Relativistic Quantum Mechanics

Question 1:

What is a mass of 1 GeV in kg? What is a cross section of 1 GeV^{-2} in barns?

Answer 1:

Use $E = mc^2$.

$$1eV = 1.6 \times 10^{-19} J \qquad \rightarrow \qquad \text{mass of } 1.78 \times 10^{-36} kg$$

So $1GeV$ is 10^9 bigger ie $1.78 \times 10^{-27} kg$

Next use $E = \hbar c/k$ to relate an energy to a wave vector (distance) (note we want to set $\hbar = 1$ so need an equation with $\hbar$ in connecting energy and length).

$$1eV \text{ corresponds to } \frac{\hbar c}{e} metres = 1.97 \times 10^{-7} m$$

So $1GeV^{-1}$ is $1.97 \times 10^{-16} m$ and a cross-section of $1GeV^{-2}$ corresponds to (1 barn $= 10^{-28} m^2$)

$$(1.97 \times 10^{-16})^2 m^2 = 3.9 \times 10^{-4} \text{barns}$$

Question 2:

Derive the expressions for probability density in the Klein Gordon and Dirac equations.

Answer 2:

Multiply the KG equation by ϕ^*:

$$\phi^* \left[\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right] \phi = 0$$

Multiply the $*$ of the KG equation by ϕ:

$$\left(\left[\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right] \phi^* \right) \phi = 0$$

Take the second from the first:

$$\phi^* \frac{\partial^2 \phi}{\partial t^2} - \phi \frac{\partial^2 \phi^*}{\partial t^2} - \phi^* \nabla^2 \phi + (\nabla^2 \phi^*) \phi = 0$$

which is

$$\frac{\partial}{\partial t}\left(\phi^*\frac{\partial\phi}{\partial t}-\frac{\partial\phi^*}{\partial t}\phi\right)-\vec{\nabla}.\left(\phi^*\vec{\nabla}\phi-(\vec{\nabla}\phi^*)\phi\right)=0$$

Which is of the form

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}.\vec{J}=0$$

with

$$\rho=\left(\phi^*\frac{\partial\phi}{\partial t}-\frac{\partial\phi^*}{\partial t}\phi\right),\qquad \vec{J}=-\left(\phi^*\vec{\nabla}\phi-(\vec{\nabla}\phi^*)\phi\right)$$

You can multiply by i as I did in the notes - then the density is real for a plane wave solution.

Next we multiply the Dirac equation by $\psi^\dagger$ ($\dagger$ means complex conjugate and transpose)

$$i\psi^\dagger\frac{\partial\psi}{\partial t}=\psi^\dagger(-i\vec{\alpha}.\vec{\nabla}+\beta m)\psi$$

Take the dagger of the Dirac equation and multiply onto ψ:

$$-i\frac{\partial\psi^\dagger}{\partial t}\psi=\psi^\dagger(-i\vec{\alpha}.\vec{\nabla}+\beta m)^\dagger\psi$$

here we used $(AB)^\dagger=B^\dagger A^\dagger$. Now use $\alpha^\dagger=\alpha$ and $\beta^\dagger=\beta$

$$-i\frac{\partial\psi^\dagger}{\partial t}\psi=\psi^\dagger(i\vec{\alpha}.\vec{\nabla}+\beta m)\psi$$

Note here ∇ acts to the left not right! Now take this from the first eqn

$$i\psi^\dagger\frac{\partial\psi}{\partial t}+i\frac{\partial\psi^\dagger}{\partial t}\psi=\psi^\dagger(-i\vec{\alpha}.\vec{\nabla}_{left}-i\vec{\alpha}.\vec{\nabla}_{right})\psi$$

Again one ∇ acts on ψ the other on $\psi^\dagger$ as indicated.

We finally have

$$i\frac{\partial\psi^\dagger\psi}{\partial t}=-i\vec{\nabla}.(\psi^\dagger\vec{\alpha}\psi)$$

We which is again a continuity equation and we conclude $\rho=\psi^\dagger\psi$ and $\vec{J}=\psi^\dagger\vec{\alpha}\psi$.

Question 3:

How would you introduce a potential into the Klein Gordon equation? Think about dimensions and the non-relativistic limit!

Answer 3:

The KG equation is based on

$$E^2 = p^2c^2 + m^2c^4$$

Expanding in the non-rel limit

$$\begin{aligned} E &= mc^2\left(1+\tfrac{p^2}{2m}\right)^{1/2} \\ &= mc^2 + \tfrac{p^2}{2m} + ... \end{aligned}$$

Now we want to adjust things so that

$$E - V = \frac{p^2}{2m} + \text{constant}$$

We must use

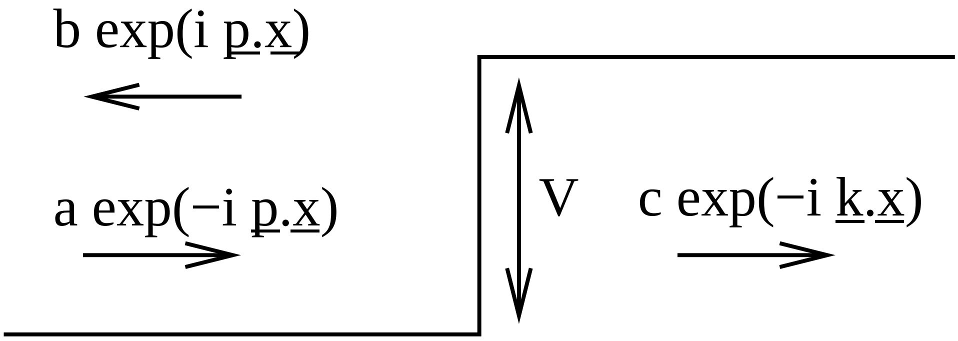
$$(E - V)^2 = p^2c^2 + m^2c^4$$

So the KG equation becomes

$$\left[(i\frac{\partial}{\partial t} - V)^2 + \nabla^2\right]\phi = m^2\phi$$

Question 4: (involved)

Consider the wave incident on a potential step shown below.



Show that if $V > m+E_p$ then one cannot avoid getting negative currents and densities. Hint: use the fact that $\phi(x)$ and $\partial\phi(x)/\partial x$ are continuous at $x = 0$, and ensure that the group velocity $v_g = \partial E/\partial k$ is positive for $x > 0$. Interpret the solution.

Answer 4:

Each wave must also have a temporal piece of the form e^{-iEt}. We must require this temporal behaviour to match at the barrier (set to be at $x = 0$):

$$ae^{-iE_pt} + be^{-iE_pt} = ce^{-iE_kt}$$

We must have $E_p = E_k$ for this equation to work for all t. We learn, from the KG equation in the barrier that

$$(E_p - V)^2 = k^2 + m^2$$

We next match the amplitudes and x derivatives at $x = 0$

$$a + b = c$$

$$-ap + bp = -ck$$

Solving together we get

$$b = \frac{p-k}{p+k}a$$

Next we turn to the group velocity in the barrier ($v_g = \frac{\partial E_p}{\partial k}$)

$$(E_p - V)^2 = k^2 = m^2$$

$$2\frac{\partial E_p}{\partial k}(E_p - V) = 2k$$

So

$$\frac{\partial E_p}{\partial k} = \frac{k}{E_p - V}$$

Now.. since $V > E_p$ we deduce that for a positive group velocity k must be negative.

What does this mean? Look at the relation between a and b above - we see

$$b > a \ \ !!$$

There are more particles reflected than incident!

The probability density in the barrier is given by

$$\rho = \phi^*(i\frac{\partial}{\partial t} - V)\phi + \phi(-i\frac{\partial}{\partial t} - V)\phi^* \quad = \quad 2(E_p - V)|d|^2$$

which is again -ve. There are anti-particles in the barrier.

Conclusion: we are seeing pair creation at the barrier edge!

Question 5:

Prove that the matrices $\vec{\alpha}$ and β in the Dirac equation are traceless with eigenvalues ± 1.Hence argue that they must be even dimensional.

Answer 5:

We use $\beta^2 = 1$ to write

$$Tr\alpha^i = Tr(\alpha^i\beta^2) = -Tr(\beta\alpha^i\beta)$$

the last following from the basic algebra of the αs and β: $\alpha^i\beta = -\beta\alpha^i$.

Now we use the property of Traces that they are cyclic

$$-Tr(\beta\alpha^i\beta) = -Tr(\beta^2\alpha^i) = -Tr\alpha^i$$

We can only have

$$Tr\alpha^i = 0$$

.

We square the eigenvalue equation:

$$\alpha^i v = \lambda v \qquad \rightarrow \qquad (\alpha^i)^2 v = \lambda^2 v$$

Since $(\alpha^i)^2 = 1$ we find $\lambda = \pm 1$.

The Trace of a matrix is given by the sum of its eigenvalues. A 2x2 matrix could have $\lambda = +1, -1$, a 4x4 $\lambda = +1, +1, -1, -1...$ we must have even dimension for the Trace to vanish though.

Question 6:

Show that for the Dirac equation solution spinors

$$u^\dagger(r,p)\ u(s,p) \ = \ v^\dagger(r,p)\ v(s,p) \ = \ 2E\delta^{rs}$$

Answer 6:

We use

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}.\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

and in detail

$$\vec{\sigma}.\vec{p}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_3 \\ p_1 + ip_2 \end{pmatrix}$$

Thus

$$u_1^\dagger = \sqrt{E+m}\left(1, 0, \frac{p_3}{E+m}, \frac{p_1 - ip_2}{E+m}\right)$$

So finally we can perform

$$u_1^\dagger u_1 = (E+m)\left[1 + \frac{1}{(E+m)^2}(p_3^2 + p_1^2 + p_2^2)\right] = (E+m) + \frac{p^2}{E+m}$$

Writing $p^2 = E^2 - m^2 = (E+m)(E-m)$ we get

$$u_1^\dagger u_1 = 2E$$

Next

$$u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}.\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_1-ip_2}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix}$$

So

$$u_1^\dagger u_2 = (E+m)\left[\frac{1}{(E+m)^2}(p_3(p_1-ip_2)+(p_1-ip_2)(-p_3)\right] = 0$$

I'll let you check the others! :-)

Question 7:

Show using the properties of $\vec{\alpha}$ and β that the γ^μ must satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Answer 7:

Remember $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$.

- $\{\gamma^0, \gamma^0\} = 2\beta^2 = 2 = 2g^{00}$
- $\{\gamma^0, \gamma^i\} = \beta\beta\alpha^i + \beta\alpha^i\beta = \beta^2\alpha^i - \beta^2\alpha^i = 0$
- $\{\gamma^i, \gamma^j\} = \beta\alpha^i\beta\alpha^j + \beta\alpha^j\beta\alpha^i = -\{\alpha^i, \alpha^j\} = -2\delta^{ij}$

We've used: $\beta\alpha^i = -\alpha^i\beta$ and $\beta^2 = 1$.

Question 8:

Show that

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

Answer 8:

Using the relations in question 7:

$\mu = 0:$ $\qquad \gamma^0\gamma^0\gamma^0 = \beta^3 = \beta = \beta^\dagger = \gamma^{0\dagger}$

We've used that β is hermitian (as is α for below).

$\mu = i$: $\qquad \gamma^0\gamma^i\gamma^0 = \beta\beta\alpha^i\beta = -\beta\alpha^i$

Now note

$$(\gamma^i)^\dagger = (\beta\alpha^i)^\dagger = \alpha^{i\dagger}\beta^\dagger = \alpha^i\beta = -\beta\alpha^i$$

So we find $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.