Quantum Chromodynamics

Question 1:

Calculate the R ratio below the strange quark mass threshold and above the bottom quark threshold. Check your latter answer against the data in the notes.

Answer 1:

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_f Q_f^2$$

$N_c = 3$ the number of colours and the sum over f tells us to include all quarks we have enough energy to create.

Below the strange quark mass threshold there are up and down quarks only so

$$R = 3 \times \left(\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 \right) = \frac{5}{3}$$

Above the bottom quark threshold there are up, down, strange, charm and bottom quarks in the theory so

$$R = 3 \times \left(2 \times \left(\frac{2}{3}\right)^2 + 3 \times \left(-\frac{1}{3}\right)^2 \right) = \frac{11}{3}$$

Question 2:

The SU(3) gauge group generators satisfy an algebra under commutation

$$[T^a, T^b] = i f_{abc} T^c$$

Compute the numbers f_{abc}.

Answer 2:

You need to be pretty dedicated to get all of them! For example, though if we define

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note the normalizations are a free choice - here I've picked them so that $Tr T^a T^b = 1/2$. Then

$$
\begin{aligned}
[T^1, T^2] = &\ \tfrac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \tfrac{1}{4}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
= &\ \tfrac{1}{4}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} - \tfrac{1}{4}\begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
= &\ \tfrac{i}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = iT^3
\end{aligned}
$$

From this we deduce that

$$f_{123} = 1, \qquad \text{and} \quad f_{213} = -1$$

The full set of non-zero f_{abc}s are:

$$f_{123} = 1, \quad f_{147} = 1/2, \quad f_{156} = -1/2, \quad f_{246} = 1/2, \quad f_{257} = 1/2, \quad f_{345} = 1/2$$

$$f_{367} = -1/2, \quad f_{458} = \sqrt{3}/2, \quad f_{678} = \sqrt{3}/2$$

Note there are other non-zero guys related to these - in particular f_{abc} is totally anti-symmetric so if you switch two labels you get a minus sign.

Question 3:

The gauge group SU(2) has 3 generators

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Show that in and SU(2) version of QCD there is a singlet in the product 2×2. Show that SU(2) gluon exchange is attractive for the singlet state.

Answer 3:

Let's call the two colours white and black. In $2 \otimes 2$ there are four states which one should divide as

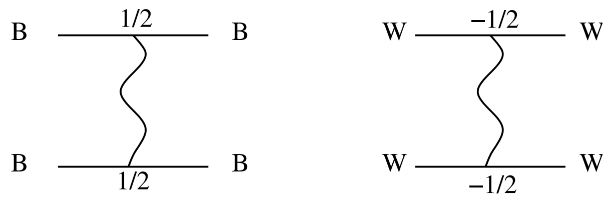
$$BW,\ WB,\ \frac{1}{\sqrt{2}}(BB + WW), \qquad \frac{1}{\sqrt{2}}(BB - WW)$$

The symmetric combination and anti-symmetric combination can't mix under an SU(2) transformation. Thus

$$2 \otimes 2 = 3 \oplus 1$$

There is a singlet.

In fact I believe that at the level of one gluon exchange there is no net force in the singlet channel. I get two diagrams using the diagonal generator:



mapping $1/\sqrt{2}(BB - WW)$ to itself gives factors of

$$\left(\frac{1}{\sqrt{2}}\right)^2 2 \times \left(\frac{1}{4} - \frac{1}{4}\right) = 0$$

Question 4:

Calculate Λ_{QCD} using the β function expression and the measured value of $\alpha_s(M_Z)$ in the notes and explicitly varying the number of quark flavours as you pass through each mass threshold.

Answer 4:

Eqn 21 in the notes missed a factor and should have read

$$\alpha_s(Q^2) = \frac{\alpha_s(M^2)}{1 + \frac{\beta_0}{4\pi}\alpha_s(M^2)\ln(Q^2/M^2)}$$

We begin with the measured value $\alpha_s(M_Z^2 = 91GeV) = 0.118$ (I think I said 0.2 in the lectures and notes but that's actually a very old measurement). Between M_Z and the bottom mass (5 GeV) there are 5 quarks and

$$\beta_0 = 11 - \frac{2}{3}N_f = \frac{23}{3}$$

So we have that

$$\alpha_s(m_b^2) = \frac{0.118}{1 + \frac{23\times 0.118}{12\pi}\ln(5^2/91^2)} = 0.20$$

Now we move down to the charm quark mass (1.5 Gev, $N_f = 4$, $\beta_0 = 25/3$)

$$\alpha_s(m_b^2) = \frac{0.20}{1 + \frac{25\times 0.20}{12\pi}\ln(1.5^2/5^2)} = 0.29$$

Next we move down to the strange quark mass (150 MeV, $N_f = 3$, $\beta_0 = 27/3$)

$$\alpha_s(m_b^2) = \frac{0.29}{1 + \frac{27\times 0.29}{12\pi}\ln(0.15^2/1.5^2)} = 7.25$$

So we see that by the time we have reached the strange quark mass energy scale the theory has become non-perturbative. At this point it's up to you to decide what you mean by Λ_{QCD}... the pole in the one-loop result is at

$$\frac{1}{\alpha_s(\Lambda)} = \frac{1 + \frac{27\times 0.29}{12\pi}\ln(\Lambda^2/1.5^2)}{0.29} = 0$$

ie the top line vanishes when

$$\Lambda = 1.5 \;\; \exp\left(\frac{-6\pi}{27\times 0.29}\right) = 135 MeV$$