Coarse geometry and scalar curvature

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Gaussian Curvature

For motivation we will begin by considering curvature in 2-d.

For a 2-d surface in 3-d space, the Gaussian curvature K is defined by

$$\partial_u n \times \partial_v n = K \partial_u p \times \partial_v p$$

for $p(u,v)$ a parametrization of the surface and $n(u,v)$ the normal vector.

Theorem (Gauss-Bonnet). *For a closed Riemannian 2-manifold M with Gauss curvature K*

$$\int_M K dA = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic.

Notes:
– Curvature is quite rigid in 2-d.
– There is a maximum ‘curvature content’ for surfaces of a given size (area). In particular there always exist points where $K \leq \frac{4\pi}{\text{Area } M}$.

Curvature in dimension $n \geq 3$

In dimension $n \geq 3$ there are several types of curvature on a Riemannian manifold.

Scalar curvature κ: total curvature over all directions, this is the weakest form of curvature

Ricci curvature

Sectional curvature: this is the strongest form of curvature

Note: In dimension 2, $K = \kappa/2$.

The curvatures are defined in terms of derivatives of the Riemannian metric g. Therefore two metric tensors which are uniformly close may nonetheless have very different curvatures.

Negative curvature

There are no (large scale) obstructions to negative curvature.

Theorem (Lohkamp). *Given a Riemannian manifold* (M^n, g)*,* $(n \geq 3)$*, a smooth function* f *on* M *with* $f < \kappa_g$ *and* $\varepsilon > 0$*, there exists a metric* g_ε *on* M *such that* $f - \varepsilon \leq \kappa_{g_\varepsilon} \leq f$ *and* $|g - g_\varepsilon| < \varepsilon$ *on the unit tangent bundle (for* g*). Moreover this can be done locally.*

There is a similar result for the Ricci curvature.

Theorem (Lohkamp). *There is a local and functorial process for reducing the sectional curvature of a space (at the cost of changing the local topology).*

For example in dimension 2 we can add negatively curved handles locally.

However there are obstructions to *positive curvature*, even to the weakest form – positive scalar curvature.

The Dirac operator

We will use index theory to study obstructions to positive scalar curvature. For suitable manifolds (orientable, spin) there is a differential operator D called the *Dirac* operator on sections of a bundle S over M, which encapsulates information about the geometry of the manifold.

Weitzenbock formula

If D is the Dirac operator for a spin manifold M, ∇ is the connection on the spin bundle S, and κ is the scalar curvature then

$$D^2 = \nabla^* \nabla + \frac{1}{4}\kappa$$

where κ acts on the sections of S by pointwise multiplication.

Index theory

Definition. An operator D is *Fredholm* if it has finite dimensional kernel and cokernel.

The index of a Fredholm operator D is

$$\text{index}(D) = \dim\ker D - \dim\ker D^*$$

If M is a closed manifold then the Dirac operator on M is Fredholm.

We will also be interested in open manifolds. The idea is to define a *higher index* which reduces to the Fredholm index in the case of a closed manifold. This higher index will belong to a K-theory group, and we may think of it as a formal difference

$$[\ker D] - [\ker D^*].$$

Coarse geometry

A map ϕ between two metric spaces is *coarse* if

- ϕ^{-1}(bounded set) is bounded, and
- for all $R > 0$ there exists $S > 0$ such that if $d(x, y) \leq R$ then $d(\phi(x), \phi(y)) \leq S$.

Two maps ϕ, ψ are *close* if there exists $S > 0$ such that $d(\phi(x), \psi(x)) \leq S$ for all x.

This gives rise to a notion of coarse equivalence of spaces, defined by a pair of coarse maps which are inverse to one another up to closeness.

The Roe algebra

We will study coarse geometry by means of operator algebras. Let X be an open manifold and S a bundle over X. We will consider operators on $L^2(X,S)$ of the form

$$\xi(\cdot) \mapsto \int_X k(\cdot,x)\xi(x)dx$$

where the kernel k takes values $k(y,x) \in \mathrm{End}(S)$.

Such an operator has *finite propagation* if there exists R such that $k(y,x) = 0$ when $d(x,y) > R$.

$\sup\{d(x,y) : k(y,x) \neq 0\}$ is called the propagation of the operator.

Definition (Roe). $C^*(X)$ is the completion of the algebra of finite propagation bounded operators arising from kernels in this way.

The coarse higher index

Let D be a Dirac-type operator and consider the wave equation $\frac{d\xi}{dt} = iD\xi$. This has solution operator e^{itD}.

Lemma. *The waves travel with speed at most 1. The wave solution operator e^{itD} has propagation at most $|t|$.*

By Fourier theory we can therefore construct finite propagation operators out of D. This gives rise to the coarse higher index

$$\text{index}(D) \in K_*(C^*X).$$

Theorem (Roe). *If D is invertible then* index(D) *vanishes.*

The argument involves an exact sequence

$$\begin{array}{ccccc} K_{*+1}(D^*X) & \to & K_{*+1}(D^*X/C^*X) & \to & K_*(C^*X) \\ ? & \mapsto & [D] & \mapsto & \text{index}(D) \end{array}$$

If D is invertible then $[D]$ can be lifted to $K_{*+1}(D^*X)$.

Obstruction to positive curvature

Theorem (Roe). *Let* $X = \tilde{M}$ *the universal cover of a closed spin manifold* M *(with metric pulled back from* M*), and let* D *be the Dirac operator for* X*. If* index$(D) \neq 0$ *then* M *admits no metric of positive scalar curvature.*

Proof. As M is compact, any two metric on M yield coarsely equivalent metrics on X. Hence the (non-)vanishing of index(D) is independent of the choice of metric on M. But by the Weitzenbock formula $D^2 = \nabla^*\nabla + \frac{1}{4}\kappa$ so

$$\begin{aligned} \langle D\xi, D\xi \rangle &= \langle (\nabla^*\nabla + \frac{1}{4}\kappa)\xi, \xi \rangle \\ &= \langle \nabla\xi, \nabla\xi \rangle + \frac{1}{4}\langle \kappa\xi, \xi \rangle \geq \varepsilon \langle \xi, \xi \rangle \end{aligned}$$

with $\varepsilon > 0$ so D is invertible. □

Example. For $X = \mathbb{R}^n$, $M = \mathbb{T}^n$, index$(D) \neq 0$ so $\mathbb{T}^n$ admits no metric of positive scalar curvature.

Properly positive scalar curvature

Question. Suppose M admits a positive scalar curvature metric. Can the curvature of M be increased arbitrarily for small changes of the metric?

Let $X = M \sqcup M \sqcup M \sqcup \ldots$ where each copy of M is equipped with the given metric. The question amounts to the following:

Is there a 'small' distortion of the metric on X for which κ is *properly positive* (i.e. $\kappa \to +\infty$)?

To make this second formulation precise we refine our notion of coarse geometry.

If $g_1, g_2, \ldots$ are metrics on M converging integrally to g, then $(X, g \sqcup g \sqcup \ldots)$ is coarsely equivalent to $(X, g_1 \sqcup g_2 \sqcup \ldots)$ in the sense of C_0 *coarse geometry*.

C_0 coarse geometry

A map $\phi\colon X \to Y$ between two metric spaces is C_0 *coarse* if

- ϕ^{-1}(bounded set) is bounded, and
- for all $r \in C_0^+(X \times X)$ there is an $s \in C_0^+(Y \times Y)$ such that if $d(x, x') \leq r(x, x')$ then $d(\phi(x), \phi(x')) \leq s(\phi(x), \phi(x'))$.

Two maps ϕ, ψ are C_0-*close* if there is an $s \in C_0^+(Y \times Y)$ such that $d(\phi(x), \psi(x)) \leq s(\phi(x), \psi(x))$ for all x.

This gives rise to a notion of C_0 coarse equivalence of spaces, defined by a pair of C_0 coarse maps which are inverse to one another up to C_0 closeness.

C_0 coarse geometry

This is coarse geometry not with bounded errors, but with errors tending to zero at infinity.

An operator on $L^2(X,S)$ given by

$$\xi(\cdot) \mapsto \int_X k(\cdot,x)\xi(x)dx$$

is C_0 controlled if $\sup\{d(x,y) : x \in X, k(y,x) \neq 0\} \to 0$ as $y \to \infty$, and likewise for x, y interchanged.

Definition. $C_0^*(X)$ is the completion of the algebra of bounded operators given by C_0 controlled kernels k.

Note that $C_0^*(X)$ is a subalgebra of $C^*(X)$.

There is a C_0-coarse higher index, $\text{index}_0(D) \in K_*(C_0^*(X))$. This maps to the coarse higher index index(D) under the inclusion $C_0^*(X) \hookrightarrow C^*(X)$.

Fredholm operators and the C_0 higher index

Theorem. *If X has properly positive scalar curvature ($\kappa \to +\infty$) then the Dirac operator for X is Fredholm, indeed it has discrete spectrum.*

Physical interpretation: If the potential well of κ is infinitely deep, then all energy eigenstates of the wave equation are bound states.

Theorem. *If D is invertible and has discrete spectrum then* $\mathrm{index}_0(D) = 0$*.*

$$\begin{array}{ccc} K_{*+1}(D_0^*(X)/\mathscr{K}) & \longrightarrow & K_*(\mathscr{K}) \\ \downarrow & & \downarrow \\ K_{*+1}(D_0^*(X)/C_0^*(X)) & \longrightarrow & K_*(C_0^*(X)) \\ & & \\ [D] & \longrightarrow & \mathrm{index}_0(D) \end{array}$$

If D has discrete spectrum then $[D]$ lifts to $K_{*+1}(D_0^* X/\mathscr{K})$ and its image in $K_*(\mathscr{K})$ is the Fredholm index. But if D is invertible then the Fredholm index vanishes.

Maximum curvature content

Theorem. *For any closed Riemannian spin manifold (M, g) there is a bound R and $\varepsilon > 0$ such that every metric ε-close to g (as a length metric) has points with $\kappa \leq R$.*

Proof. For such a manifold M, $X = M \sqcup M \sqcup \ldots$ and D the Dirac operator for X, the C_0 higher index of D is non-zero. Hence no metric on X, C_0-equivalent to $g \sqcup g \sqcup \ldots$ has properly positive scalar curvature. □