

Relativistic Quantum Mechanics

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- Klein Gordon Equation
- Feynman Stueckelburg Interpretation
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Learning Outcomes

- Be able to derive KG equation and explain physical meaning of -ve E solutions.
- Be able to derive Dirac eqn and algebra for α^i, β
- Be able to solve the free Dirac equation and interpret the solutions in terms of energy and spin
- Be able to use the covariant notation γ^μ

Reference Books

- Halzen and Martin
- many other QM books cover this material

In this section of the course we will study relativistic quantum mechanics. In a particle accelerator we are interested in, for example, the interactions of highly energetic electrons so the need to combine relativity and quantum mechanics is pressing. Some remarkable results will come out of this synthesis. In particular we will theoretically predict the existence of anti-particles and also fermion spin. Sit back and enjoy!

1 Non-Relativistic Wave Equation Review

Let's review how wave equations describe non-relativistic quantum particles. Experimentally we learnt that a particle with definite momentum \vec{p} and energy E could be associated with a plane wave

$$\psi = e^{i(\vec{k}\cdot\vec{x}-wt)}, \quad \vec{k} = \frac{\vec{p}}{\hbar}, \quad w = \frac{E}{\hbar} \quad (1)$$

To extract E and \vec{p} from the wave we use *operators*

$$E\psi = i\hbar \frac{d}{dt}\psi, \quad \vec{p}\psi = -i\hbar \vec{\nabla}\psi \quad (2)$$

Classically we know that $E = p^2/2m + V$ so we arrive at the Schroedinger equation

$$i\hbar \frac{d}{dt}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \quad (3)$$

Equally we can write the Schroedinger equation quite generally (with $\hbar = 1$) as

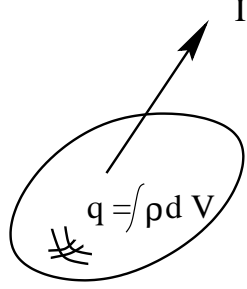
$$H\psi = i\hbar \frac{\partial\psi}{\partial t} \quad (4)$$

where H is the Hamiltonian (i.e. the energy operator).

1.1 Probability Density

Remember that we interpret $\psi^*\psi dx$ as the probability that the particle will be found between x and $x + \delta x$. Probability is a conserved quantity - there is always probability one that the particle will be somewhere!

This means that if the probability of the particle being in some area decreases then the probability that it lies outside must increase. In other words there is a flow of probability/current density satisfying the usual conservation equation (cf electric charge)



$$\int_S \vec{J} \cdot d\vec{A} = - \int \frac{\partial \rho}{\partial t} dV \quad (5)$$

Using the divergence theorem ($\int \vec{A} \cdot d\vec{S} = \int \vec{\nabla} \cdot \vec{A} dV$) we have the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad (6)$$

wherein $\rho(\vec{J})$ is the probability density(current). Now we can show using the Schroedinger equation that $\rho = \psi^* \psi$ satisfies such a relation. We add two copies of the Schroedinger equation as follows

$$-i\psi^*(SE) + (SE)^*i\psi \quad (7)$$

This gives

$$\begin{aligned} \hbar\psi^* \frac{\partial \psi}{\partial t} + \hbar\psi \frac{\partial \psi^*}{\partial t} &= \frac{i\hbar^2}{2m} \psi^* \nabla^2 \psi - i\psi^* V \psi \\ &\quad - \frac{i\hbar^2}{2m} \psi \nabla^2 \psi^* + i\psi^* V \psi \end{aligned} \quad (8)$$

and hence

$$\frac{\partial}{\partial t}(\psi^* \psi) = \frac{i\hbar}{2m} \vec{\nabla} \cdot (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) \quad (9)$$

which indeed has the form of a conservation equation with $\rho = \psi^* \psi$.

2 Relativity Review

In relativity an event is described by the four coordinates of a four-vector

$$x^\mu = (ct, \vec{x}) \quad (10)$$

Under Lorentz transformations (LT) it transforms - a familiar example of a LT is a boost along the z -axis, for which

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix},$$

with, as usual, $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. LT's can be thought of as generalized rotations.

x^μ is then a (contravariant) 4-vector since it transforms as

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \quad (11)$$

The Greek labels $\mu, \nu \dots \in \{0, 1, 2, 3\}$ denote Lorentz indices and the summation convention is used.

The “length” of the 4-vector ($c^2 t^2 - |\vec{x}|^2$) is invariant to LTs. In general we define the Minkowski scalar product of two 4-vectors x and y as

$$x \cdot y = x^\mu y^\nu g_{\mu\nu} = x^\mu y_\mu \quad (12)$$

where the metric

$$g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1, -1, -1, -1), \quad g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad (13)$$

has been introduced. The last step in (12) is nothing but the definition of a covariant 4-vector (sometimes referred to as co-vector)

$$x_\mu \equiv g_{\mu\nu} x^\nu \quad (14)$$

To formulate a coherent relativistic theory of dynamics we define kinematic variables that are also 4-vectors (ie transform as described above). For example, we define a 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} \quad (15)$$

where τ is the *proper time* measured by a clock moving with the particle. Everyone will agree as to what the clock says at a particular event so this measure of time is Lorentz invariant and u^μ transforms as x^μ . Note

$$u^\mu = \frac{dt}{d\tau} \frac{dx^\mu}{dt} = \gamma(c, \vec{v}) \quad (16)$$

and has invariant length

$$u^\mu u_\mu = \gamma^2(c^2 - |\vec{v}|^2) = c^2 \quad (17)$$

Similarly 4-momentum provides a relativistic definition of energy and momentum

$$p^\mu = m u^\mu \equiv (E/c, \vec{p}) \quad (18)$$

The invariant length gives us the crucial relation

$$p^\mu p_\mu = E^2/c^2 - |\vec{p}|^2 = m^2 c^2 \quad (19)$$

Note that ∂_μ is a covariant 4-vector,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \quad \partial_\mu x^\nu = \delta_\mu^\nu, \quad (20)$$

so $\nabla^i = -\partial^i$ and $\partial^\mu = (\partial^0, -\vec{\nabla})$.

I will use natural units in this course. This firstly means redefining the unit of distance so that $c = 1$. Secondly I shall redefine the unit of energy so that $E = h\nu = 2\pi\hbar\nu$ ie set $\hbar = 1$. So mass, energy, inverse length and inverse time all have the same dimensions. Generally think of energy E as the basic unit, e.g mass m has units of GeV and distance x has unit GeV^{-1} .

3 The Klein Gordon Equation

For a free relativistic particle the total energy E is no longer given by the equation we used to derive the Schroedinger equation. Instead it is given by the Einstein equation

$$E^2 = \vec{p}^2 + m^2. \quad (21)$$

In position space we write the energy-momentum operator as

$$p^\mu \rightarrow i\partial^\mu, \quad (E, \vec{p}) = (i\frac{\partial}{\partial t}, -i\vec{\nabla}) \quad (22)$$

so that the KG equation becomes

$$(\square + m^2)\phi(x) = 0 \quad (23)$$

where we have introduced the box notation,

$$\square = \partial_\mu \partial^\mu = \partial^2 / \partial t^2 - \nabla^2 \quad (24)$$

and x is the 4-vector (t, \vec{x}) .

The Klein-Gordon equation has plane wave solutions:

$$\phi(x) = N e^{-i(Et - \vec{p} \cdot \vec{x})} \quad (25)$$

where N is a normalization constant and $E = \pm\sqrt{\vec{p}^2 + m^2}$.

There are two problems with this equation though. Firstly there are both positive and negative energy solutions. The negative energy solutions pose a severe problem if you try to interpret ϕ as a wave function (as indeed we are trying to do). The spectrum is no longer bounded from below, and you can extract arbitrarily large amounts of energy from the system by driving it into ever more negative energy states. Any external perturbation capable of pushing a particle across the energy

gap of $2m$ between the positive and negative energy continuum of states can uncover this difficulty. Furthermore, we cannot just throw away these solutions as unphysical since they appear as Fourier modes in any realistic solution of (23).

A second problem with the wave function interpretation arises when trying to find a probability density. Since ϕ is Lorentz invariant, $|\phi|^2$ does not transform like a density so we will not have a Lorentz covariant continuity equation $\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0$.

To search for a candidate we derive such a continuity equation. Start with the Klein-Gordon equation multiplied by ϕ^* and subtract the complex conjugate of the KG equation multiplied by ϕ . Defining ρ and \vec{J} by

$$\rho \equiv i \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right), \quad (26)$$

$$\vec{J} \equiv -i (\phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^*) \quad (27)$$

you obtain a covariant conservation equation

$$\partial_\mu J^\mu = 0 \quad (28)$$

where J is the 4-vector (ρ, \vec{J}) . It is thus natural to interpret ρ as a probability density and \vec{J} as a probability current. However, for a plane wave solution (25), $\rho = 2|N|^2 E$, so ρ is not positive definite since we've already found E can be negative.

3.1 Feynman Stueckelberg Interpretation

We have found that the Klein-Gordon equation, a candidate for describing the quantum mechanics of spinless particles, admits unacceptable negative energy states when ϕ is interpreted as the single particle wave function. There is another way forward (this is the way followed in the textbook of Halzen & Martin) due to Stueckelberg and Feynman. Causality forces us to ensure that positive energy states propagate forwards in time. But if we force the negative energy states only to propagate backwards in time then we find a theory that is consistent with the requirements of causality and that has none of the aforementioned problems. In fact, the negative energy states cause us problems only so long as we think of them as real physical states propagating forwards in time. Therefore, we should interpret the emission (absorption) of a negative energy particle with momentum p^μ as the absorption (emission) of a positive energy antiparticle with momentum $-p^\mu$.

In order to get more familiar with this picture, consider a process with a π^+ and a photon in the initial state and final state. In figure 2.1(a) the π^+ starts from the point A and at a later time t_1 emits a photon at the point \vec{x}_1 . If the energy of the π^+ is still positive, it travels on forwards in time and eventually will absorb the initial state photon at t_2 at the point \vec{x}_2 . The final state is then again a photon and a (positive energy) π^+ .

There is another process however, with the same initial and final state, shown in figure 2.1(b). Again, the π^+ starts from the point A and at a later time t_1 emits a

photon at the point \vec{x}_1 . But this time, the energy of the photon emitted is bigger than the energy of the initial π^+ . Thus, the energy of the π^+ becomes negative and it is forced to travel backwards in time. Then at an *earlier* time t_2 it absorbs the initial state photon at the point \vec{x}_2 , thereby rendering its energy positive again. From there, it travels forward in time and the final state is the same as in figure 2.1(a), namely a photon and a (positive energy) π^+ .

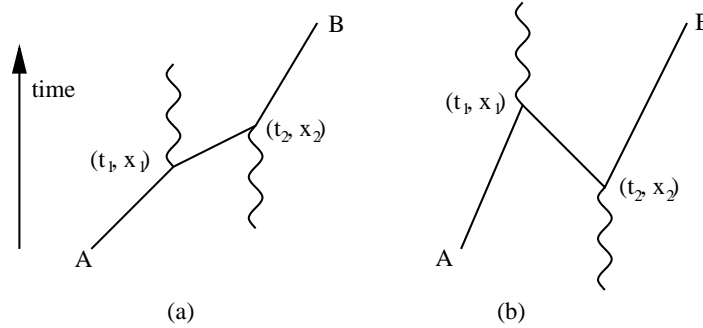


Figure 2.1 - Interpretation of negative energy states

In today's language, the process in figure 2.1(b) would be described as follows: in the initial state we have an π^+ and a photon. At time t_2 and at the point \vec{x}_2 the photon creates a $\pi^+-\pi^-$ pair. Both propagate forwards in time. The π^+ ends up in the final state, whereas the π^- is annihilated at (a later) time t_1 at the point \vec{x}_1 by the initial state π^+ , thereby producing the final state photon. To someone observing in real time, the negative energy state moving backwards in time looks to all intents and purposes like a negatively charged pion with positive energy moving forwards in time. We have discovered anti-matter!

4 Dirac Equation

Historically the Klein Gordon equation was believed to be sick although now we understand it is telling us about anti-particles. To try to solve the problem of negative energy solutions Dirac wanted an equation first order in time derivatives. His starting point was to assume a Hamiltonian of the form,

$$H_D = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \beta m \quad (29)$$

where P_i are the three components of the momentum operator \vec{p} , and α_i and β are some unknown (constant) quantities, which, as will be seen below, cannot simply be commuting numbers.

We should write the momentum operators explicitly in terms of their differential operators, using equation (22). Then the Dirac equation becomes, using the Dirac Hamiltonian in equation (29),

$$i \frac{\partial \psi}{\partial t} = (-i \vec{\alpha} \cdot \vec{\nabla} + \beta m) \psi \quad (30)$$

which is the position space Dirac equation.

If ψ is to describe a free particle it must though satisfy the Klein-Gordon equation so that it has the correct energy-momentum relation. This requirement imposes relationships among $\alpha_1, \alpha_2, \alpha_3$ and β . To see these, apply the operator on each side of equation (30) twice, i.e. iterate the equation,

$$-\frac{\partial^2 \psi}{\partial t^2} = [-\alpha^i \alpha^j \nabla^i \nabla^j - i(\beta \alpha^i + \alpha^i \beta) m \nabla^i + \beta^2 m^2] \psi$$

with an implicit sum over i and j from 1 to 3. The Klein-Gordon equation by comparison is

$$-\frac{\partial^2 \psi}{\partial t^2} = [-\nabla^i \nabla^i + m^2] \psi \quad (31)$$

If we do not assume that the α^i and β commute then the KG will be satisfied if

$$\begin{aligned} \alpha_i \alpha_j + \alpha_j \alpha_i &= 2\delta_{ij} \\ \beta \alpha_i + \alpha_i \beta &= 0 \\ \beta^2 &= 1 \end{aligned} \quad (32)$$

for $i, j = 1, 2, 3$. The α_i and β cannot be ordinary numbers, but it is possible to give them a realization as matrices. In this case, ψ must be a multi-component *spinor* on which these matrices act.

In two dimensions a natural set of matrices for the $\vec{\alpha}$ would be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (33)$$

However, there is no other independent 2×2 matrix with the right properties for β , so the smallest number of dimensions for which the Dirac matrices can be realized is four. One choice is the *Dirac representation*:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (34)$$

Note that each entry above denotes a two-by-two block and that the 1 denotes the 2×2 identity matrix.

There is a theorem due to Pauli that states that all sets of matrices obeying the relations in (32) are equivalent. Since the hermitian conjugates $\vec{\alpha}^\dagger$ and β^\dagger clearly obey the relations, you can, by a change of basis if necessary, assume that $\vec{\alpha}$ and β are hermitian. All the common choices of basis have this property. Furthermore, we would like α_i and β to be hermitian so that the Dirac Hamiltonian (29) is hermitian.

Continuity Equation

If we define

$$\rho = J^0 = \psi^\dagger \psi, \quad \vec{J} = \psi^\dagger \vec{\alpha} \psi, \quad (35)$$

then it is a simple exercise using the Dirac equation to show that this satisfies the continuity equation $\partial_\mu J^\mu = 0$. Simply add $\psi DE^* + \psi^* DE$ and rearrange it.

Note that ρ is now positive definite - this seemed like a major achievement to Dirac.

4.1 Solutions to the Dirac Equation

The wave function in the Dirac equation is a four component vector. To shed light on what this means lets look at free particle solutions.

We look for plane wave solutions of the form

$$\psi = \begin{pmatrix} \chi(\vec{p}) \\ \phi(\vec{p}) \end{pmatrix} e^{-i(Et - \vec{p} \cdot \vec{x})} \quad (36)$$

where $\phi(\vec{p})$ and $\chi(\vec{p})$ are two-component spinors that depend on momentum \vec{p} but are independent of \vec{x} . Using the Dirac representation of the matrices, and inserting the trial solution into the Dirac equation gives the pair of simultaneous equations

$$E \begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -m \end{pmatrix} \begin{pmatrix} \chi \\ \phi \end{pmatrix}. \quad (37)$$

There are two simple cases for which equation (37) can readily be solved, namely

1. $\vec{p} = 0$, $m \neq 0$, which might represent an electron in its rest frame.
2. $m = 0$, $\vec{p} \neq 0$, which describes a massless particle or a particle in the ultra-relativistic limit ($E \gg m$).

4.1.1 Particle at Rest

For case (1), an electron in its rest frame, the equations (37) decouple and become simply,

$$E\chi = m\chi, \quad E\phi = -m\phi. \quad (38)$$

So, in this case, we see that χ corresponds to solutions with $E = m$, while ϕ corresponds to solutions with $E = -m$. Dirac had therefore failed to remove these solutions. In light of our earlier discussions, we no longer need to recoil in horror at the appearance of these negative energy states.

4.1.2 Dirac's Interpretation of Negative Energy

As an aside, it is interesting to see how Dirac coped with the negative energy states.

Dirac interpreted the negative energy solutions by postulating the existence of a "sea" of negative energy states. The vacuum or ground state has all the negative energy states full. An additional electron must now occupy a positive energy state since the Pauli exclusion principle forbids it from falling into one of the filled negative energy states. On promoting one of these negative energy states to a positive energy one, by supplying energy, an electron-hole pair is created, i.e. a positive energy electron and a hole in the negative energy sea. The hole is seen in nature as a positive energy positron. This was a radical new idea, and brought pair creation and antiparticles into physics.

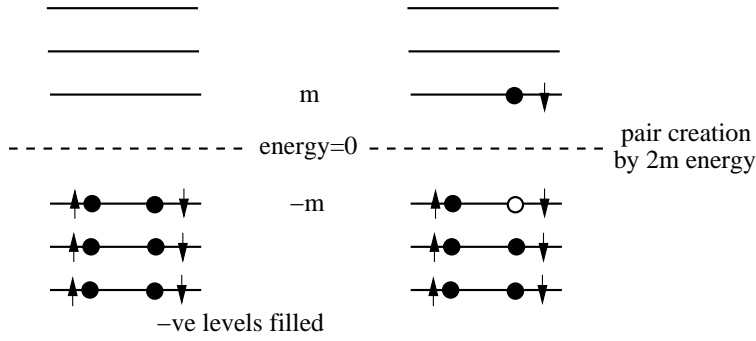


Figure 2.2 - Dirac's filled negative energy states

The problem with Dirac's hole theory is that it does not work for bosons. Such particles have no exclusion principle to stop them falling into the negative energy states, releasing their energy.

4.1.3 General Solutions

The negative energy solutions persist for an electron with $\vec{p} \neq 0$ for which the solutions to equation (37) are

$$\phi = \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi, \quad \chi = \frac{\vec{\sigma} \cdot \vec{p}}{E-m} \phi. \quad (39)$$

Now we can substitute one of these equations into the other and use $(\vec{\sigma} \cdot \vec{p})^2 = \vec{p}^2$. Explicitly

$$\begin{aligned} (\vec{\sigma} \cdot \vec{p})^2 &= \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}^2 \\ &= \begin{pmatrix} (p_1)^2 + (p_2)^2 + (p_3)^2 & 0 \\ 0 & (p_1)^2 + (p_2)^2 + (p_3)^2 \end{pmatrix} = \vec{p}^2 I \end{aligned} \quad (40)$$

We find that

$$\phi = \frac{(\vec{\sigma} \cdot \vec{p})^2}{E^2 - m^2} \phi = \frac{\vec{p}^2}{E^2 - m^2} \phi \quad (41)$$

from which we deduce that $E = \pm |\sqrt{\vec{p}^2 + m^2}|$. medskip

We write the positive energy solutions with $E = +|\sqrt{\vec{p}^2 + m^2}|$ as

$$\psi(x) = \begin{pmatrix} \chi \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi \end{pmatrix} e^{-i(Et - \vec{p} \cdot \vec{x})}, \quad (42)$$

while the general negative energy solutions with $E = -|\sqrt{\vec{p}^2 + m^2}|$ are

$$\psi(x) = \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E-m} \phi \\ \phi \end{pmatrix} e^{-i(Et - \vec{p} \cdot \vec{x})}, \quad (43)$$

for arbitrary constant ϕ and χ . Clearly when $\vec{p} = 0$ these solutions reduce to the positive and negative energy solutions discussed previously.

We want to rewrite the solutions, eqs. (42) and (43), introducing the spinors $u_\alpha(s, \vec{p})$ and $v_\alpha(s, \vec{p})$. The label $\alpha \in \{1, 2, 3, 4\}$ is a spinor index that often will be suppressed. Take the positive energy solution equation (42) and define

$$\sqrt{E+m} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi_r \end{pmatrix} e^{-ip \cdot x} \equiv u(s, p) e^{-ip \cdot x}. \quad (44)$$

For the negative energy solution of equation (43), change the sign of the energy, $E \rightarrow -E$, and the three-momentum, $\vec{p} \rightarrow -\vec{p}$, to obtain,

$$\sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \phi_r \\ \phi_r \end{pmatrix} e^{ip \cdot x} \equiv v(s, p) e^{ip \cdot x}. \quad (45)$$

In these two solutions E is now (and for the rest of the course) always positive and given by $E = (\vec{p}^2 + m^2)^{1/2}$. The argument s takes the values 1, 2, with

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (46)$$

The u -spinor solutions will correspond to particles and the v -spinor solutions to antiparticles. The role of the two χ 's will become clear in the following section, where it will be shown that the two choices of s are spin labels. Note that each spinor solution depends on the three-momentum \vec{p} , so it is implicit that $p^0 = E$.

Orthogonality and Completeness

Our solutions to the Dirac equation take the form

$$\psi = N u_s e^{-ip \cdot x}, \quad \psi = N v_r e^{ip \cdot x}, \quad r, s = 1, 2 \quad (47)$$

The N is a normalization factor. We have already included a normalization factor $\sqrt{E+m}$ in our spinors. With this factor,

$$u^\dagger(r, p) u(s, p) = v^\dagger(r, p) v(s, p) = 2E \delta^{rs}. \quad (48)$$

This corresponds to the standard relativistic normalization of $2E$ particles per unit volume - this makes $u^\dagger u$ and hence $\psi^* \psi$ transforms like the time component of a 4-vector under Lorentz transformations, as it must to be the zeroth component of J^μ . Note that the spinors are *orthogonal*.

We must further normalize the spatial wave functions. In fact a plane wave is not normalizable in an infinite space so we will work in a large box of volume V

$$\int \psi_a^\dagger \psi_b d^3x = 2E N^2 V \delta_{ab} \quad (49)$$

where a, b run over the possible values of r, s and the value of p . Note again the orthogonality of the states. To normalize to $2E$ particles per unit volume we must

set $N = 1/\sqrt{V}$. Sometimes its helpful to normalize so that $\int \psi_a^\dagger \psi_b d^3x = \delta_{ab}$ (so that there is one particle per unit volume) in which case $N = 1/\sqrt{2EV}$ - this is not a Lorentz invariant normalization so must be done in a particular frame.

Remember that the solutions to the wave equation form a complete set of states meaning that we can expand (like a Fourier expansion) an arbitrary function $\chi(x)$ in terms of them

$$\chi(x) = \sum_n a_n \psi_n(x) \quad (50)$$

The a_n are the equivalent of Fourier coefficients and if χ is a wave function in some quantum mixed state then $|a_n|^2$ is the probability of being in the state ψ_n .

4.2 Spin

Now it is time to justify the statements we have been making that the Dirac equation describes spin-1/2 particles. We will see that the two components of each of the positive and negative energy solutions describe spin up and spin down states.

Conserved Quantities: Remember that a conserved quantity in quantum mechanics is described by a time independent operator that commutes with the Hamiltonian. To prove this we evaluate

$$\begin{aligned} \frac{d\langle f \rangle}{dt} &= \frac{d}{dt} \int \psi^* F \psi dx \\ &= \int \frac{\partial \psi^*}{\partial t} F \psi dx + \int \psi^* F \frac{\partial \psi}{\partial t} dx \end{aligned} \quad (51)$$

Note that F is time independent here. Now we use the Schroedinger equation

$$H\psi = i\hbar \frac{\partial \psi}{\partial t} \quad (52)$$

to find

$$\frac{d\langle f \rangle}{dt} = \frac{i}{\hbar} \int \psi^* (HF - FH)\psi dx \quad (53)$$

So if the commutator $[F, H]$ vanishes the expectation value is conserved.

Now the Dirac Hamiltonian in momentum space is given in equation (29) as

$$H_D = \vec{\alpha} \cdot \vec{p} + \beta m \quad (54)$$

and the orbital angular momentum operator is

$$\vec{L} = \vec{R} \times \vec{p}.$$

\vec{L} and H_D may not commute because they contain x and p which do not commute ($[x_i, p_j] = i\delta_{ij}$). Evaluating the commutator of \vec{L} with H_D ,

$$\begin{aligned} [\vec{L}, H_D] &= [\vec{R} \times \vec{p}, \vec{\alpha} \cdot \vec{p}] \\ &= [\vec{R}, \vec{\alpha} \cdot \vec{p}] \times \vec{p} \\ &= i\vec{\alpha} \times \vec{p}, \end{aligned} \quad (55)$$

we see that the orbital angular momentum is not conserved (otherwise the commutator would be zero).

Now we would like to find a *total* angular momentum \vec{J} that *is* conserved, by adding an additional operator \vec{S} to \vec{L} ,

$$\vec{J} = \vec{L} + \vec{S}, \quad [\vec{J}, H_D] = 0. \quad (56)$$

To this end, consider the three matrices,

$$\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = -i\alpha_1\alpha_2\alpha_3\vec{\alpha}, \quad (57)$$

where the first equivalence is merely a definition of $\vec{\Sigma}$ and the last equality can be verified by an explicit calculation. The $\vec{\Sigma}/2$ have the correct commutation relations to represent angular momentum, since the Pauli matrices do, and their commutators with $\vec{\alpha}$ and β are,

$$[\vec{\Sigma}, \beta] = 0, \quad [\Sigma_i, \alpha_j] = 2i\varepsilon_{ijk}\alpha_k. \quad (58)$$

Here ε_{ijk} is a totally anti-symmetric tensor which is zero if any of the three indices are the same - ε_{123} is +1, and we get a minus sign if we interchange any two indices so $\varepsilon_{213} = -1$. From the relations in (58) we find that

$$[\vec{\Sigma}, H_D] = -2i\vec{\alpha} \times \vec{p}. \quad (59)$$

Comparing equation (59) with the commutator of \vec{L} with H_D in equation (55), you see that

$$[\vec{L} + \frac{1}{2}\vec{\Sigma}, H_D] = 0,$$

and we can identify

$$\vec{S} = \frac{1}{2}\vec{\Sigma}$$

as the additional quantity that, when added to \vec{L} in equation (56), yields a conserved total angular momentum \vec{J} . We interpret \vec{S} as an angular momentum *intrinsic* to the particle. Now

$$\vec{S}^2 = \frac{1}{4} \begin{pmatrix} \vec{\sigma} \cdot \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \cdot \vec{\sigma} \end{pmatrix} = \frac{3}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and, recalling that the eigenvalue of \vec{J}^2 for spin j is $j(j+1)$, we conclude that \vec{S} represents spin-1/2 and the solutions of the Dirac equation have spin-1/2 as promised. We worked in the Dirac representation of the matrices for convenience, but the result is necessarily independent of the representation.

Now consider the u -spinor solutions $u(s, p)$ of equation (44). Choose $\vec{p} = (0, 0, p_z)$ and write

$$u_{\uparrow} \equiv u(1, p) = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix}, \quad u_{\downarrow} \equiv u(2, p) = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix}. \quad (60)$$

With these definitions, we get

$$S_z u_\uparrow = \frac{1}{2} u_\uparrow, \quad S_z u_\downarrow = -\frac{1}{2} u_\downarrow.$$

So, these two spinors represent spin up and spin down along the z -axis respectively. For the v -spinors, with the same choice for \vec{p} , write,

$$v_\downarrow = v(1, p) = \begin{pmatrix} \sqrt{E-m} \\ 0 \\ \sqrt{E+m} \\ 0 \end{pmatrix}, \quad v_\uparrow = v(2, p) = \begin{pmatrix} 0 \\ -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} \end{pmatrix}, \quad (61)$$

where now,

$$S_z v_\downarrow = \frac{1}{2} v_\downarrow, \quad S_z v_\uparrow = -\frac{1}{2} v_\uparrow.$$

This apparently perverse choice of up and down for the v 's is actually quite sensible when one realizes that a negative energy electron carrying spin $+1/2$ backwards in time looks just like a positive energy positron carrying spin $-1/2$ forwards in time.

4.3 Lorentz Covariant Notation

There is a more compact way of writing the Dirac equation, which requires that we get to grips with some more notation. Define the γ -matrices,

$$\gamma^0 = \beta, \quad \vec{\gamma} = \beta \vec{\alpha}. \quad (62)$$

In the Dirac representation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \quad (63)$$

In terms of these, the relations between the $\vec{\alpha}$ and β in equation (32) can be written compactly as,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (64)$$

.

Combinations like $a_\mu \gamma^\mu$ occur frequently and are conventionally written as,

$$\not{a} = a_\mu \gamma^\mu = a^\mu \gamma_\mu,$$

pronounced “a slash.”

Observe that using the γ -matrices the Dirac equation (30) becomes

$$(i\not{\partial} - m)\psi = 0, \quad (65)$$

or, in momentum space,

$$(\not{p} - m)\psi = 0. \quad (66)$$

The spinors u and v satisfy

$$(\not{p} - m)u(s, p) = 0, \quad (67)$$

$$(\not{p} + m)v(s, p) = 0, \quad (68)$$

since for $v(s, p)$, $E \rightarrow -E$ and $\vec{p} \rightarrow -\vec{p}$.

We want the Dirac equation (65) to preserve its form under Lorentz transformations eq. (11). We've just naively written the matrices in the Dirac equation as γ_μ however this does not make them a 4-vector! They are just a set of numbers in four matrices and there's no reason they should change when we do a boost. However, the notation is deliberately suggestive, for when combined with Dirac fields you can construct quantities that transform like vectors and other Lorentz tensors (we won't show this here). Since ∂^μ does transform, for the equation to be Lorentz covariant we are led to propose that ψ transforms too. As an example of such a transformation lets look at Parity transformations

Parity

Consider parity (space inversion) transformations, $P: t, \vec{x} \rightarrow t, -\vec{x}$

We wouldn't expect physics to change because of such a redefinition of our axes labelling. For the Dirac equation to remain the same though we must also transform ψ as

$$\psi \rightarrow \psi' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi \quad (69)$$

To see this works note that under Parity

$$\not{\partial} = \begin{pmatrix} \frac{\partial}{\partial t} & \vec{\sigma} \cdot \vec{\nabla} \\ -\vec{\sigma} \cdot \vec{\nabla} & -\frac{\partial}{\partial t} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\partial}{\partial t} & -\vec{\sigma} \cdot \vec{\nabla} \\ \vec{\sigma} \cdot \vec{\nabla} & -\frac{\partial}{\partial t} \end{pmatrix} \quad (70)$$

So

$$\not{\partial} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{\partial}{\partial t} & \vec{\sigma} \cdot \vec{\nabla} \\ \vec{\sigma} \cdot \vec{\nabla} & \frac{\partial}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \not{\partial} \quad (71)$$

This means we can write the parity transformed Dirac equation

$$(i\not{\partial} - m)\psi' = 0 \quad (72)$$

as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (i\not{\partial} - m)\psi = 0 \quad (73)$$

which has the same solutions as the Dirac equation before the transformation as we require.

The upshot is that we have discovered that particles and anti-particles have opposite intrinsic parity.

4.4 Massless (Ultra-relativistic) Fermions

At very high energies we may neglect the masses of particles ($E^2 \simeq |\vec{p}|^2$). Let us look, therefore, at solutions of the Dirac equation with $m = 0$, on the basis that this will be an extremely good approximation for many situations.

From equation (37) we have in this case

$$E\phi = \vec{\sigma} \cdot \vec{p}\chi, \quad E\chi = \vec{\sigma} \cdot \vec{p}\phi. \quad (74)$$

These equations can easily be decoupled by taking linear combinations and defining the two component spinors N_L and N_R ,

$$N_R \equiv \chi + \phi, \quad N_L \equiv \chi - \phi \quad (75)$$

which leads to

$$EN_R = \vec{\sigma} \cdot \vec{p}N_R, \quad EN_L = -\vec{\sigma} \cdot \vec{p}N_L. \quad (76)$$

The system is in fact described by two entirely separated two component spinors. If we take them to be moving in the z direction, and noting that $\sigma_3 = \text{diag}(1, -1)$, we see that there is one positive and one negative energy solution in each.

Further since $E = |\vec{p}|$ for massless particles, these equations may be written

$$\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}N_L = -N_L, \quad \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}N_R = N_R \quad (77)$$

Now, $\frac{1}{2} \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$ is known as the *helicity* operator (i.e. it is the spin operator projected in the direction of motion of the momentum of the particle). We see that the N_L corresponds to solutions with negative helicity, while N_R corresponds to solutions with positive helicity. In other words N_L describes a left-handed particle while N_R describes a right-handed particle, and each type is described by a two-component spinor.

Note that under parity transformations $\vec{\sigma} \rightarrow \vec{\sigma}$ (like $\vec{R} \times \vec{p}$), $\vec{p} \rightarrow -\vec{p}$, therefore $\vec{\sigma} \cdot \vec{p} \rightarrow -\vec{\sigma} \cdot \vec{p}$, i.e. the spinors transform into each other:

$$N_L \leftrightarrow N_R. \quad (78)$$

So a theory in which N_L has different interactions to N_R (such as the standard model in which the weak force only acts on left handed particles) manifestly violates parity.

Although massless particles can be described very simply using two component spinors as above, they may also be incorporated into the four-component formalism as follows. From equation (30) we have, in momentum space,

$$|\vec{p}| \psi = \vec{\alpha} \cdot \vec{p} \psi.$$

For such a solution,

$$\gamma^5 \psi = \gamma^5 \frac{\vec{\alpha} \cdot \vec{p}}{|\vec{p}|} \psi = 2 \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|} \psi,$$

using the spin operator $\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\gamma^5\vec{\alpha}$, with $\vec{\Sigma}$ defined in equation (57). But $\vec{S}\cdot\vec{p}/|\vec{p}|$ is the projection of spin onto the direction of motion, i.e. the helicity, and is equal to $\pm 1/2$. Thus $(1+\gamma^5)/2$ projects out the particle with helicity $1/2$ (right handed) and $(1-\gamma^5)/2$ projects out the particle with helicity $-1/2$ (left handed):

$$\frac{(1+\gamma^5)}{2}\psi \equiv \psi_R, \quad \frac{(1-\gamma^5)}{2}\psi \equiv \psi_L, \quad (79)$$

define the four-component spinors ψ_R and ψ_L .