Property A, partial translations and extensions

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14th May, 2008

Property A

Definition: (Yu). A metric space X has property A if for all $R, \varepsilon > 0$ there is a family $A_x, x \in X$ of finite subsets of $X \times \mathbb{N}$ such that

(i) there exists S such that $A_x \subseteq B_S(x) \times \mathbb{N}$ for all x,

(ii) $|A_x \Delta A_y| / |A_x| < \varepsilon$ for $d(x, y) \leq R$.

Theorem: (Yu). If X has property A then the **coarse Baum-Connes conjecture** holds for X. If G is a group with property A then the **Novikov conjecture** holds for G.

Block-Weinberger gave a homological characterisation of amenability.

Question: Is there a homological characterisation of property A?

Exactness

A C^*-algebra B is **exact** if for every exact sequence

$$0 \to J \to A \to A/J \to 0$$

of C^*-algebras, the spatial tensor product with B preserves exactness.

A C^*-algebra B is **nuclear** if its tensor products are unique.

A group G is **exact** if for every exact sequence

$$0 \to J \to A \to A/J \to 0$$

of G-C^*-algebras, the reduced crossed product with G preserves exactness.

Equivalently $C_r^*(G)$ is exact, where $C_r^*(G)$ is the completion of $\mathbb{C}[G]$ acting on $\ell^2(G)$ (on the right).

Ozawa Kernels

Theorem: (Higson, Roe, Ozawa, Tu). X has property A iff for all $R, \varepsilon > 0$ there exists a positive kernel $k(x,y)$ such that

(i) if $d(x,y) \leq R$ then $|k(x,y) - 1| < \varepsilon$,

(ii) there exists S such that if $d(x,y) \geq S$ then $k(x,y) = 0$.

These are called **Ozawa kernels**.

Let $C_u^*(G)$ be the completion of the algebra of finite propagation operators on $\ell^2(G)$.

Theorem: (Ozawa). The following are equivalent for a group G

(i) $C_u^*(G)$ is nuclear,

(ii) $C_r^*(G)$ is exact,

(iii) G admits Ozawa kernels.

Proof: (ii) $\Rightarrow$ (iii)

Exactness of $C_r^*(G)$ is equivalent to nuclearity of the inclusion $C_r^*(G) \hookrightarrow B(\ell^2(G))$.

That is for $\varepsilon > 0$ and $E \subset C_r^*(G)$ finite, there exists $\theta : C_r^*(G) \to B(\ell^2(G))$, finite rank and completely positive, such that $\|\theta(T) - T\| < \varepsilon$ for T in E.

The map θ can be chosen to depend only on finitely many matrix entries $< v_i T, v_j >$ of T.

Now define $k(x, y) = \langle \delta_x, \theta([y^{-1}x])\delta_y \rangle$. This has the desired properties.

Key point: the matrix $y^{-1}x$ is positive.

Partial translations

A *partial bijection* of X is a triple (U, V, t) where U, V are subsets of X and $t : U \to V$ is a bijection.

Equivalently, a partial bijection is given by a partial isometry of $\ell^2(X)$, $t : \delta_x \mapsto \delta_{t(x)}$ for $x \in U$, $t : \delta_x \mapsto 0$ for $x \notin U$.

A *partial translation* of X is a partial bijection such that for some R we have $d(t(x), x) \leq R$ for all $x \in U$.

Equivalently a partial translation is given by a partial isometry in $C^*_u(X)$.

A *partial translation structure* $\mathcal{T}$ on X is a collection of partial translations satisfying certain axioms.

Define $C^*(X, \mathcal{T})$ to be the subalgebra of $C^*_u(X)$ generated by $t \in \mathcal{T}$.

Idea: A space with a partial translation structure is a generalisation of a group, or a space with a group action.

Theorem: (Brodzki, Niblo, W). Let $\mathcal{T}$ be a free partial translation structure on X. Then the following are equivalent:

(i) $C^*_u(X)$ is nuclear,

(ii) $C^*(X, \mathcal{T})$ is exact,

(iii) X admits Ozawa kernels.

Theorem: (B,N,W). If X admits a uniform embedding into a group G, then the translation structure on G pulls back to a free partial translation structure on X.

In particular, any *subset* of a group is naturally equipped with a free partial translation structure.

The Toeplitz extension

The space $L^2(S^1)$ is $\overline{\mathrm{span}}\{e^{2\pi inx} : n \in \mathbb{Z}\}$, and the Hardy space is

$$\mathcal{H}^2 = \overline{\mathrm{span}}\{e^{2\pi inx} : n = 0, 1, 2, \ldots\}.$$

For f in $C(S^1)$, let $T_f = P_{\mathcal{H}_2} M_f P_{\mathcal{H}_2}$.

The Toeplitz algebra $\mathscr{T}$ is the algebra generated by T_f. This fits into an extension

$$0 \to \mathscr{K} \to \mathscr{T} \to C(S^1) \to 0.$$

Let $G = \mathbb{Z}$, let $X = \{0, 1, 2, \ldots\}$ and let $\mathcal{T}_X$ denote the partial translation structure on X inherited from $\mathbb{Z}$.

Then $X \hookrightarrow \mathbb{Z}$ induces a map $C^*(X, \mathcal{T}_X) \to C^*_r(\mathbb{Z})$.

$C^*(X, \mathcal{T}_X) \cong \mathscr{T}$, $C^*_r(\mathbb{Z}) \cong C(S^1)$ and the map between these gives the Toeplitz extension.

Translations on the primes

The twin primes conjecture asserts that there are infinitely many pairs p, q of primes such that $q - p = 2$.

De Polignac's conjecture asserts that for any even number $2k$, there are infinitely many pairs p, q of primes such that $q - p = 2k$.

Let $P \subset \mathbb{Z}$ denote the set of primes, and $\mathcal{T}_P$ the partial translation structure inherited from $\mathbb{Z}$. The algebra $C^*(P, \mathcal{T}_P)$ contains $\widetilde{\mathscr{K}}$.

Theorem: (B,N,W). $C^*(P, \mathcal{T}_P) \ncong \widetilde{\mathscr{K}}$ if and only if de Polignac's conjecture holds for some k.

The Cuntz algebra

Theorem: (Cuntz). There is a unique algebra $\mathcal{O}_2$ generated by two isometries V_1, V_2 such that $V_1V_1^* + V_2V_2^* = 1$.

Let $G = F_2 =< a, b >$. Let $X \in G$ be the set of non-negative words (i.e. words in a, b), and $\mathcal{T}_X$ the inherited partial translation structure.

Theorem: (B,N,W). $C^*(X, \mathcal{T}_X)$ is the universal algebra on two isometries with orthogonal range.

Let Y be the set of words of the form $a^{-n}x$ for some x in X, and $\mathcal{T}_Y$ the inherited partial translation structure.

Theorem: (B,N,W). $C^*(Y, \mathcal{T}_Y) \cong \mathcal{O}_2$ and the extension

$$0 \to \mathscr{K} \to C^*(X, \mathcal{T}_X) \to C^*(Y, \mathcal{T}_Y) \to 0.$$

induced by $X \hookrightarrow Y$ is the Cuntz extension.

Embedding $C_r^*(F_2)$ in $\mathcal{O}_2$

Choi showed indirectly that $C_r^*(F_2)$ can be embedded as a subalgebra of $\mathcal{O}_2$.

Theorem: (B,N,W). For Y as above there is a quasi-isometry $\phi : F_2 \to Y$ such that the image of $C_r^*(F_2)$ under ϕ_* lies in $C^*(Y, \mathcal{T}_Y) \cong \mathcal{O}_2$.

Outline:

A word in $F_2 =< a, b >$ which does not begin with a^{-1} can be described geometrically by a sequence of moves

$$L = \text{turn left, then move forward one step}$$
$$R = \text{turn right, then move forward one step}$$
$$F = \text{move forward one step}$$

where initially we face East (the a direction).

Allowing initial backwards moves B we obtain arbitrary elements of F_2.

Encode L, R, F, B by $u_L = b^2, u_R = ab, u_F = a, u_B = a^{-1}$.

Encode the directions N, E, S, W as $v_N = b^2, v_E = a^2, v_S = ba, v_W = ab$.

For an element $g \in F_2$ described by $B^n\, w(L, R, F)$ with a heading $h \in \{N, E, S, W\}$, we define

$$\phi(g) = u_B^n\, w(u_L, u_R, u_F)\, v_h.$$

This embeds F_2 into the 3-regular tree Y.

Key point: the **partial** nature of partial translations allows them to act conditionally.

For example $a^3(a^*)^2$ can be interpreted as the instruction:

if a word ends with a^2 then replace it with a^3
(heading$=E$) $\mapsto$ (move$=F$, heading$=E$).

Thus the action of F_2 on Y can be encoded by partial translations. The image of $C_r^*(F_2)$ therefore lies in the partial translations. □