Property A and CAT(0) cube complexes

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Amenability

Question: Can a bounded function f on a group G be averaged over the group?

Yes if G is finite (or compact).

An *invariant mean* μ is a linear map $f \mapsto \mu(f)$ with the property that

$$g \cdot f \mapsto \mu(g \cdot f) = \mu(f).$$

A group G which admits an invariant mean is *a-mean-able*.

Example: $\mathbb{Z}$ is amenable.

$$\mu(f) = \lim_{k \in \omega} \frac{1}{2k+1} \sum_{n=-k}^{k} f(n)$$

Let $f_m(n) = f(m+n)$. Then

$$\mu(f_m) = \lim_{k\in\omega} \frac{1}{2k+1} \sum_{n=-k}^{k} f(m+n)$$
$$= \lim_{k\in\omega} \frac{1}{2k+1} \sum_{n=m-k}^{m+k} f(n)$$

so

$$\mu(f) - \mu(f_m)$$
$$= \lim_{k\in\omega} \frac{1}{2k+1} \left(\sum_{n=-k}^{m-k-1} f(n) + \sum_{n=k+1}^{m+k} f(n) \right)$$

Theorem: (Følner) G is amenable iff there exists a sequence of families $\{A_{n,g} = g \cdot A_n : g \in G\}$ of non-empty finite subsets of G such that for all R

$$|A_{n,g} \Delta A_{n,h}| / |A_{n,g}| \to 0$$

uniformly on $\{(g,h) : d(g,h) \leq R\}$.

Property A

Let (X, d) be a metric space.

Definition: (Yu) X has property A iff there exists a sequence $S_n > 0$ and a sequence of families $\{A_{n,x} : x \in X\}$ of non-empty finite sets with $A_{n,x} \subseteq B_{S_n}(x) \times \mathbb{N}$ such that for all R

$$|A_{n,x} \Delta A_{n,y}| / |A_{n,x}| \to 0$$

uniformly on $\{(x, y) : d(x, y) \leq R\}$.

A group G is *exact* is for every exact sequence

$$0 \to J \to A \to A/J \to 0$$

of G-C^*-algebras, the reduced crossed product with G preserves exactness.

Theorem: (Higson, Roe, Guentner, Kaminker, Ozawa, Anantharaman-Delaroche, Renault) A group G is exact iff its underlying metric space has property A.

Example: A tree has property A. We use the following

Reformulation of property A

Let $f_{n,x}(y) = |A_{n,x} \cap \{y\} \times \mathbb{N}|$. Then $f_{n,x}$ is supported in a finite subset of $B_{S_n}(x)$ and for all R

$$\|f_{n,x} - f_{n,y}\|_1 / \|f_{n,x}\|_1 \to 0$$

uniformly on $\{(x, y) : d(x, y) \leq R\}$.

Fix an origin O in the tree T. For x in T we define weights $f_{n,x}(y)$ to be

- 1 for n steps from x towards O,
- $n - d(x, O)$ at $y = O$, if the distance is less than n,
- 0 everywhere else.

Note that $\|f_{n,x} - f_{n,y}\|_1 \leq 2d(x, y)$.

CAT(0) cube complexes

A CAT(0) cube complex is a higher dimensional analogue of a tree. It is a cell complex built out of Euclidean cubes, satisfying the *CAT(0) condition* (non-positive curvature.

The *edge-path metric* sometimes called the l^1 *metric* is the metric on vertices x, y given by the minimum number of edges on a path from x to y.

Minimal paths are called *geodesics*.

The convex hull of all geodesics from x to y is called the *interval from x to y*, denoted $[x, y]$.

Example: $\mathbb{R}^d$ is a CAT(0) cube complex. An interval in $\mathbb{R}^d$ is a rectangle.

Finite dimensional CAT(0) cube complexes have property A

Theorem: (Brodzki, Campbell, Guentner, Niblo, W) The above statement is true.

We will construct explicit weights on a finite dimensional CAT(0) cube complex.

First case: $X = \mathbb{R}^d$.

Fix $N \geq d$, and define the deficiency of a point y in $\mathbb{R}^d$ to be

$$\delta(y) = N - \dim([O, y]).$$

Define weights

$$f^N_{n,x}(y) = \begin{cases} \binom{n-d(x,y)+\delta(y)-1}{\delta(y)} & y \in [O,x] \\ 0 & y \notin [O,x] \end{cases}$$

Lemma: $\|f^N_{n,x}\|_1 = \binom{n+N-1}{N}$

‘Proof’ of Lemma

$$
\begin{array}{ccccccccc}
\binom{n-4}{1} & - & \binom{n-4}{0} & - & \binom{n-3}{0} & - & \binom{n-2}{0} & - & \binom{n-1}{0} \\
| & & | & & | & & | & & | \\
\binom{n-5}{1} & - & \binom{n-5}{0} & - & \binom{n-4}{0} & - & \binom{n-3}{0} & - & \binom{n-2}{0} \\
| & & | & & | & & | & & | \\
\binom{n-5}{2} & - & \binom{n-5}{1} & - & \binom{n-4}{1} & - & \binom{n-3}{1} & - & \binom{n-2}{1}
\end{array}
$$

$$\binom{n-5}{0} \quad \binom{n-5}{1} \quad \binom{n-5}{2} \ \ldots$$

$$\binom{n-4}{0} \quad \binom{n-4}{1} \quad \binom{n-4}{2} \ \ldots$$

$$\binom{n-3}{0} \quad \binom{n-3}{1} \quad \binom{n-3}{2} \ \ldots$$

$$\binom{n-2}{0} \quad \binom{n-2}{1} \quad \binom{n-2}{2} \ \ldots$$

$$\binom{n-1}{0} \quad \binom{n-1}{1} \quad \binom{n-1}{2} \ \ldots$$

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \ldots$$

$$\binom{n+1}{0} \quad \binom{n+1}{1} \quad \binom{n+1}{2} \ \ldots$$

Almost invariance for $\mathbb{R}^d$

Suppose x_0 and x_1 are adjacent vertices, with x_1 closer to O.

$$\begin{array}{ccccccccc}
 & - & \dots & - & x_1 & - & x_0 \\
| & & | & & | & & | \\
 & - & \dots & - & & - & \\
| & & | & & | & & | \\
O & - & \dots & - & & - &
\end{array}$$

On a codimension 1 subspace f_{n,x_1}^N vanishes. Restricting f_{n,x_0}^N to this subspace gives the same result as taking f_{n,x_0}^{N-1} for the subspace.

On the interval $[O, x_1]$ we have $f_{n,x_0}^N(y) =$

$$\binom{n-d(x_0,y)+\delta(y)-1}{\delta(y)} = \binom{n-(d(x_1,y)+1)+\delta(y)-1}{\delta(y)}$$

so $f_{n,x_1}^N(y) - f_{n,x_0}^N(y) = \binom{n-d(x_1,y)+\delta(y)-2}{\delta(y)-1}$. This is $f_{n,x_1}^{N-1}(y)$.

Thus $f^N_{n,x_0} - f^N_{n,x_1}$ is

$$\begin{array}{ccccccc}
-f^{N-1,\mathbb{R}^d}_{n,x_1} & - & \cdots & - & -f^{N-1,\mathbb{R}^d}_{n,x_1} & - & f^{N-1,R^{d-1}}_{n,x_0} \\
| & & | & & | & & | \\
-f^{N-1,\mathbb{R}^d}_{n,x_1} & - & \cdots & - & -f^{N-1,\mathbb{R}^d}_{n,x_1} & - & f^{N-1,R^{d-1}}_{n,x_0} \\
| & & | & & | & & | \\
-f^{N-1,\mathbb{R}^d}_{n,x_1} & - & \cdots & - & -f^{N-1,\mathbb{R}^d}_{n,x_1} & - & f^{N-1,R^{d-1}}_{n,x_0}
\end{array}$$

So $\|f^N_{n,x_0} - f^N_{n,x_1}\|_1 = \|f^{N-1,\mathbb{R}^d}_{n,x_1}\|_1 + \|f^{N-1,\mathbb{R}^{d-1}}_{n,x_0}\|_1 = 2\binom{n+N-2}{N-1}$

For general x_0, x_1 we deduce that

$$\|f^N_{n,x_0} - f^N_{n,x_1}\|_1 \leq 2d(x_0, x_1)\binom{n+N-2}{N-1}$$

$$\frac{\|f^N_{n,x_0} - f^N_{n,x_1}\|_1}{\|f^N_{n,x_0}\|_1} \leq 2d(x_0, x_1)\frac{\binom{n+N-2}{N-1}}{\binom{n+N-1}{N}}$$

$$= 2d(x_0, x_1)\frac{N}{n+N-1} \to 0$$

General CAT(0) cube complexes

Let X be a CAT(0) cube complex of dimension at most d, and pick a basepoint O.

Fix $N \geq d$, and define the deficiency of a point y in X to be

$\delta(y) = N-$ the dimension of the first normal cube in the path from y to O. (This is the number of edges from y pointing towards O.)

Define weights

$$f_{n,x}^{N,X}(y) = \begin{cases} \binom{n-d(x,y)+\delta(y)-1}{\delta(y)} & y \in [O,x] \\ 0 & y \notin [O,x] \end{cases}$$

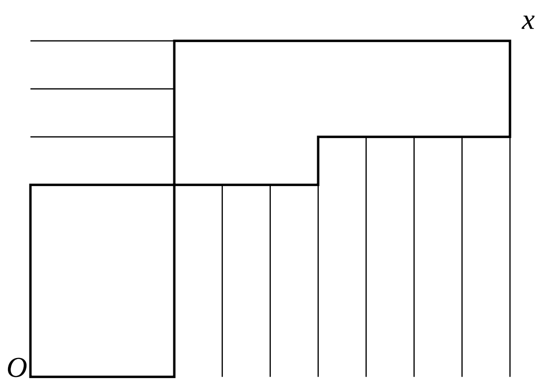
Theorem: $\|f_{n,x}^{N,X}\|_1 = \binom{n+N-1}{N}$

Theorem: (Brodzki, Niblo) Every interval in a CAT(0) cube complex of dimension d embeds into an interval in $\mathbb{R}^d$.

Fibring the interval

Let I be the image of an interval of X in an interval $[O, x]$ of R^d.

For $y \in I$ we define the fibre over y to be those points in the interval $[O, y]$ which are separated from O by all hyperplanes crossing the normal cube from y to O.



The fibres over I partition $[O, x]$.

Lemma: $f_{n,x}^{N,X}(y)$ is the sum of $f_{n,x}^{N,\mathbb{R}^d}(z)$ for z in the fibre of y.

It follows that

$$\sum_{y \in I} f_{n,x}^{N,X}(y) = \sum_{z \in [0,x]} f_{n,x}^{N,\mathbb{R}^d}(z) = \binom{n+N-1}{N}.$$

We can now proceed as for $\mathbb{R}^d$.

If x_0 and x_1 are adjacent vertices then

$$\|f_{n,x_0}^{N,X} - f_{n,x_1}^{N,X}\|_1 = \|f_{n,x_1}^{N-1,X}\|_1 + \|f_{n,x_0}^{N-1,X_1}\|_1$$

where X_1 is a codimension 1 subcomplex of X.

For general x_0, x_1 we deduce that

$$\begin{aligned}\frac{\|f_{n,x_0}^{N,X} - f_{n,x_1}^{N,X}\|_1}{\|f_{n,x_0}^{N,X}\|_1} &\leq 2d(x_0,x_1)\frac{\binom{n+N-2}{N-1}}{\binom{n+N-1}{N}} \\ &= 2d(x_0,x_1)\frac{N}{n+N-1} \to 0\end{aligned}$$

This proves that X has property A.