PROPERTY A AND $CAT(0)$ CUBE COMPLEXES

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ABSTRACT. Yu's property A is a non-equivariant analogue of amenability, which is defined for metric spaces. A tree is a typical example of space with property A. In this paper we show that finite dimensional $CAT(0)$ cube complexes, higher dimensional analogues of trees, also have property A.

INTRODUCTION

This paper is devoted to the study of property A for finite dimensional $CAT(0)$ cube complexes. These spaces, which are generalizations of trees, appear naturally in many problems in geometric group theory. We prove that finite dimensional $CAT(0)$ cube complexes satisfy Yu's property A. For a cube complex X, we will do this by explicitly constructing a sequence $f_{n,x}^{N,X}$ of families of weighting functions, which are ℓ^1-functions on the set of vertices of X.

The weighting functions we construct satisfy the following remarkable property (Theorem 3.8).

Theorem. *Let X be a $CAT(0)$ cube complex of dimension at most d, and take $N \geq d-1$. For x a vertex of X, the ℓ^1-norm of the function $f_{n,x}^{N,X}$ is $\binom{n+N}{N}$. In particular it depends only on n and N, and not on the vertex x or the complex X.*

From this we deduce our main result (Theorem 3.10).

Theorem. *Let X be a finite dimensional $CAT(0)$ cube complex. Then X has property A.*

1. PRELIMINARIES

1.1. **Property A.** We begin by recalling the definition of amenability for a discrete group.

Definition 1.1. A countable discrete group G is *amenable* if there exists a sequence of finite subsets A_n of G such that for each g in G the sequence

$$\frac{|A_n \Delta g A_n|}{|A_n|}$$

tends to zero as n tends to infinity.

Such a sequence is called a *Følner sequence.*

Amenability can equivalently be described in terms of the characteristic functions of the sets A_n. A discrete group G is amenable if there exists a sequence of finitely supported $\{0,1\}$-valued functions f_n on G such that for each g in G we have

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$\frac{\|f_n - g\cdot f_n\|_1}{\|f_n\|_1} \to 0$, where $\|f\|_1$ denotes the ℓ^1-norm on the space of finitely supported functions $f\colon G \to \{0,1\}$, i.e. $\|f\|_1 = \sum_{h\in G} |f(h)|$.

We note that if G is equipped with a proper left-invariant metric, then f_n is supported in some ball about the identity $B_{S_n}(e)$. Hence each function $g \cdot f_n$ is supported in the ball $B_{S_n}(g)$.

We will now define property A. The following definitions are due to Yu, [5].

Definition 1.2. A discrete metric space X has *property A* if for all $R, \varepsilon > 0$ there exists $S > 0$ and a family of finite non-empty subsets A_x of $X \times \mathbb{N}$, indexed by x in X, such that

- for all x, x' with $d(x,x') < R$ we have $\frac{|A_x \Delta A_{x'}|}{|A_x|} < \varepsilon$;
- for all (x', n) in A_x we have $d(x,x') \leq S$.

Definition 1.3. An arbitrary metric space X has property A if it contains a discrete coarsely dense subset with property A.

Implicit in the definition in the general case are the following facts. Firstly, every metric space X contains some discrete coarsely dense subspace: this is a straightforward Zorn's lemma argument. Secondly, if any one coarsely discrete subset has property A then they all do. These subsets are coarsely equivalent to the space X and hence to each other, therefore this second requirement amounts to the fact that property A is preserved by coarse equivalence. The argument is short, and for the sake of completeness we give it here. Suppose X_1 and X_2 are discrete spaces with $\phi\colon X_1 \to X_2$ and $\psi\colon X_2 \to X_1$ defining a coarse equivalence, and that X_1 has property A. Given $R, \varepsilon > 0$ there exists R_1 such that if $y, y' \in X_2$ with $d(y,y') < R$ then $d(\psi(y), \psi(y')) < R_1$. Given a family A_x, $x \in X_1$ satisfying the conditions of 1.2 for R_1, ε one defines B_y for $y \in X_2$ by

$$B_y = \big\{(y', n) \in X_2 \times \mathbb{N} : n \leq |\{(x', m) \in A_{\psi(y)} : \phi(x') = y'\}|\big\}.$$

As $d(x', \psi(y))$ is bounded for $(x', m) \in A_{\psi(y)}$, so also is $d(y', y)$ for (y', n) in B_y, since $y' = \phi(x')$ for some x'. Moreover if $d(y,y') \leq R$ then $d(\psi(y), \psi(y')) \leq R_1$ and hence $\frac{|B_y \Delta B_{y'}|}{|B_y|} \leq \frac{|A_{\psi(y)} \Delta A_{\psi(y')}|}{|A_{\psi(y)}|} < \varepsilon$. One thus deduces that X_2 also has property A.

Note that this argument does not require the spaces to be discrete. Hence one could equivalently define property A for arbitrary metric spaces by the existence of families A_x as in Definition 1.2.

Proposition 1.4. *For an arbitrary metric space X, the following are equivalent.*

(1) *X has property A.*
(2) *For all $R, \varepsilon > 0$ there exists $S > 0$ and a family of finitely supported functions $f_x\colon X \to \mathbb{N} \cup \{0\}$, indexed by x in X, such that f_x is supported in $B_S(x)$, and if $d(x,x') \leq R$ then*

$$\frac{\|f_x - f_{x'}\|_1}{\|f_x\|_1} < \varepsilon,$$

where $\|\cdot\|_1$ denotes the ℓ^1-norm on the space of finitely supported functions on X.
(3) *There exists a sequence of families $f_{n,x}$, indexed by x in X, of finitely supported functions from X to $\mathbb{N} \cup \{0\}$, and a sequence $S_n > 0$, such that*

for each n, x the function $f_{n,x}$ is supported in $B_{S_n}(x)$, and for any $R > 0$

$$\frac{\|f_{n,x} - f_{n,x'}\|_1}{\|f_{n,x}\|_1} \to 0$$

uniformly on the set $\{(x, x') : d(x, x') \leq R\}$ as $n \to \infty$.

We will make use of condition 3, and will explicitly define functions $f_{n,x}$ on $CAT(0)$ cube complexes, satisfying the above conditions. We will refer to the functions $f_{n,x}$ as weighting functions.

The proof of the proposition is straightforward. The equivalence of 1 and 2 is given by mapping A_x to $f_x(y) = |A_x \cap \{y\} \times \mathbb{N}|$, and conversely by mapping f_x to $A_x = \{(y, n) : n \leq f_x(y)\}$. The equivalence of 2 and 3 is elementary.

Note that in passing from groups to metric spaces we replace the equivariant family of functions $g \cdot f_n$ with a family of functions $f_{n,x}$ each of which is merely required to be supported in the ball $B_{S_n}(x)$ for some S_n. We also allow the functions to take any non-negative integer value.

The extra flexibility of allowing weights which are greater than 1 is not required in the case of amenability: if there exists a sequence $f_n : G \to \mathbb{N} \cup \{0\}$ with $\frac{\|f_n - g \cdot f_n\|_1}{\|f_n\|_1} \to 0$ for each g, then in fact there is also a sequence of $\{0, 1\}$-valued functions with the same property. Indeed if one writes each function f_n as a sum $\sum_i f_n^i$ where each f_n^i is $\{0, 1\}$-valued and $f_n^1 \geq f_n^2 \geq \dots$ then one has

$$\frac{\|f_n - g \cdot f_n\|_1}{\|f_n\|_1} = \frac{\left\|f_n^1 - g \cdot f_n^1\right\|_1 + \left\|f_n^2 - g \cdot f_n^2\right\|_1 + \dots}{\|f_n^1\|_1 + \|f_n^2\|_1 + \dots}.$$

If this is at most ε then for some i one must have $\frac{\left\|f_n^i - g \cdot f_n^i\right\|_1}{\|f_n^i\|_1} < \varepsilon$. In general for any collection $g_1, \dots, g_k$, if $\frac{\|f_n - g_j \cdot f_n\|_1}{\|f_n\|_1} < \frac{\varepsilon}{k+1}$ for each j then there is some i such that $\frac{\left\|f_n^i - g_j \cdot f_n^i\right\|_1}{\|f_n^i\|_1} < \varepsilon$ for all j, and hence one can extract a subsequence of $\{0, 1\}$-valued functions such that $\frac{\|f_n - g \cdot f_n\|_1}{\|f_n\|_1} \to 0$ for all g.

For property A however it would be too restrictive to insist that all weights are either 0 or 1. The problem is that one would need to find a single value i such that $\frac{\left\|f_{n,x}^i - f_{n,x'}^i\right\|_1}{\left\|f_{n,x}^i\right\|_1} < \varepsilon$ for all x, x' in the set $\{(x, x') : d(x, x') \leq R\}$; this could be achieved for any finite set of pairs, but without the equivariance of the family it would not in general be possible to achieve it over this infinite set.

1.2. $CAT(0)$ **cube complexes.** A $CAT(0)$ cube complex is a higher dimensional analogue of a tree. It is a cell complex built out of Euclidean cubes of side-length 1, satisfying the *CAT(0) condition*. Equipped with the Euclidean path-length metric d_2.

Definition 1.5. A geodesic metric space (X, d) is $CAT(0)$ if for every three points $x, y, z \in X$ and for every triangle T in X having these points as vertices, the map from a Euclidean triangle with edge-lengths $d(x, y), d(y, z), d(z, x)$ to T is contractive.

For more details on the foundations of $CAT(0)$ cube complexes see [1].

The *edge-path metric* (or ℓ^1-metric) on the set of vertices of X is the metric $d_1(x, y)$ defined to be the minimum number of edges on a path from x to

y. This metric extends to the whole space X: for two points x, y in the same cube, having coordinates $(x_1, \dots, x_d), (y_1, \dots, y_d)$, the distance is defined to be $d_1(x, y) = \sum_{i=1}^{d} |x_i - y_i|$. This extends to a path-metric on X.

Proposition 1.6. *For any $CAT(0)$ cube complex X the restrictions of d_1 and d_2 to the vertex set are coarsely equivalent. If moreover the cube complex is finite dimensional then the vertex set with either of these metrics is coarsely equivalent to X with either metric.*

Proof. Given any two vertices x, y one has $d_1(x, y) = \sum_{i=1}^{l} d_1(x_{i-1}, x_i)$ where $x_0 = x, x_l = y$ and x_{i-1}, x_i are adjacent for all i. Now $d_2(x, y) \leq \sum_{i=1}^{l} d_2(x_{i-1}, x_i)$ by the triangle inequality. Since x_{i-1}, x_i are adjacent we have $d_1(x_{i-1}, x_i) = d_2(x_{i-1}, x_i) = 1$, hence $d_2(x, y) \leq d_1(x, y)$.

On the other hand if $d_1(x, y) = l$ then the Euclidean geodesic from x to y lies in a subcomplex of dimension at most l, and hence $d_1(x, y) \leq \sqrt{l} d_2(x, y)$. Hence $d_2(x, y) \geq \sqrt{l} = \sqrt{d_1(x, y)}$. This proves that the metrics are coarsely equivalent.

If the cube complex is finite dimensional with dimension d, then the vertex set is $d/2$-dense in the metric d_1, and is $\sqrt{d/2}$-dense in the metric d_2. This proves that the vertex set is coarsely equivalent to X for each metric. □

We will prove that the vertex set of a $CAT(0)$ cube complex has property A, and hence making use of the proposition, we will deduce that the complex X itself has property A. As we will only make use of the metric d_1 we shall henceforth drop the subscript and denote this by d.

For x and y vertices of X, the convex hull of x and y i.e. the set of points lying on geodesics from x to y is called the *interval* from x to y, and is denoted $[x, y]$. Note that intervals in X naturally carry the structure of a subcomplex of X.

For example if $\mathbb{R}^d$ is viewed as a $CAT(0)$ cube complex in the obvious way, that is we view the integer lattice $\mathbb{Z}^d$ as a set of vertices, then the interval is a cuboid. More precisely it is a subcomplex of $\mathbb{R}^d$ having vertex set

$$\{a_1, a_1 + 1, a_1 + 2, \dots, b_1\} \times \{a_2, a_2 + 1, \dots, b_2\} \times \cdots \times \{a_d, a_d + 1, \dots, b_d\}$$

for some a_i, b_i, with cubes of the form $\{x_1, x_1 + 1\} \times \dots \{x_d, x_d + 1\}$ (and all faces of this) for each $x_1, \dots, x_d$ with $a_i \leq x_i < b_i$.

We will make use of the following useful result.

Theorem 1.7. *Every interval in a CAT(0) cube complex of dimension d embeds into an interval in $\mathbb{R}^d$.*

Proof. We fix a base point x_0 and let x be an arbitrary vertex in V. Let $\mathcal{H}_x$ denote the set of hyperplanes in X that separate x_0 from x. Let us choose a cube C of the largest dimension which is contained in the interval $[x_0, x]$; let us assume that the dimension of this cube is k, $0 \leq k \leq n = \dim X$. Let V_C be the set of vertices of X contained in C and let $\mathcal{H}_C \subset \mathcal{H}_x$ be the set of hyperplanes which separate at least two vertices in V_C; these are the hyperplanes that intersect the cube C. The set $\mathcal{H}_C$ has k elements which we label $t_1, \dots, t_k$. (We could choose a geodesic from x_0 to x which intersects each of the hyperplanes in $\mathcal{H}_C$ and then number those hyperplanes according to the order in which they are intersected by the chosen geodesic.)

Recall that each hyperplane h of X corresponds to a pair of half spaces $h^{\pm}$ and the set of all half spaces is partially ordered set under inclusion. We say that a family of hyperplanes $\{h_i\}$ forms a *chain* if they are disjoint and for each disjoint triple h_i, h_j, h_k in the family one of the hyperplanes separates the other 2. This is equivalent to the statement that we may choose one half space h_i^+ corresponding to each hyperplane so that these half spaces form a chain under inclusion. If a disjoint family of hyperplanes all cross a given interval $[x_0, x]$ in the cube complex then the CAT(0) geodesic from x_0 to x imposes a linear order on the hyperplanes (the order in which it crosses them) exhibiting the fact that they form a chain.

Let s_1 be a maximal chain of hyperplanes on $\mathcal{H}_x$ which contains the hyperplane t_1. Let then s_2 be a maximal chain of hyperplanes in $\mathcal{H}_x \setminus s_1$ which contains the hyperplane s_2. By construction, s_2 is disjoint from s_1. Continuing in this way, we obtain pairwise disjoint chains $s_1, \ldots, s_k$, where s_k is the maximal chain taken in $\mathcal{H}_x$ in the complement of $s_1 \cup \cdots \cup s_{k-1}$ containing t_k.

The following result is due to Chatterji and Ruane [2].

Proposition 1.8. *The subsets $s_1, \ldots, s_k$ form a partition of the set $\mathcal{H}_x$:*

$$\mathcal{H}_x = s_1 \sqcup s_2 \cdots \sqcup s_k$$

We now use this partition to construct an embedding

$$j : [x_0, x] \to \mathbb{R}^k$$

of the interval $[x_0, x]$ in $\mathbb{R}^k$ as follows.

Let $y \in [x_0, x]$ be a vertex. Let $j(y)_m$, $1 \leq m \leq k$ denote the m-th coordinate of $j(y)$. We put $j(y)_m = l$, $0 \leq l \leq |\mathcal{H}_x|$, if the first l hyperplanes in the chain s_m do not separate y from x_0, while the $m+1$-st hyperplane does separate y from x_0. Repeating this process for all values of m defines a unique image $j(y) \in \mathbb{R}^k$ of the vertex y.

It is clear that distinct points must have different images in $\mathbb{R}^k$. Indeed, if two points x, y are different then there exists a hyperplane h that separates them, which has to belong to one of the components, say s_m, of the partition of $\mathcal{H}_x$. But then h separates either x or y from x_0 and so the m-th coordinates of $j(x)$ and $j(y)$ will be different.

Assume now that h is the only hyperplane separating the two points x and y, and that $h \in s_m$. Assume also that h separates x, but not y, from x_0. If $j(x)_m = l$ then clearly $j(y)_m = l + 1$. Given that h is the only hyperplane separating the two points, if h' is any other hyperplane in $\mathcal{H}_x$ then x and y have to belong to the same half-space determined by h'. Thus all coordinates of $j(x)$ and $j(y)$ are equal, except for the m-th coordinates, which differ by 1.

These two properties imply that the embedding j maps cubes in $[x_0, x]$ to cubes in $\mathbb{R}^k$. It will map vertices of the interval $[x_0, x]$ to vertices of the integer lattice in $\mathbb{R}^k$. The map j preserves the partial ordering of hyperplanes in $\mathcal{H}_{[x,y]}$. Hyperplanes that belong to the component s_1 in the partition will be mapped to coordinate hyperplanes orthogonal to the first coordinate axis in $\mathbb{R}^k$, and so on.

The injective map j induces an isometric embedding

$$i : \ell^2[x_0, x] \to \ell^2(\mathbb{R}^k)$$

such that for every $y \in [x_0, x]$

$$i : \delta_y \mapsto \delta_{j(y)}.$$

□

There is one further important construction we will require, the notion of the *median* of three vertices of a CAT(0) cube complex. Given any three vertices x, y, z

there is a unique vertex m referred to as the median of x, y, z characterised by the property that it lies on all three intervals $[x, y], [y, z], [z, x]$. The fact that m exists and that it is unique is a classical fact and may be found in [4].

Finally we require the definition of a normal cube path. A normal cube path from a vertex x to a vertex y distance d apart is a sequence of cubes $C_0, C_1, \ldots, C_{i\leq d-1}$ obtained such that each hyperplane separating x and y is crossed once, with the maximum number of hyperplanes being crossed at each stage. Any normal cube path is unique but not necessarily reversible [3].

1.3. **Combinations.** The weights that we give to vertices in a $CAT(0)$ cube complex will be defined in terms of the function $\binom{n}{r}$. *A priori* this function is defined on pairs (n, r) with $0 \leq r \leq n$. It is uniquely determined by the following properties.

- $\binom{n}{0} = \binom{n}{n} = 1$ for $n \geq 0$.
- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $1 \leq r \leq n$.

In fact the function $\binom{n}{r}$ can be defined for all $n, r \in \mathbb{Z}$: it is the unique function on $\mathbb{Z} \times \mathbb{Z}$ with the following properties.

- $\binom{n}{0}$ for $n \geq 0$, and $\binom{n}{n} = 1$ for all $n \in \mathbb{Z}$.
- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for all $n, r \in \mathbb{Z}$.

It follows that $\binom{n}{r}$ vanishes when $r > n$ or $r < 0 \leq n$. Moreover it satisfies the identity $\binom{n}{r} = (-1)^{n+r}\binom{-1-r}{-1-n}$, which allows one to compute $\binom{n}{r}$ for $r < 0$.

We will make use of $\binom{n}{r}$ for $r \geq -1$ and $n \in \mathbb{Z}$, where the function only takes non-negative values, thus ensuring that we only get non-negative weights. In particular note that $\binom{n}{-1} = (-1)^{n-1}\binom{0}{-1-n}$ which is 1 if $n = -1$ and otherwise vanishes.

2. The Euclidean case

In this section we will view $\mathbb{R}^d$ as a $CAT(0)$ cube complex with vertex set $\mathbb{Z}^d$. We will give a proof of the well known result that $\mathbb{Z}^d$ has property A, and hence by coarse invariance of property A, so also does $\mathbb{R}^d$. However we will do this in a way that generalises to the vertex set of an arbitrary finite dimensional $CAT(0)$ cube complex.

We view $\mathbb{R}^d$ as a $CAT(0)$ cube complex in the obvious way, that is we view the integer lattice $\mathbb{Z}^d$ as a set of vertices, and tile $\mathbb{R}^d$ with d-dimensional unit cubes.

Let us begin by briefly recalling the standard proof that $\mathbb{Z}^d$ has property A. One takes a Følner sequence $A_n = B_n(O)$, where B_n denotes the ball of radius n in the ℓ^1-metric. Equivalently, one takes the sequence of functions $f_n(y)$ defined to be 1 if $d(O, y) \leq n$ and zero otherwise. For fixed dimension d, the ℓ^1-norm of f_n, or equivalently the cardinality of the ball, is a polynomial $p(n)$ of degree d. In fact, one has $p(n) = 1 + 2^d\binom{n+d-1}{d}$.

Now for any x in $\mathbb{Z}^d$, let $f_{n,x}$ denote the translate of f_n by x. Fix $R \in \mathbb{N}$. For x, x' in $\mathbb{Z}^d$ with $d(x, x') \leq R$ we have $\|f_{n,x} - f_{n,x'}\|_1 = \|f_n - f_{n,x'-x}\|_1$, and this is bounded above by $2\|f_{n+R} - f_n\|_1$. Since $\|f_n\|_1$ is a polynomial of degree d in n, and

$$\|f_{n+R} - f_n\|_1 = \|f_{n+R}\|_1 - \|f_n\|_1 = p(n + R) - p(n),$$

we deduce that $\|f_{n,x} - f_{n,x'}\|_1$ is bounded above by a polynomial of degree $d-1$ in n. Now dividing by $p(n)$ we find that the limit as $n \to \infty$ is 0.

Our proof will differ from this in the several ways. Our weighting functions will be supported on a certain subset of the n-ball about a point x, rather than the whole ball. We will also have more variation in the weights themselves, rather than just taking values 0 and 1. Finally each function $f_{n,x}$ will be defined separately, rather than being translates of a single sequence of functions f_n. This last point is crucial for generalising the argument to arbitrary CAT(0) cube complexes, which do not admit an action by an amenable group.

Throughout the section we will fix the dimension d and a number N with $N \geq d-1$. We view N as the 'ambient dimension,' that is, we imagine that $\mathbb{R}^d$ lies inside a space of dimension N. When we prove that $\mathbb{R}^d$ has property A we will take $N \geq d$; it will nonetheless be useful to note that the definitions and results also work when the 'codimension' is -1, i.e. $N = d-1$.

2.1. **Construction of $f_{n,x}$.** The proof that a tree has property A runs as follows. Fix a basepoint O, and for each vertex x in the tree place weights on the intersection of the interval $[O,x]$ with the ball of radius n about x. For y a vertex in $[O,x]$ the weight is defined to be

$$f_{n,x}(y) = \begin{cases} 1 & \text{if } y \neq O \text{ and } d(x,y) \leq n \\ n - d(x,y) + 1 & \text{if } y = O \text{ and } d(x,y) \leq n \\ 0 & \text{if } d(x,y) > n. \end{cases}$$

It is interesting to note that the weight $f_{n,x}(y)$ can be defined more succinctly as $\binom{n-d(x,y)}{0}$ for $y \neq O$, and $\binom{n-d(x,y)+1}{1}$ for the origin. The origin is special in that having reached the origin, the dimension of the interval drops from 1 to 0, and hence the weights must 'pile up' at the origin. Our definition of weighting functions for CAT(0) cube complexes will generalise what happens for a tree. Motivated by these ideas we make the following definitions.

Fix a basepoint $O = (0, 0, \ldots, 0)$ of $\mathbb{R}^d$. Again we will place weights on the intersection of the interval $[O,x]$ with the ball of radius n.

Definition 2.1. For y a vertex of $\mathbb{R}^d$, define the *deficiency*, $\delta(y)$, of y to be N minus the dimension of the first cube on the normal cube path from O to y.

Note that interval $[O,y]$ has the same dimension as the first cube on the normal cube path from O to y. Thus we could equivalently define $\delta(y)$ to be N minus the dimension of $[O,y]$, i.e.

$$\delta(y) = N - \dim([O,y]).$$

This however is special to the case of $\mathbb{R}^d$; for a general $CAT(0)$ cube complex it is the local dimension, i.e. the dimension of the first cube on the normal cube path, rather than the global dimension of the interval that will be important.

Definition 2.2. For x a vertex of $\mathbb{R}^d$, we define a function $f_{n,x} = f_{n,x}^{N,\mathbb{R}^d}$ from the vertices of $\mathbb{R}^d$ to $\mathbb{N} \cup \{0\}$ by

$$f_{n,x}(y) = \binom{n - d(x,y) + \delta(y)}{\delta(y)}$$

for $y \in [O,x]$ and $f_{n,x}(y) = 0$ otherwise.

Note that as $N \geq d-1$, we have $\delta(y) \geq -1$ for all y, and hence $f_{n,x}$ is non-negative integer valued. Moreover $f_{n,x}(y)$ vanishes at all but finitely many vertices y, hence $f_{n,x}$ lies in the space of finitely supported functions on the vertex set.

The definitions are motivated by the following geometric intuition. Imagine a vertex x in the ambient $\mathbb{R}^N$, all of whose coordinates exceed n. In this case the intersection of the interval from x to the origin with the ball of radius n will be an N-dimensional tetrahedron, and the number of points of $\mathbb{Z}^N$ contained in this is counted by $\binom{n+N}{N}$. If one projects $\mathbb{R}^N$ onto a subspace $\mathbb{R}^d$ (supposing $d \leq N$) then the image will be a d-dimensional tetrahedron, and the fibre over a vertex y will be an $N-d$-dimensional tetrahedron, the sides of which have length $n - d(x,y)$. Hence each fibre will contain $\binom{n-d(x,y)+N-d}{N-d}$ points of $\mathbb{Z}^N$. We will thus take a weighting of $\binom{n-d(x,y)+N-d}{N-d}$ on each point of the image tetrahedron in $\mathbb{Z}^d$.

Now suppose that the coordinates of x do not all exceed n. Then the tetrahedron will cross outside the interval from x to the origin, and hence we must further project points of the tetrahedron onto the faces of the interval. It is this that results in higher deficiencies than the standard $N-d$.

We will show that the norm of $f_{n,x}$ depends only on n and N, in particular it does not depend on x or d. Indeed the norm is exactly the number of points of $\mathbb{Z}^N$ contained in a tetrahedron whose sides have length n.

Lemma 2.3. *For $N \geq d-1$ and $x \in \mathbb{Z}^d$, the ℓ^1-norm of $f_{n,x}$ is $\binom{n+N}{N}$.*

Proof. Since the entries of $f_{n,x}$ are all positive, $\|f_{n,x}\|_1$ is the sum of the entries of $f_{n,x}$. It is at this point that we make use of the fact that $N \geq d-1$. We will show that this sum is $\binom{n+N}{N}$. We will prove this by induction on d. If $d = 0$ then $x = O$ and $f_{n,x}$ has only one entry, taking the value $\binom{n+N}{N}$, since the deficiency is N. Now for $d > 0$, suppose x has coordinates $(x_1, \ldots, x_d)$. If x_1 vanishes then $[O,x]$ can be identified with the interval from $(x_2, \ldots, x_d)$ to the origin in $\mathbb{R}^{d-1}$. Hence by induction the result holds. We will therefore assume that $x_1 \neq 0$.

For $x_1 > 0$, fix some vertex $y = (x_1, y_2, y_3, \ldots, y_d)$ in the interval $[O,x]$, and define $y^i = (i, y_2, y_3, \ldots, y_d)$ for $i \leq x_1$. We claim that

$$\sum_{j=0}^{i} f_{n,x}(y^j) = \binom{n - d(x,y^i) + \delta(y) + 1}{\delta(y)+1}.$$

We will prove this by a further induction on i.

Note that the interval $[O,y^i]$ has the same dimension as $[O,y]$ for all $i > 0$ and has dimension 1 lower if $i = 0$. Hence $\delta(y^i) = \delta(y)$ if $i > 0$ while $\delta(y^0) = \delta(y)+1$. The latter formula proves the claim when $i = 0$. We now proceed by induction. Note that y^{i+1} lies on a geodesic between x and y^i, hence $d(x,y^i) = d(x,y^{i+1}) + d(y^{i+1},y^i) = d(x,y^{i+1}) + 1$. We deduce that

$$\begin{aligned}\binom{n - d(x,y^{i+1}) + \delta(y) + 1}{\delta(y)+1} &= \binom{n - (d(x,y^i) - 1) + \delta(y) + 1}{\delta(y)+1} \\ &= \binom{n - d(x,y^i) + \delta(y) + 1}{\delta(y)+1} + \binom{n - (d(x,y^i) - 1) + \delta(y)}{\delta(y)}.\end{aligned}$$

The last term is $f_{n,x}(y^{i+1})$, hence if $\sum_{j=0}^{i} f_{n,x}(y^j) = \binom{n-d(x,y^i)+\delta(y)+1}{\delta(y)+1}$ then $\sum_{j=0}^{i+1} f_{n,x}(y^j) = \binom{n-d(x,y^{i+1})+\delta(y)+1}{\delta(y)+1}$. By induction we establish the claim for all i.

In particular we deduce that $\sum_{j=0}^{x_1} f_{n,x}(y^j) = \binom{n-d(x,y)+\delta(y)+1}{\delta(y)+1}$. Similarly if $x_1 < 0$ we get that $\sum_{j=x_1}^{0} f_{n,x}(y^j) = \binom{n-d(x,y)+\delta(y)+1}{\delta(y)+1}$.

Now let $x' = (x_2, \dots, x_d), y' = (y_2, \dots, y_d)$. The point y' lies in the interval from x' to the origin in $\mathbb{R}^{d-1}$, and taking the same ambient dimension N, y' has a deficiency of $\delta(y)+1$. Since $d(x', y') = d(x, y)$ we get $f_{n,x'}^{N,\mathbb{R}^{d-1}}(y') = \binom{n-d(x,y)+\delta(y)+1}{\delta(y)+1}$. We thus deduce that the sum of the entries of $f_{n,x}$ over $[O, x]$ is the same as the sum of the entries of $f_{n,x'}^{N,\mathbb{R}^{d-1}}$. By induction on d this is $\binom{n+N}{N}$ which completes the proof. □

2.2. **Almost invariance for $\mathbb{R}^d$.** We will now show that $f_{n,x}$ is 'almost invariant' in x, in the sense that for any $R > 0$ the sequence

$$\frac{\|f_{n,x} - f_{n,x'}\|_1}{\|f_{n,x}\|_1} \to 0$$

uniformly on the set $\{(x, x') : d(x, x') \leq R\}$ as $n \to \infty$.

We begin with the following result.

Proposition 2.4. *Suppose $N \geq d$. If x and x' are adjacent vertices, i.e. $x, x' \in \mathbb{Z}^d$ with $d(x, x') = 1$, then $\|f_{n,x} - f_{n,x'}\|_1 = 2\binom{n+N-1}{N-1}$.*

Proof. Without loss of generality suppose that x' is closer to the origin than x. Then x' lies on a geodesic from x to O, and the interval $[O, x']$ is contained in $[O, x]$. Moreover for any $y \in [O, x']$, the point x' lies on a geodesic from x to y, so $d(x, y) = d(x', y) + 1$.

We will prove that the sum of $f_{n,x'} - f_{n,x}$ over $[O, x']$ is $\binom{n+N-1}{N-1}$. Note that

$$\begin{aligned} f_{n,x'}(y) - f_{n,x}(y) &= \binom{n - d(x', y) + \delta(y)}{\delta(y)} - \binom{n - (d(x', y) + 1) + \delta(y)}{\delta(y)} \\ &= \binom{n - d(x', y) + \delta(y) - 1}{\delta(y) - 1}. \end{aligned}$$

Replacing N by $N - 1$ has the effect of reducing all deficiencies by 1, hence we deduce that $\binom{n-d(x',y)+\delta(y)-1}{\delta(y)-1} = f_{n,x'}^{N-1,\mathbb{R}^d}(y)$. Note that $N - 1 \geq d - 1$ as required, hence $f_{n,x'}^{N-1,\mathbb{R}^d}$ is non-negative, and so the ℓ^1-norm of $f_{n,x'} - f_{n,x}$ restricted to $[O, x']$ is $\left\| f_{n,x'}^{N-1,\mathbb{R}^d} \right\|_1$ which is $\binom{n+N-1}{N-1}$ by Lemma 2.3.

We note that $f_{n,x'} - f_{n,x}$ is supported in $[O, x]$ (since $[O, x'] \subset [O, x]$). We have seen that it takes non-negative values on $[O, x']$, with sum $\binom{n+N-1}{N-1}$, and it clearly takes non-positive values $-f_{n,x}(y)$ for y in $[O, x] \setminus [O, x']$. Since $\sum_y f_{n,x'}(y) = \sum_y f_{n,x}(y)$ (by Lemma 2.3), the sum of $f_{n,x'}(y) - f_{n,x}(y)$ vanishes. It thus follows that the sum of the non-positive values is

$$\sum_{y \in [O,x] \setminus [O,x']} -f_{n,x}(y) = -\binom{n + N - 1}{N - 1}.$$

Hence we deduce that $f_{n,x'} - f_{n,x}$ has norm $2\binom{n+N-1}{N-1}$. □

We conclude this section by deducing the following result.

Theorem 2.5. *The Euclidean space $\mathbb{R}^d$ has property A for all d.*

Proof. We prove that the vertex set $\mathbb{Z}^d$ has property A, and hence $\mathbb{R}^d$ has property A as it is coarsely equivalent to this. We take the sequence of families $f_{n,x}$ as above for x in the vertex set. Since $\binom{n-d(x,y)+\delta(y)}{\delta(y)}$ vanishes if $n - d(x,y) + \delta(y) < \delta(y)$ it follows that $f_{n,x}$ is supported in $B_n(x)$.

We have noted that if $d(x,x') \leq 1$ then $\|f_{n,x} - f_{n,x'}\|_1 = 2\binom{n+N-1}{N-1}$. Since for any two vertices x, x' we can take a sequence of adjacent vertices going from x to x' we deduce that in general $\|f_{n,x} - f_{n,x'}\|_1 \leq 2d(x,x')\binom{n+N-1}{N-1}$ by the triangle inequality. Hence as $\|f_{n,x}\|_1 = \binom{n+N}{N}$ we have

$$\frac{\|f_{n,x} - f_{n,x'}\|_1}{\|f_{n,x}\|_1} \leq 2d(x,x')\frac{\binom{n+N-1}{N-1}}{\binom{n+N}{N}} = \frac{2d(x,x')N}{n+N}.$$

This tends to zero uniformly on $\{(x,x') : d(x,x') \leq R\}$ as $n \to \infty$, which completes the proof. ☐

3. Property A for $CAT(0)$ cube complexes

In this section we will generalise the techniques of the previous section to prove that if X is any finite dimensional $CAT(0)$ cube complex then X has property A. It is relatively straightforward to generalise the definition of the functions $f_{n,x}$. The main technical step of the proof is to achieve a computation of the norm as in Lemma 2.3. To do so we will make use of a notion of fibring.

3.1. **Construction of $f_{n,x}$.** Let X be a $CAT(0)$ cube complex of dimension $d < \infty$. We fix a basepoint O in X and an ambient dimension $N \geq d-1$ and define functions $f_{n,x} = f_{n,x}^{N,X}$ on the vertices of the interval $[O,x]$ for x a vertex of X. For y a vertex of X let $\mathcal{N}_X(y)$ denote the first cube in the normal cube path from y to O.

Definition 3.1. For y a vertex of X, define the *deficiency*, $\delta(y)$, of y to be the difference between the ambient dimension N and the dimension of $\mathcal{N}_X(y)$, i.e.

$$\delta(y) = N - \dim(\mathcal{N}_X(y)).$$

Equivalently this is N minus the number of vertices in $[O,y]$ which are adjacent to y

Definition 3.2. For x a vertex of X, we define a function $f_{n,x} = f_{n,x}^{N,X}$ from the vertices of X to $\mathbb{N} \cup \{0\}$ by

$$f_{n,x}(y) = \binom{n - d(x,y) + \delta(y)}{\delta(y)}$$

for $y \in [O,x]$ and $f_{n,x}(y) = 0$ otherwise.

As in the case of $\mathbb{R}^d$, we have $\delta(y) \geq -1$ for all y, as $N \geq d-1$. Hence $f_{n,x}$ is a non-negative function.

3.2. **Fibres.** Given a vertex x in X, consider the interval $[O,x]$. By Theorem 1.7 we can embed this as a subset J of an interval I in $\mathbb{R}^d$. We can arrange that O maps to the origin of $\mathbb{R}^d$. The embedding provides an identification of hyperplanes and half-spaces in $[O,x]$ with hyperplanes and half-spaces in I. This identification has the property that the image in I of a half-space in $[O,x]$ is precisely the intersection of J with the corresponding half-space in I. For a vertex z in I we denote by $[O,z]_I$ the subinterval of I from O to z. We view $[O,x]$ as being identified with its image J in I, and we will define a retraction of I onto J, or equivalently onto $[O,x]$.

Definition 3.3. A hyperplane h in J is *adjacent* to a point $y \in J$ if y lies in an edge which crosses h.

For $y \in J$, let $\mathcal{H}_y$ denote the set of hyperplanes crossing the first cube $\mathcal{N}_J(y)$ in the normal cube path from y to O in J. Working within the interval from y to O in J, the intersection of all half spaces containing y and bounded by some h in $\mathcal{H}_y$ is $\{y\}$.

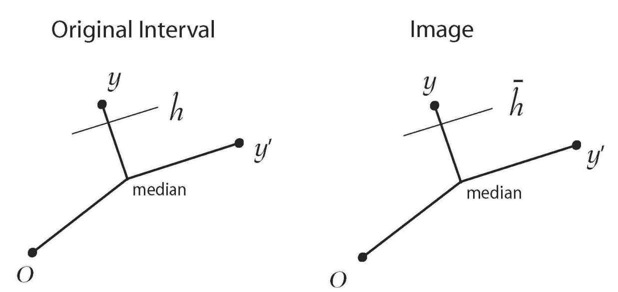
Definition 3.4. For $y \in J$ the *fibre of I over y*, denoted I_y, is the set of vertices in $[O, y]_I$ lying in the intersection of the half-spaces of I containing y and bounded by a hyperplane $\overline{h}$ corresponding to some h in $\mathcal{H}_y$.

Note that if a hyperplane is adjacent to y then either it lies in $\mathcal{H}_y$, or it is one of the boundary hyperplanes of the interval from O to y. Hence the fibre I_y lies in the intersection of all half-spaces in I containing y and bounded by a hyperplane which is adjacent to y in J. Indeed the fibre is exactly the intersection of this with $[O, y]_I$.

As the terminology suggests we will show that the fibres are disjoint, and indeed partition I. Since I_y clearly contains y, the map taking each z in I to the unique y such that $z \in I_y$ will then define a retraction of I onto J.

Lemma 3.5. *For any $y \neq y'$ in J the fibres over y, y' are disjoint.*

Proof. Let m be the median of O, y and y', and note that as O, y, y' lie in J, it follows that m also lies in J. Consider the following diagram.



Without loss of generality we may assume that y does not lie on a geodesic from y' to O (exchanging the roles of y, y' if necessary). Hence it follows that y is not equal to m, and hence the first hyperplane crossed on the path from y to m separates y from both O and y'. Denote this hyperplane by h and let $\overline{h}$ be the corresponding hyperplane in I. Since the hyperplane h is adjacent to y in J it follows that the fibre over y lies on the same side of $\overline{h}$ as y. However as $\overline{h}$ separates y from both y' and O, it follows that the interval $[O, y']_I$ is separated from y by $\overline{h}$. Since the fibre over y' lies in $[O, y']_I$ we conclude that the fibres over y, y' lie on different sides of $\overline{h}$, and hence are disjoint. □

Lemma 3.6. *For any z in I there is a point y in J such that the fibre over y contains z.*

Proof. Let y be a point in the intersection $[z, x]_I \cap J$ minimising the distance from z. We will show that z lies in I_y; it will then follow from the above lemma that y must be unique.

First we note that z lies in $[O,y]_I$. Indeed this is true for any point y in $[z,x]_I$: the interval $[O,y]_I$ is the intersection of all half-spaces in I containing both O and y, or equivalently those half-spaces containing y but not x. But such half-spaces must all contain z, since a half-space containing neither z nor x is by definition disjoint from $[z,x]_I$.

To show that z lies in the fibre over y we must therefore show that no hyperplane $\overline{h}$ corresponding to $h \in \mathcal{H}_y$ separates y from z. Suppose that such an h and $\overline{h}$ exist. Since h lies in $\mathcal{H}_y$ it follows that h is adjacent to y, so there is an edge yy' in J which crosses h, and hence crosses $\overline{h}$. Thus y' and z lie on the same side of $\overline{h}$, and as there are no other hyperplanes separating y from y' we deduce that $d(z,y') = d(z,y) - 1$. Certainly y' lies in J. We note moreover that y' lies in the interval $[z,x]_I$. This follows from the fact that y lies in $[z,x]_I$, and the hyperplane $\overline{h}$ separates z from x (since it separates y from O) so it does not bound a half space containing both x and z.

We have shown that y' lies in the intersection $[z,x]_I \cap J$ and is closer to z than y is, hence contradicting minimality of y. This contradiction proves that there is no hyperplane $\overline{h}$ corresponding to some $h \in \mathcal{H}_y$ and separating y from z. Hence z lies in the fibre over y. □

Fix some y in J and let L be the dimension of $\mathcal{N}_J(y)$ or equivalently the cardinality of $\mathcal{H}_y$. Let F be the fibre over y, and equip this with basepoint O_F closest to O in I. Equivalently O_F is defined by the property that for z in F, every hyperplane in I which meets F and separates z from O also separates z from O_F.

Lemma 3.7. *For each z in $F = I_y$, the first cube on the normal cube path from z to O_F has dimension* $\dim \mathcal{N}_I(z) - |\mathcal{H}_y|$.

Proof. The dimension of $\mathcal{N}_F(z)$ is the number of hyperplanes adjacent to z in F and separating z from O_F. Such hyperplanes also separate z from O, and hence we deduce that $\dim \mathcal{N}_I(z) \geq \dim \mathcal{N}_F(z)$, and moreover the difference is the number of hyperplanes adjacent to z which separate z from O but not from O_F. We will show that these are precisely those hyperplanes corresponding to elements of $\mathcal{H}_y$, hence the difference is the cardinality of $\mathcal{H}_y$.

Firstly suppose that $\bar{k}$ is a hyperplane in I adjacent to z, which separates z from O but not from O_F. Let z' be the vertex of I adjacent to z over the hyperplane $\bar{k}$. If z' lies in F then the median of O, O_F and z' lies in F, since F is an interval, but this contradicts minimality of O_F, since the median is closer to the origin than O_F; indeed it lies on the geodesic from O_F to O and is separated from O_F by $\bar{k}$. We deduce that z' does not lie in the fibre. Note that as $\bar{k}$ separates z from O it must also separate y from O, otherwise z would not lie in $[O,y]_I$. Hence as $\bar{k}$ crosses $[O,y]_I$, and z lies in $[O,y]_I$, so does z'. Since z' does not lie in the fibre F there must be some h in $\mathcal{H}_y$ such that the corresponding hyperplane $\overline{h}$ separates y and z'. But $\overline{h}$ cannot separate y and z, since z lies in the fibre, hence it separates z and z'. As $\bar{k}$ is the only such hyperplane, we deduce that $\overline{h} = \bar{k}$.

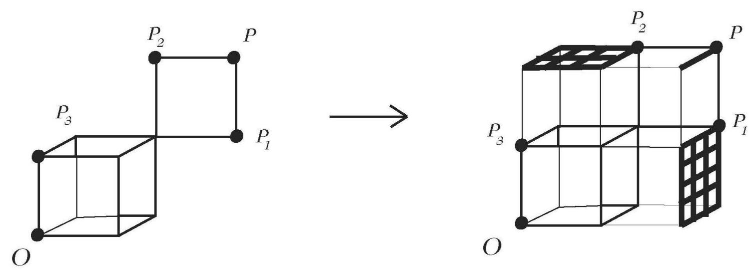
Now conversely take any h in $\mathcal{H}_y$ and corresponding hyperplane $\overline{h}$. Let $\overline{h}'$ be the hyperplane parallel to $\overline{h}$ which is adjacent to y but does not separate it from O. The fibre lies between these two hyperplanes, and any point sandwiched between two such hyperplanes is adjacent to both. Hence z is adjacent to $\overline{h}$. By definition, $\overline{h}$ separates every point of the fibre from O, hence it separates z from O but not from O_F. We conclude that the hyperplanes adjacent to z which separate z from

O but not from O_F are precisely those which correspond to elements of $\mathcal{H}_y$ as required. $\square$

Note that since the $\mathcal{N}_F(z)$ will have greatest dimension when $z = y$ it follows that the dimension of the fibre is $\dim \mathcal{N}_I(y) - \dim \mathcal{N}_J(y)$, or equivalently it is the difference between the deficiencies of y viewed as an element of J or I.

The following diagram illustrates this. It represents the fibres of P, P_1, P_2 and P_3. Note that as an element of J the point P has deficiency 1 while both P_1 and P_2 have deficiency 2. However as elements of I, all have deficiency 0. As expected, the fibre in the case of P has dimension 1 and the others dimension 2.

The point P_3 has deficiency 2 in both J and I, so its fibre has dimension 0 and is P_3 itself.



Theorem 3.8. *Let X be a $CAT(0)$ cube complex of dimension at most d, and take $N \geq d-1$. For x a vertex of X, the ℓ^1-norm of the function $f^{N,X}_{n,x}$ defined above is $\binom{n+N}{N}$. In particular it depends only on n and N, and not on the vertex x or the complex X.*

Proof. Fix x and an identification of $[O, x]$ as a subset J of an interval I in $\mathbb{R}^d$. We begin by proving that for $y \in [O, x]$, and $F = I_y$ we have

$$f^{N,X}_{n,x}(y) = \sum_{z \in F} f^{N,I}_{n,x}(z).$$

Let $\delta^{N,X}(y)$ denote the deficiency of y as a vertex of X (or equivalently as a vertex of J), let $\delta^{N,I}(z)$ denote the deficiency of z as a vertex of I, and let $\delta^{N,F}(z)$ denote the deficiency of z viewed as a vertex of F with basepoint O_F. Then by Lemma 3.7 we have

$$\delta^{N,F}(z) - \delta^{N,I}(z) = L = \dim \mathcal{N}_J(y).$$

Hence it follows that $\delta^{N,I}(z) = \delta^{N-L,F}(z)$ (note that F has dimension at most $d - L$ so the latter makes sense).

As each z in F lies in the interval from O to y in I, equivalently y lies in the interval from z to x, and hence y lies on a geodesic from z to x. Thus $d(x, z) = d(x, y) + d(y, z)$. We conclude that

$$\begin{aligned} f^{N,I}_{n,x}(z) &= \binom{n - d(x,z) + \delta^{N,I}(z)}{\delta^{N,I}(z)} \\ &= \binom{(n - d(x,y)) - d(y,z) + \delta^{N-L,F}(z)}{\delta^{N-L,F}(z)} = f^{N-L,F}_{n-d(x,y),y}(z) \end{aligned}$$

for z in F. Summing $f_{n,x}^{N,I}(z)$ over z in F we thus get

$$\left\| f_{n-d(x,y),y}^{N-L,F} \right\|_1 = \binom{n-d(x,y)+N-L}{N-L}$$

by Lemma 2.3. But by definition $N-L = \delta^{N,X}(y)$. Hence the sum $\sum_{z\in F} f_{n,x}^{N,I}(z)$ is $f_{n,x}^{N,X}(y)$ as claimed.

Since the fibres partition I it now follows that the sum of $f_{n,x}^{N,I}(z)$ over all z in I is the sum of $f_{n,x}^{N,X}(y)$ over all y in J. Since J is identified with $[O,x]$ this is the ℓ^1-norm of $f_{n,x}^{N,X}$. We hence conclude that $\left\|f_{n,x}^{N,X}\right\|_1 = \left\|f_{n,x}^{N,I}\right\|_1$, and this is $\binom{n+N}{N}$ by Lemma 2.3. This completes the proof. □

The above proof demonstrates a remarkable fact. Although the embedding of an interval of X into a Euclidean interval is not canonical, and although the construction of the fibres relies heavily upon this embedding, the process of summing the weights over each fibre gives a quantity which is independent of these choices. Specifically, summing over the fibre one gets the value of $f_{n,x}$ at the point, and this is defined intrinsically with no reference to an embedding.

3.3. **Almost invariance for $CAT(0)$ cube complexes.** We will now show that every finite dimensional $CAT(0)$ cube complex has property A.

Proposition 3.9. *Let X be a $CAT(0)$ cube complex of dimension at most d, and suppose $N \geq d$. If x and x' are adjacent vertices of X then $\|f_{n,x} - f_{n,x'}\|_1 = 2\binom{n+N-1}{N-1}$.*

Theorem 3.10. *Let X be a finite dimensional $CAT(0)$ cube complex. Then X has property A.*

The proofs of these two results are now identical to the proofs of 2.4 and 2.5, except making use of Theorem 3.8 in place of Lemma 2.3.

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