ \gamma_{k'}$ lies in a single simplex of $N_{\mathcal{U}_{i'}}$. Then for any y, y' in the image of $\gamma_{k'}$ the coefficients in the sums $\beta(y)$,$\beta(y')$ each differ by at most $\varepsilon d(y,y')/(m+1)$ and there are at most $m+1$ coefficients, hence $d(\beta(y),\beta(y')) < \varepsilon d(y,y')$. It follows that the length of $\beta \circ \gamma$ is at most ε times the length of γ, and hence $d(\beta(x), \beta(x')) \leq \varepsilon d(x, x')$ for all $x, x' \in N_{\mathcal{U}_i}$.

Now compare $\beta(x)$ with $\Phi_{i'}(x)$. If x lies in a simplex spanned by $[V_1], \ldots, [V_j]$ then $\Phi_{i'}(x)$ lies in a simplex spanned by $[U_1], \ldots, [U_j]$ (not necessarily all distinct) with $V_{j'} \subseteq U_{j'}$ for all $j' \leq j$. On the other hand $\beta(x)$ lies in a simplex spanned by $[U'_1], \ldots, [U'_k]$ where from the lemma we know that each $U'_{k'}$ contains $V_1 \cap \cdots \cap V_j$. As $U_1 \cap \cdots \cap U_j$ contains $V_1 \cap \cdots \cap V_j$ it therefore follows that $[U_1], \ldots, [U_j], [U'_1], \ldots, [U'_k]$ span a simplex in $N_{\mathcal{U}_{i'}}$.

We will now extend β to a map from $\pi^{-1}([i, \infty)) \to \pi^{-1}([i', \infty))$. Let ψ_i denote the retraction of $\pi^{-1}([i, i+1))$ onto $N_{\mathcal{U}_i}$ and define

$$\beta(x) = \begin{cases} (1-t)\beta(\psi_i(x)) + t\Phi_{i'}(x) & \text{for } x \in X,\ \pi(x) = i + {}^1\!/_2 t \in [i, i + {}^1\!/_2] \\ \Phi_{i'}(x) & \text{for } \pi(x) \in [i + {}^1\!/_2, \infty) \end{cases}.$$

We will now establish that β is Lipschitz. Without loss of generality we will assume that $\varepsilon \leq 2/\pi$. It suffices to establish $d(\beta(x), \beta(x')) \leq 3\pi d(x, x')$ for $x, x' \in \pi^{-1}([i, i + {}^1\!/_2])$. The retraction ψ_i restricted to $\pi^{-1}([i, i + {}^1\!/_2])$ increases distances by at most $\pi/2$, and hence as β has Lipschitz constant at most $2/\pi$ on $N_{\mathcal{U}_i}$ it follows that $\beta \circ \psi_i$ is a contraction. As $\Phi_{i'}$ is also a contraction, it follows from lemma 5.47 that $d(\beta(x), \beta(x')) \leq \pi(d(x, x') + |t - t'|)$ where $\pi(x) = i + {}^1\!/_2 t$ and $\pi(x') = i + {}^1\!/_2 t'$. Note that $|t - t'| \leq 2d(x, x')$ so $d(\beta(x), \beta(x')) \leq 3\pi d(x, x')$.

The remaining two assertions of the theorem are immediate from the construction, and the above observation that for $x \in N_{\mathcal{U}_i}$ the images $\beta(x)$ and $\Phi_{i'}(x)$ lie in a common simplex. □

Theorem 5.50. *Let W be a uniformly discrete bounded geometry metric space of asymptotic dimension m, and let $\mathcal{U}_*$ be an anti-Čech sequence for W with degrees bounded by $m+1$. Then the groups $K_*(C^*(X(W,\mathcal{U}_*)_h))$ are trivial.*

Proof. We will apply lemma 3.2. The idea of the proof is to use proposition 5.49 to construct a sequence of maps $\beta_1, \beta_2, \ldots$ which can be composed to form a sequence of maps $\beta_1, \beta_2 \circ \beta_1, \beta_3 \circ \beta_2 \circ \beta_1, \ldots$ with the property that for each i and each $\varepsilon > 0$, all distances between points in X_i are contracted by a factor of ε by all but finitely many of the maps. Such a sequence would satisfy the first two hypotheses of the lemma, however the maps would not get closer and closer together, and hence the third hypothesis would not be satisfied. We will therefore construct a family α_t of maps with the same contractive properties as the compositions $\cdots \circ \beta_2 \circ \beta_1$, and then pick a sequence $t_1, t_2, \ldots$ tending to infinity, such that α_{t_k} and $\alpha_{t_{k+1}}$ get closer together as k tends to infinity.

Let $i_1 = 1$ and let β_1 be a map from $X = \pi^{-1}([i_1, \infty))$ to $\pi^{-1}([i_2, \infty))$ with Lipschitz constant 1 as provided by proposition 5.49, where $i_2 > i_1$ is provided by the theorem. Then inductively, let β_j from $\pi^{-1}([i_j, \infty))$ to $\pi^{-1}([i_{j+1}, \infty))$ with Lipschitz constant $1/j$ be provided by proposition 5.49. We define

$$\alpha_{i_j} = \Phi_{i_j} \circ \beta_{j-2} \circ \cdots \circ \beta_1 \colon X \to \pi^{-1}([i_j, \infty)),$$

and we will use a homotopy to construct the family α_t for $i_j < t < i_{j+1}$. We know from the theorem that Φ_{i_j} is linearly homotopic to β_{j-1}. Let $\gamma_{j,t}$ be such a homotopy where $t \in [i_j, i_{j+1}]$ and $\gamma_{j,i_j} = \Phi_{i_j}, \gamma_{j,i_{j+1}} = \beta_{j-1}$. We can now define α_t for $t \in [1, \infty)$ by

$$\alpha_t = \Phi_t \circ \gamma_{j,t} \circ \beta_{j-2} \circ \cdots \circ \beta_1 \colon X \to \pi^{-1}([t, \infty)), \text{ for } t \in [i_j, i_{j+1}].$$

Note that this agrees with the previous definition when $t = i_j$.

Let t_k be a sequence tending to infinity with $t_{k+1} - t_k \to 0$ as $k \to \infty$. We will show that the sequence of maps α_{t_k} satisfies the hypotheses of lemma 3.2; this will complete the proof of the theorem. As any bounded subset of X lies in some X_i, and for $t_k > i$ the range of α_{t_k} does not meet X_i, it is clear that the maps are properly supported. To show that they have uniformly close steps, note that $d(\Phi_t(x), \Phi_{t'}(x)) \leq |t - t'|$ for all x, and as $i_{j+1} - i_j \geq 1$ the homotopy $\gamma_{j,t}$ has the property that $d(\gamma_{j,t}(x), \gamma_{j,t}(x)) \leq \pi|t - t'|$ for all x. Thus $d(\alpha_{t_k}(x), \alpha_{t_{k+1}}(x)) \leq (\pi + 1)|t_k - t_{k+1}| \to 0$ as $k \to \infty$. Hence given $\varepsilon > 0$ there is a k_0 such that for $k \geq k_0$ the pair $(\alpha_{t_k}(x), \alpha_{t_{k+1}}(x))$ lies within ε of the diagonal, while for each $k < k_0$ either $(\alpha_{t_k}(x), \alpha_{t_{k+1}}(x))$ lies on the diagonal or $\alpha_{t_k}(x), \alpha_{t_{k+1}}(x)$ both lie in $\pi^{-1}([1, t_{k+1}])$. Thus the collection of all such pairs is a controlled set for the hybrid structure, and the third condition is satisfied.

It remains to show that the sequence of maps is uniformly controlled. Given a hybrid controlled set A, we must show that

$$B_A = \{(\alpha_{t_k}(x), \alpha_{t_k}(x')) : k = 1, 2, \ldots, \text{ and } (x, x') \in A\}$$

is hybrid controlled. We already know that each β_j is Lipschitz with constant independent of j. We will show that the composition $\beta_j \circ \cdots \circ \beta_1$ is λ-Lipschitz for some constant λ independent of j. Let us for the moment assume this. As Φ_{i_j} and β_{j-1} are both Lipschitz, having constants respectively 1 and 3π it follows from lemma 5.47 that $\gamma_{j,t}$ is Lipschitz for all j, t with constant independent of j, t. We thus conclude that α_{t_k} is λ'-Lipschitz for some constant λ' independent of k.

Given $\varepsilon > 0$, as A is hybrid controlled it is a union $A_\varepsilon \cup A'_\varepsilon$ where A_ε lies in an ε/λ'-neighbourhood of the diagonal, and A'_ε is a subset of $X_i \times X_i$ lying within R of the diagonal, for some i, R sufficiently large. It is clear that $(\alpha_{t_k}(x), \alpha_{t_k}(x'))$ lies in an ε-neighbourhood of the diagonal for all k and for all $(x, x') \in A_\varepsilon$. On the other hand if $R\lambda/j < \varepsilon/\lambda'$ and $i_j \geq i$ then for $(x, x') \in A'_\varepsilon$, the map $\alpha_{i_{j+2}}$ is the composition of the contraction $\Phi_{i_{j+2}}$, the map β_j which contracts by a factor of $1/j$ on $N_{\mathcal{U}_{i_j}}$ and the composition $\beta_{j-1} \circ \cdots \circ \beta_1$ which is λ-Lipschitz and maps X_i onto $N_{\mathcal{U}_{i_j}}$. Hence $d(\alpha_{i_{j+1}}(x), \alpha_{i_{j+1}}(x')) < \varepsilon/\lambda'$ for $(x, x') \in A'_\varepsilon$. Thus for all k with $t_k \geq i_{j+2}$, the pair $(\alpha_{t_k}(x), \alpha_{t_k}(x'))$ lies in an ε-neighbourhood of the diagonal for $(x, x') \in A'_\varepsilon$. To show that B_A is hybrid controlled it remains to observe that the set of pairs $(\alpha_{t_k}(x), \alpha_{t_k}(x'))$ with $t_k < i_{j+2}$ and $(x, x') \in A'_\varepsilon$ lies in some $X_{i'} \times X_{i'}$ for i' sufficiently large.

To complete the proof we must show that the composition $\beta_j \circ \cdots \circ \beta_1$ is λ-Lipschitz with λ independent of j. Given $x, x' \in X$, let x_j, x'_j be their images under $\beta_j \circ \cdots \circ \beta_1$. First suppose $|\pi(x) - \pi(x')| \geq 1$. Then as x_j must lie in a simplex containing $\Phi_{i_{j+1}}(x)$, and similarly for x'_j it follows that $d(x_j, x'_j) \leq d(\Phi_{i_{j+1}}(x), \Phi_{i_{j+1}}(x')) + \pi \leq (1 + \pi)d(x, x')$ for all j. On the other hand if $|\pi(x) - \pi(x')| < 1$ there are at most two terms $\beta_{j'}$ in the composition which do not fix x, x' and for which $x_{j'-1}, x'_{j'-1}$ do not lie in the space $N_{\mathcal{U}_{i_{j'}}}$ on which $\beta_{j'}$ is contractive. As each $\beta_{j'}$ is Lipschitz with constant $3\pi/2$ it follows that when $|\pi(x) - \pi(x')| < 1$ we have $d(x_j, x'_j) \leq 9\pi^2 d(x, x')/4$. This inequality completes the proof. □

To complete the proof of theorem 5.50 we must prove the lemma which provided the coefficients for the maps β.

Proof of lemma 5.48. We will fix i' and a parameter k, and construct a partition of unity $\{h_U(x)\}$ depending on those choices. Having constructed the partition of unity we will see that $\sup\{|h_U(x) - h_U(x')|/d(x, x') : x \neq x'\}$ depends on i', k, and we can choose these so that this is less than ε.

For fixed i', we will begin by constructing bump functions $\tilde{h}^k_U$, depending on k. Increasing k will make the bump functions more spread out, and for an appropriate choice of k, the partition of unity will be given by

$$h_U(x) = \frac{\tilde{h}^k_U(x)}{\sum_{V \in \mathcal{U}_{i'}} \tilde{h}^k_V(x)}.$$

Let $\chi\colon \mathbb{R}^+ \to [0, 1]$ be a smooth function with $\chi \equiv 1$ on $[0, \pi/2]$, $\chi \equiv 0$ on $[\pi, \infty)$, and $|\chi'(t)| \leq 1$ for all t. We define the bump functions by

$$\tilde{h}^k_U(x) = \chi(\frac{1}{k} \min\{d(x, [V]) : V \in \mathcal{U}_i, B(V, (2k+1)\operatorname{Diam}(\mathcal{U}_i)) \subset U\}),$$

where $B(V, R) = \{w \in W : d(w, V) < R\}$. Note that for sufficiently small k, for each $x \in N_{\mathcal{U}_i}$ there is a set U with $\tilde{h}^k_U(x) = 1$. Indeed if $[V]$ is a vertex within distance $\pi/2$ of x, then let $V' = B(V, (2k+1)\operatorname{Diam}(\mathcal{U}_i))$. For any set $U \in \mathcal{U}_{i'}$ with $V' \subseteq U$, we have

$\tilde{h}^k_U(x) = 1$, and provided that $(4k+3)\operatorname{Diam}(\mathcal{U}_i) \leq \operatorname{Lebesgue}(\mathcal{U}_{i'})$ such a set U must exist. Let us assume that k satisfies this inequality.

Suppose that x lies in the simplex spanned by $[V_1], \ldots, [V_j]$ and $\tilde{h}^k_U(x) \neq 0$ for some $U \in \mathcal{U}_{i'}$. Then certainly there is some vertex $[V]$ with $B(V, (2k+1)\operatorname{Diam}(\mathcal{U}_i)) \subset U$ and $d(x, [V']) < k\pi$. For the spherical metric this allows us to conclude that there is a simplicial path from V to some vertex $[V_{j'}]$ of σ with at most $\frac{k\pi+\pi/2}{\pi/2} = 2k+1$ edges. Hence there is a sequence $V = V'_0, V'_1, \ldots, V'_{2k+1} = V_{j'}$ of elements of $\mathcal{U}_i$ with the intersection of consecutive sets non-empty. Hence $V_{j'} \subset B(V, (2k+1)\operatorname{Diam}(\mathcal{U}_i)) \subset U$. Thus whenever $x \in \sigma$ and $\tilde{h}^k_U(x) \neq 0$, we conclude that U contains some element of $\mathcal{U}_i$ defining a vertex of σ.

Note that $|\tilde{h}^k_U(x) - \tilde{h}^k_U(x')|$ is bounded by $\frac{1}{k}d(x, x')$. Consider the function $h_U(x) = \frac{\tilde{h}^k_U(x)}{\sum_{V \in \mathcal{U}_{i'}} \tilde{h}^k_V(x)}$. The numerator is Lipschitz with constant at most $1/k$ and is bounded by 1. From the bound on the degree of $\mathcal{U}_*$ the denominator has at most $m+1$ terms. Hence it is Lipschitz with constant at most $(m+1)/k$, and it is bounded between 1 and $m+1$. It follows that h_U is Lipschitz with constant as most $2(m+1)/k$.

Now choosing k sufficiently large, we have $2(m+1)/k < \varepsilon$, and given this choice of k, choosing i' sufficiently large we have $(4k+3)\operatorname{Diam}(\mathcal{U}_i) \leq \operatorname{Lebesgue}(\mathcal{U}_{i'})$. For these choices of k, i', the partition of unity $\{h_U\}$ is well defined, and each function is ε-Lipschitz. The second assertion of the lemma now follows immediately from the corresponding property of $\tilde{h}^k_U$. □

To complete the proof of the coarse Baum-Connes conjecture for spaces of finite asymptotic dimension we need the following result:

Theorem 5.51. *Let W be a uniformly discrete bounded geometry metric space, and let $\mathcal{U}_*$ be an anti-Čech sequence for W with bounded degrees. Then the forgetful map $C^*X(W,\mathcal{U}_*)_0/I_0 \to C^*X(W,\mathcal{U}_*)_h/I_h$ induces an isomorphism on K-theory.*

This is a 'homology uniqueness' result in the same vein as the isomorphism of the C_0 coarse assembly map.

Proof. Decompose $X = X(W,\mathcal{U}_*)$ as $X = X_{\text{ev}} \cup X_{\text{odd}}$, where

$$X_{\text{ev}} = \pi^{-1}(\bigsqcup_i [2i-1, 2i]), X_{\text{odd}} = \pi^{-1}(\bigsqcup_i [2i, 2i+1]).$$

Using the collapsing maps Φ_t, these spaces will coarse homotopy retact onto $\pi^{-1}(2\mathbb{N})$ and $\pi^{-1}(2\mathbb{N}+1)$ respectively, and their intersection is $\pi^{-1}(\mathbb{N})$. Using a relative version of the Mayer-Vietoris sequence, and the 5-lemma, it therefore suffices to prove that the forgetful map induces isomorphisms for $\pi^{-1}(I)$ where I is a subset of $\mathbb{N}$. It now suffices to prove:

Claim. *Let I be a subset of $\mathbb{N}$, let $X^I = \pi^{-1}(I)$, and let J_0, J_h be the ideals of the Roe algebras $C^*(X_0^I), C^*(X_h^I)$ given by the direct limit of the subalgebras supported on $X^I \cap X_i$. Suppose there exists m such that each simplicial complex $N_{\mathcal{U}_i}$ has dimension at most m. Then the coarsening $C^*(X_0^I)/J_0 \to C^*(X_h^I)/J_h$ induces an isomorphism on K-theory.*

The proof of this claim is modelled on the proof of the C_0 coarse Baum-Connes conjecture for uniform metric simplicial complexes. We write X^I as a union $Y_0 \cup \cdots \cup$

Y_m, where Y_j is the union of stars about the barycentres of j-simplices in the second barycentric subdivision of X^I.

We will prove inductively on k that the result holds for $C^*((Z_k)_0)/J_0 \cap C^*((Z_k)_0) \to C^*((Z_k)_h)/J_h \cap C^*((Z_k)_h)$, where $Z_k = Y_0 \cup \cdots \cup Y_k$. The space X^I is a uniform metric simplicial complex, so the induction process just repeats the C_0 coarse Baum-Connes result with the only difference being the replacement of the usual Mayer-Vietoris sequence by a relative Mayer-Vietoris sequence for the quotient algebras.

The one remaining point to justify is that the result holds for each space Y_j. The argument is as follows. By the coarse homotopy invariance theorem we may replace the space Y_j by the discrete space consisting of the j-barycentres of X^I. However with this final reduction the quotient algebras for the C_0 and hybrid structures actually agree. Any operator which is hybrid controlled is a sum of an operator supported within some X_i, and an operator of zero propagation. The hybrid algebra is therefore the direct limit $\varinjlim_i A_i$, where A_i is the subalgebra of hybrid controlled operators which have propagation zero on the complement of X_i. Similarly the C_0 Roe algebra is a direct limit $\varinjlim_i B_i$, where B_i is the subalgebra of C_0 controlled operators which have propagation zero on the complement of X_i. But $B_i/J_0 \cap B_i \cong A_i/J_h \cap A_i$, hence taking the direct limits we conclude that $C^*((Y_j)_0)/J_0 \cap C^*((Y_j)_0) \to C^*((Y_j)_h)/J_h \cap C^*((Y_j)_h)$ induces an isomorhpism on K-theory. This isomorphism proves the claim, and hence completes the proof of the coarse Baum-Connes conjecture for spaces of finite asymptotic dimension. □

Let us now gather together all the pieces.

Theorem 5.52. *If W is a bounded geometry metric space of finite asymptotic dimension, then the assembly map $\mu\colon KX_*(W) \to K_*(C^*W)$ is an isomorphism.*

Proof. Let m be the asymptotic dimension of W, let $\mathcal{U}_*$ be an anti-Čech sequence for W such that the nerves $N_{\mathcal{U}_i}$ have dimension at most m, and let $X = X(W,\mathcal{U}_*)$ be the total coarsening space. We have the following commutative diagram.

$$\begin{array}{ccc} K_{*+1}(C^*X_0/I_0) & \longrightarrow & K_{*+1}(C^*X_h/I_h) \\ \downarrow & & \downarrow \\ K_*(I_0) & \longrightarrow & K_*(I_h) \\ \uparrow & & \uparrow \\ KX_*(W) & \underset{\mu}{\longrightarrow} & K_*(C^*W) \end{array}$$

The maps $KX_*(W) \to K_*(I_0)$ and $K_*(C^*W) \to K_*(I_h)$ are isomorphisms in complete generality by theorems 5.39 and 5.42. The map $K_{*+1}(C^*X_0/I_0) \to K_*(I_0)$ is also an isomorphism in complete generality by theorem 5.37. Now using the finite dimensionality of X, the maps $K_{*+1}(C^*X_0/I_0) \to K_{*+1}(C^*X_h/I_h)$ and $K_{*+1}(C^*X_h/I_h) \to K_*(I_h)$ are isomorphisms by theorem 5.51 and 5.50 respectively. This completes the proof. □

We will conclude with a few observations about the proof. Although theorem 5.50 is the most technical step of the proof, it does not make a fundamental use of the finite asymptotic dimension hypothesis. Finite dimensionality is used only to obtain certain metric estimates for the maps α_{t_k} used in the Eilenberg swindle. In fact provided that $\dim N_{\mathcal{U}_i}$ does not grow too rapidly compared to $\operatorname{Diam}\mathcal{U}_i$ the same argument would still work. One might even suppose that for any bounded geometry space W, there is an anti-Čech sequence, for which $K_*(C^*X_h)$ vanishes. On the other hand, theorem 5.51 makes a fundamental use of finite dimensionality. The proof therefore involves a careful balancing between coarsening enough to make the Eilenberg swindle work, but not so much that we lose the finite dimensionality required for 5.51.

In the case of a uniformly contractible locally finite and finite dimensional complex W, instead of using an anti-Čech sequence to build X, we might simply take $X = W \times \mathbb{R}^+$. Again we would obtain isomorphisms from $K_*(W) = KX_*(W) \to K_{*+1}(C^*X_0/I_0)$ and $K_*(C^*W) \to K_*(I_h)$. Finite dimensionality of X would again give us the isomorphism $K_{*+1}(C^*X_h/I_h) \to K_*(I_h)$, hence in this case we find that $K_*(C^*X_h)$ is an obstruction group for the coarse Baum-Connes conjecture for W. Given some hypothesis on W, for example scaleability, an Eilenberg swindle in the spirit of theorem 5.50 could perhaps be used to show that this group vanishes.

Finally for Γ a discrete group, let $W = E\Gamma$ and suppose that this is a finite dimensional complex. Again let $X = W \times \mathbb{R}^+$ and note that Γ also acts on X. It would be interesting to see how much of the argument could be carried through equivariantly. The C_0 structure is not very well adapted to this context, which causes problems with the left hand side. On the other hand the hybrid structure appears better adapted for an equivariant theory. The left hand side should in principle be straightforward to deal with, so it might be possible to identify the K-theory of the Γ invariant part of C^*X_h as an obstruction group for the Baum-Connes conjecture for Γ.

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Vita

Nick Wright was born in 1976 in the small village of Barton-in-the-beans, just a few miles away from where Henry Tudor defeated Richard III in the battle of Bosworth. From an early age Nick was interested in Mathematics, and on getting his hands on a computer he started honing his logical tendencies by programming. He developed an interest in computers, not to mention science and especially physics, but nonetheless mathematics remained number one, and in 1994 he began studying for a BA in Mathematics at Churchill College, Cambridge. In 1997 he graduated with first class honours, and he remained at Cambridge for a further year, attaining the Certificate of Advanced Study in Mathematics more commonly known as 'Part III'. In 1998 while seeking a PhD position at Oxford he met John Roe, and joined him in moving from Oxbridge to Penn State University. While at Cambridge he received three scholarships from Churchill College, and since arriving at Penn State he has been awarded the Curry fellowship and membership of the honour society Phi Kappa Phi. Nick has a great enthusiasm for games and puzzles, which tend to bring out his otherwise dormant competitive streak. He also enjoys reading, has a great interest in classic films, mostly those which are twenty or more years older than he is, and more recently developed a taste for the art of Magritte.