Difference Equations

- Definition and Motivation
- Fibonacci numbers
- Linear maps
  - Examples
  - Some theory how to solve them ...
  - Classification
- Non-linear maps
  - Cobwebs
  - Equilibrium + stability analysis
  - The logistic map and some cool stuff

Definition

- A difference equation is an equation that defines a sequence recursively: each term of the sequence is defined as a function of previous terms of the sequence

$$X_t = f(X_{t-1}, X_{t-2}, \ldots, X_0)$$

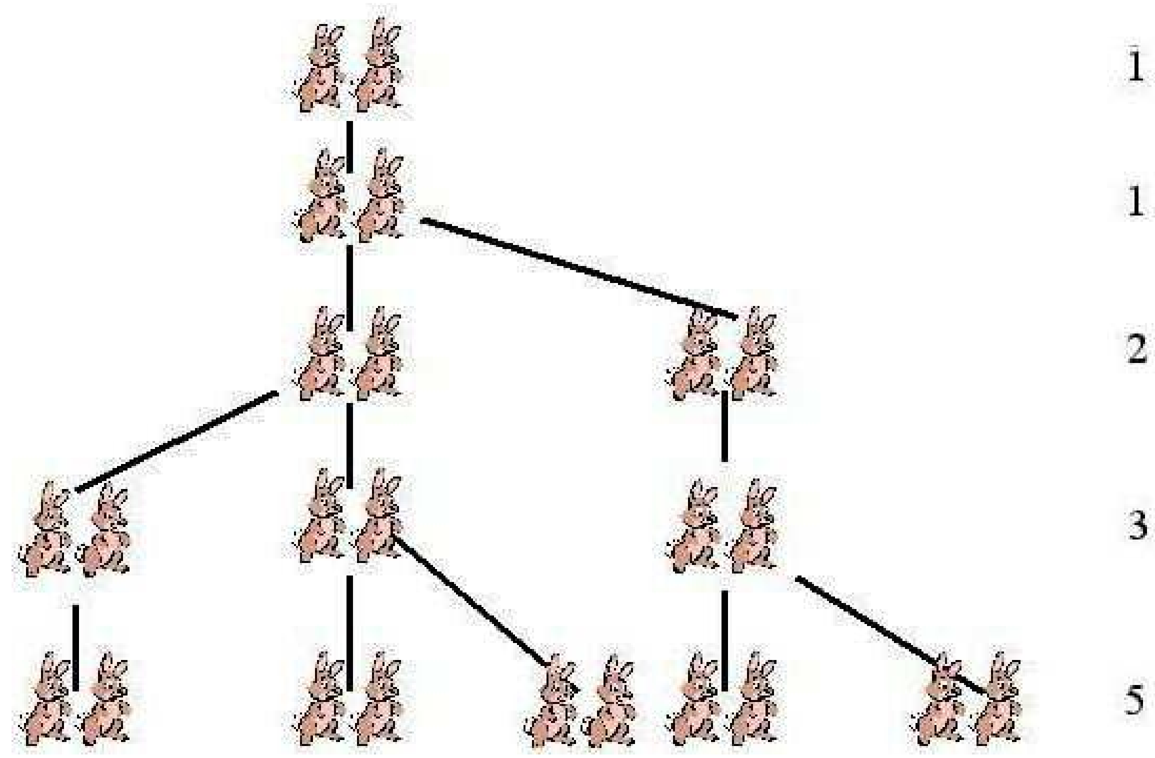
  - Some people also call this an “iterated map” or a “recursion equation”

Why Bother?

- In a way these are the simplest form of an equation that models an evolution in time through a "microscopic" principle that just states what happens at every instant of time
  - e.g. for $X_t = f(X_{t-1})$ the state of the system at time t is given as a function of the state of the system at time t-1. What we want to know is X(t) for any t.
- We will see in the next lecture how this leads on to differential equations ...
- Other reasons:
  - Recursion is very common in computer science, so often if we want to estimate time complexity we find them
  - Often found in analysis of numerical methods

Example: Fibonacci Numbers

- Model: a rabbit population. Rabbits never die. Every pair mates and then produces a new pair.



$$X_t = X_{t-1} + X_{t-2} \longrightarrow 1,1,2,3,5,8,13,21,35,\ldots$$

- More realistic: Logistic map $X_{t+1} = rX_t(1 - X_t)$

Example: Divide and Conquer

- Many algorithms break down a problem into smaller problems -> if we analyse running time we encounter recursion relations
- E.g.: searching an ordered list of n numbers
  - Naively: search from left to right ... worst case T=n
  - Binary search:
    - Always check element in the middle of the interval, then go left or right (discarding other half of interval)
    - Number of comparisons given by

$$c_1 = 1$$
$$c_n = 1 + c_{n/2} \longrightarrow c_n \propto \log_2(n)$$

Classification

- A difference equation is called linear if each term in the sequence is defined as a linear function of the preceding terms
  - $X_t = X_{t-1} + X_{t-2}$ is linear
  - $X_{t+1} = rX_t(1 - X_t)$ is non-linear
- Order of the equation = number of preceding sequence members needed in definition
  - $X_t = X_{t-1} + X_{t-2}$ is second order
  - $X_{t+1} = rX_t(1 - X_t)$ Is first order

Classification (2)

- A linear difference equation of order p has the form

$$X_t = a_{t-1} X_{t-1} + a_{t-2} X_{t-2} + \ldots + a_{t-p} X_{t-p} + a_0$$

- The equation is said to have constant coefficients if the a_i are independent of t
- The equation is homogeneous if $a_0 = 0$
- For a p-th order equation, we need p values for initial conditions, i.e. for $X_t = X_{t-1} + X_{t-2}$ two values X_0 and X_1 need to be given
- Solving the equation means finding X_t for general t and given initial conditions, e.g. for t=365

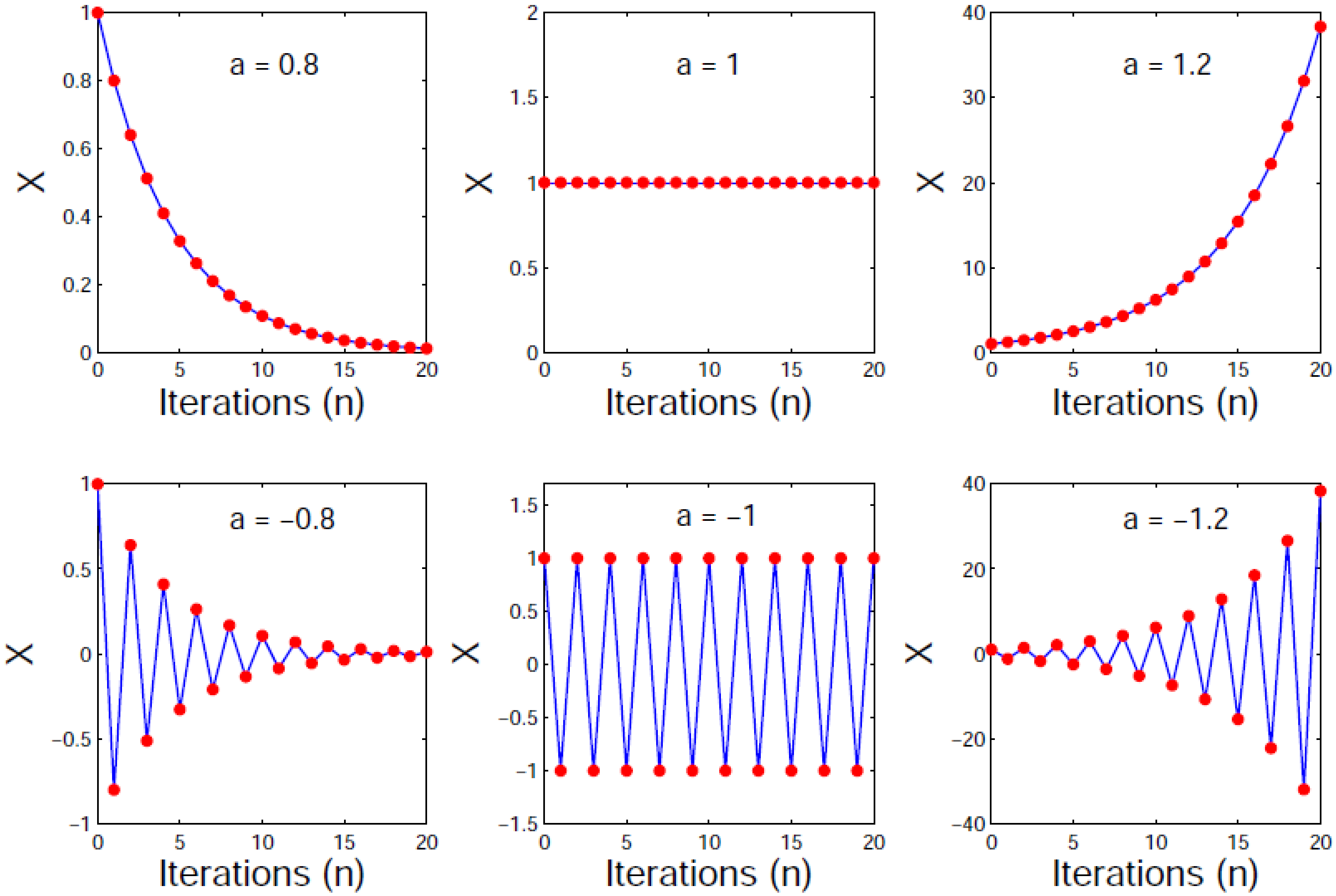
Solving Linear Homogeneous Difference Equations

- Linear difference equations with constant coefficients -> there are methods to solve them
- E.g. let us consider

$$
\begin{aligned}
X_{t+1} &= a X_t \\
&= a\left(aX_{t-1}\right) = a\left(a\left(aX_{t-2}\right)\right) = a\left(a\left(a\ldots\left(aX_0\right)\ldots\right)\right) \\
&= a^{t+1} X_0
\end{aligned}
$$

- Alternatively: could have “guessed” an ansatz $X_t = A\lambda^t$
  - Inserting into $X_{t+1} = a X_t \longrightarrow A\lambda^{t+1} = A a \lambda^t \rightarrow \lambda = a$
  - And A from $X_0 = a^0 A \rightarrow A = X_0 \longrightarrow X_t = a^t X_0$

Behaviour of the Solution



What about Inhomogeneities?

- E.g.: $X_{t+1} = a X_t + b$
  - Trick: transform variables $Y_t = X_t + c \rightarrow X_t = Y_t - c$
    (c to be determined suitably)

$$\rightarrow \quad Y_{t+1} - c = a\left(Y_t - c\right) + b$$

$$Y_{t+1} = aY_t - \underbrace{ac + b + c}_{=0} \quad \rightarrow \quad c = \frac{b+1}{a}$$

  - We already know the solution for Y, i.e. $Y_t = a^t Y_0$
  - Re-substitution:

$$X_t + c = a^t \left(X_0 + c\right)$$

$$X_t = a^t X_0 + \left(a^t - 1\right)\left(\frac{b+1}{a}\right)$$

  - This trick works for all linear diff. eq. with const. coeff.

What about higher Order Equations?

- For example consider

$$X_t = X_{t-1} + X_{t-2}, X_0 = 0, X_1 = 1 \quad (*)$$

- Try ansatz $X_t = A\lambda^t$
- Inserting into $X_t = X_{t-1} + X_{t-2}$

$$\rightarrow \quad A\lambda^t = A\lambda^{t-1} + A\lambda^{t-2}$$

$$\rightarrow \quad 0 = \lambda^2 - \lambda - 1$$ ← "characteristic equation"

- Only λ which fulfill the characteristic equation are suitable for our ansatz
- Solutions: $\lambda_{1/2} = \frac{1 \pm \sqrt{5}}{2}$

Higher Order (cont.)

- Hence:

$$X_t = c_1 \lambda_1^t \quad \text{and} \quad X_t = c_2 \lambda_2^t$$

are solutions of (*)

- Since our equations are linear, any linear combination of solutions is a solution, so solutions can have the form:

$$X_t = c_1 \lambda_1^t + c_2 \lambda_2^t$$

with coefficients to be determined from the initial conditions.

Higher Order (cont.)

- Let's calculate the c's. From initial conditions we have:

$$0 = X_0 = c_1 + c_2 \longrightarrow c_1 = -c_2$$
$$1 = X_1 = c_1\lambda_1 + c_2\lambda_2 = 0 \longrightarrow 1 = c_1(\lambda_1 - \lambda_2)$$

- i.e. $c_1 = 1/\sqrt{5}, c_2 = -1/\sqrt{5}$
- And: $0 = X_0 = c_1 + c_2$

$$\longrightarrow X_t = \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\lambda_1 > 1}^t - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{-1 < \lambda_2 < 0}^t \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^t$$

for large t.

Fibonacci (cont.)

- Why is this useful? Without explicitly iterating we can calculate that we have 7.692E64 rabbits after 1 year (t=365)
- As an aside ...
  - There also is a connection to the *Golden ratio*, i.e. the ratio between sequence members converges to it

$$X_t \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^t$$

$$\longrightarrow \quad X_{t+1} / X_t \rightarrow \left(\frac{1+\sqrt{5}}{2} \right)$$

Another Example

- Consider:

$$X_{t+2} - 2X_{t+1} + 2X_t = 0$$

- Characteristic equation:

$$\lambda^2 - 2\lambda + 2 = 0$$

- Roots:

$$\lambda_{1/2} = 1 \pm \sqrt{1-2} = 1 \pm i$$

→ Roots are complex!

- Solution ... as before

$$X_t = c_1 \lambda_1^t + c_2 \lambda_2^t$$

$$X_t = c_1 (1+i)^t + c_2 (1-i)^t$$

Example (cont.)

- Use of the complex domain can just be a help for calculations, real initial conditions -> real solution
- Suppose: $X_0 = 0, X_1 = 1$

$$\longrightarrow \quad \begin{array}{l} X_0 = 0: \quad c_1 + c_2 = 0 \\ X_1 = 1: \quad c_1(1+i) + c_2(1-i) = 1 \end{array}$$

$$\longrightarrow \quad \begin{array}{l} c_1 = -i/2, c_2 = i/2 \\ X_t = -i/2(1+i)^t + i/2(1-i)^t \end{array}$$

- To see why this is real, remember $a + bi = r\exp(i\phi)$

$$\longrightarrow \quad X_t = 1/2\, e^{-\pi/2 i}\left(\sqrt{2}\, e^{i\pi/4}\right)^t + 1/2\, e^{i\pi/4}\left(\sqrt{2}\, e^{-i\pi/4}\right)^t$$

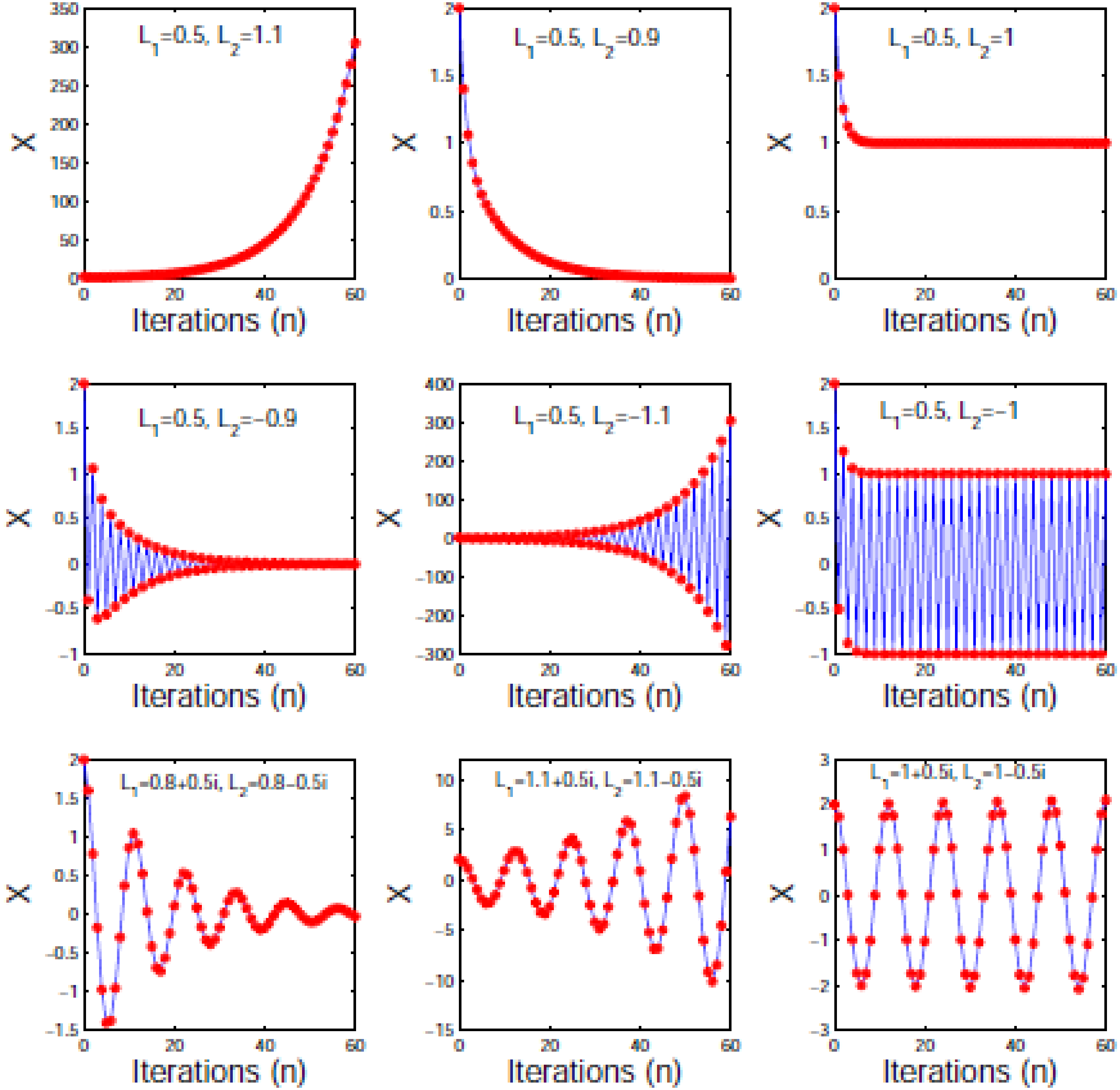
Example (cont.)

- ... and after a bit of algebra:

$$X_t = \sqrt{2^t} \sin(\pi/4\, t)$$

  ... which is oscillating and exponentially growing and real.

- Generally, the systems behaviour can be classified by the roots of the characteristic equation, roughly:
  - Complex -> sin/cos oscillations
  - |lambda| < 1 -> convergence to a fixed point
  - |lambda| > 1 -> exponential divergence


L1=0.5, L2=1.1
L1=0.5, L2=0.9
L1=0.5, L2=1
L1=0.5, L2=−0.9
L1=0.5, L2=−1.1
L1=0.5, L2=−1
L1=0.8+0.5i, L2=0.8−0.5i
L1=−1.1+0.5i, L2=−1.1−0.5i
L1=1+0.5i, L2=1−0.5i
X
Iterations (n)

Systems?

- What about a system like:

$$X_{t+1} = a_{11} X_t + a_{12} Y_t \qquad (1)$$
$$Y_{t+1} = a_{21} X_t + a_{22} Y_t \qquad (2)$$

... can be written as a single second order equation and then solved as before. To see this, e.g., add up a_{22}*(1) and $-a_{12}$*(2) and solve for Y_{t+1}:

$$Y_{t+1} = -1/a_{12}\left(\left(-a_{11} a_{22} + a_{12} a_{21}\right) X_t + a_{22} X_{t+1}\right)$$

- This gives Y_t which can be inserted into (1)

$$X_{t+1} - \left(a_{11} + a_{22}\right) X_t + \left(a_{11} a_{22} - a_{12} a_{21}\right) X_{t-1} = 0$$

A Note on Multiple Roots

- What about if roots of the characteristic equation have multiplicity != 1?
  - e.g.: $(\lambda - 2)^3 = 0$

    has the root lambda=2 with multiplicity 3.
- In this case we multiply lambda^t with increasing powers of t up to multiplicity -1
  - e.g. for the above example:

$$X_t = c_1 2^t + c_2 t 2^t + c_3 t^2 2^t$$

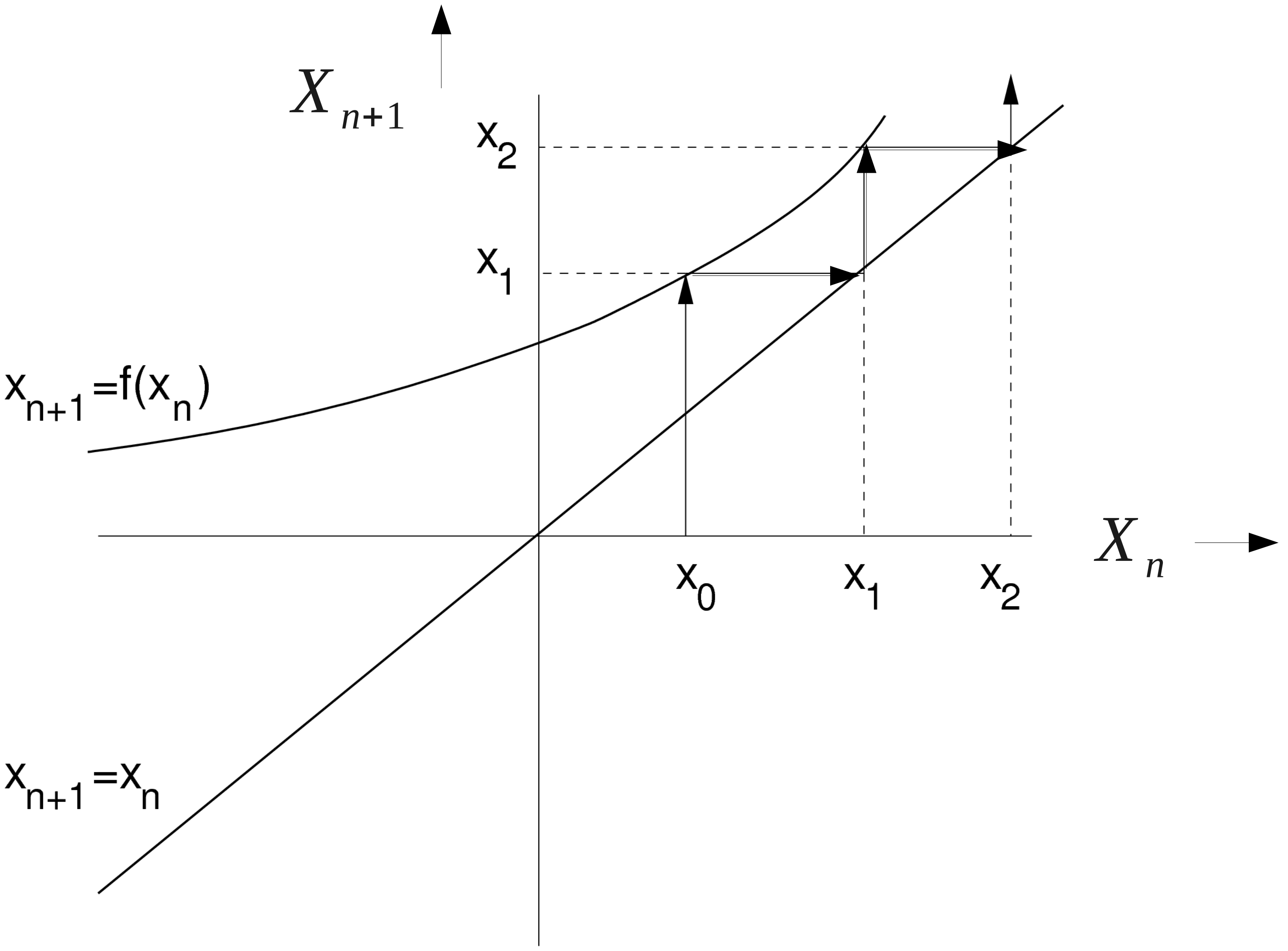
Recap: Linear Maps

- To solve a (system of) linear maps of any order, we just:
  - Determine the roots of the characteristic equation; they already determine the systems long-term dynamics if any |lambda|>1 the system “explodes” to infinity

$$X_t = c_1 \lambda_1^t + c_2 \lambda_2^t + \ldots + c_n \lambda_n^t$$

  - The coefficients c required for exact solutions can be determined from initial conditions
- In detail this might be a lot of algebra, but in principle nothing too complicated.

Cobwebs



A graphical way to illustrate the dynamics of 1d maps

Non-Linear Difference Equations

- Getting analytical results becomes much more difficult, if not impossible ...
- We can often understand something about **equilibrium points**, i.e. stationary points at which the system does not change any more and $X_t^{stat} = X_{t-1}^{stat}$
- E.g., for the logistic map

$$X_{t+1} = rX_t(1 - X_t)$$

one has: $X^{stat} = rX^{stat}(1 - X^{stat})$

i.e.: $X^{stat} = 0 \quad \text{or} \quad X^{stat} = 1$

Stability Analysis

- What is often important when analysing the convergence of numerical algorithms is what happens close to an equilibrium point
  - Say ... numerically we have not quite got it right. Will small differences blow up/die out over time?
- This is what we do in stability analysis:
  - We perturb the system a tiny bit
  - We try to figure out the fate of these perturbations
- Mathematically:
  - This often means linearizing around the equilibrium point and then using the theory of linear maps from previous slides

Fixed Points

- Fixed point: $x^{\text{stat}} = f(x^{\text{stat}})$
- Stability?
  - Consider nearby orbit $x_n = x^* + \eta_n$ Is it attracted or repelled from x*?

$$x_{n+1} = x^{\text{stat}} + \eta_{n+1} = f(x^{\text{stat}} + \eta_n) = f(x^{\text{stat}}) + f'(x^{\text{stat}})\eta_n + O(\eta_n^2)$$

$$\longrightarrow \quad \eta_{n+1} = f'(x^{\text{stat}})\eta_n + O(\eta_n^2)$$

  - Neglect $O(\eta^2)$ terms -> linearized map with eigenvalue/multiplier $\lambda = f'(x^{\text{stat}})$

$$\longrightarrow \quad \eta_n = \lambda^n \eta_0$$

  - $|f'(x^{\text{stat}})|<1$ -> **linearly stable**, =1 **marginal**, >1 **unstable**
  - $f'(x^{\text{stat}})=0$ -> **superstable** $\eta_n \propto \eta_0^{(2^n)}$

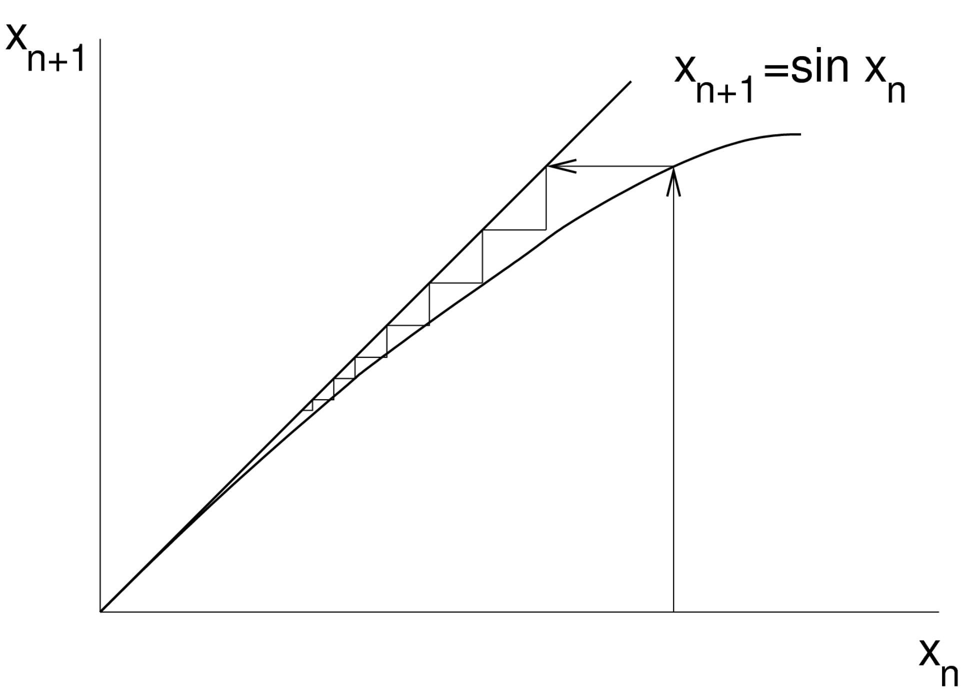
Examples

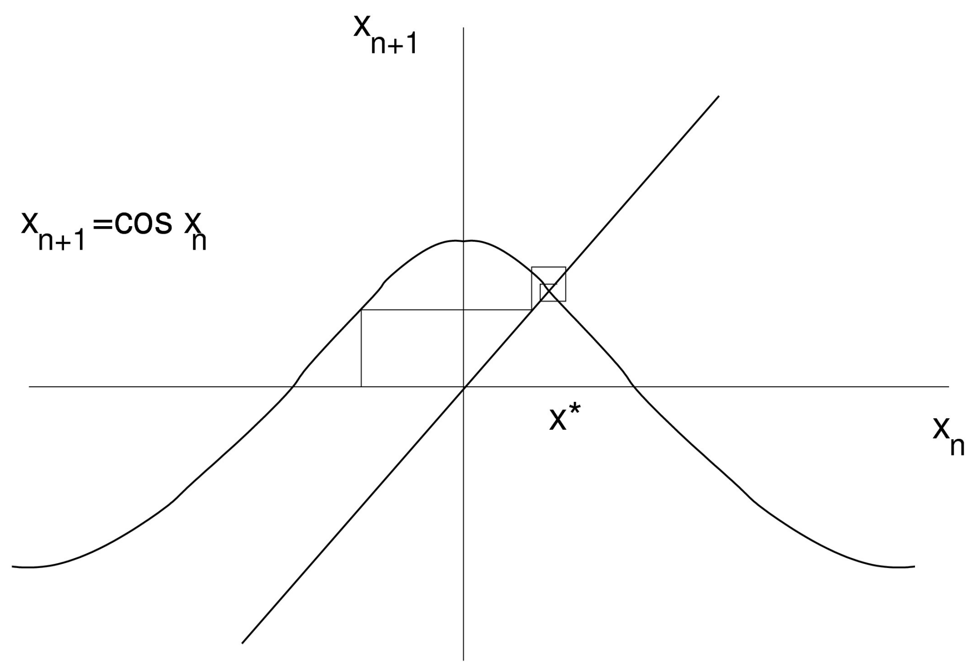
- Let's have a look at

$$x_{n+1} = \sin x_n \qquad\qquad x_{n+1} = \cos x_n$$

$$x^{\text{stat}} = 0 \qquad\qquad x^{\text{stat}} = 0.739\ldots$$

$$\lambda = f'(0) = \cos(0) = 1 \qquad\qquad \lambda = -\sin(0.739\ldots), 0 > \lambda > -1$$





However ...

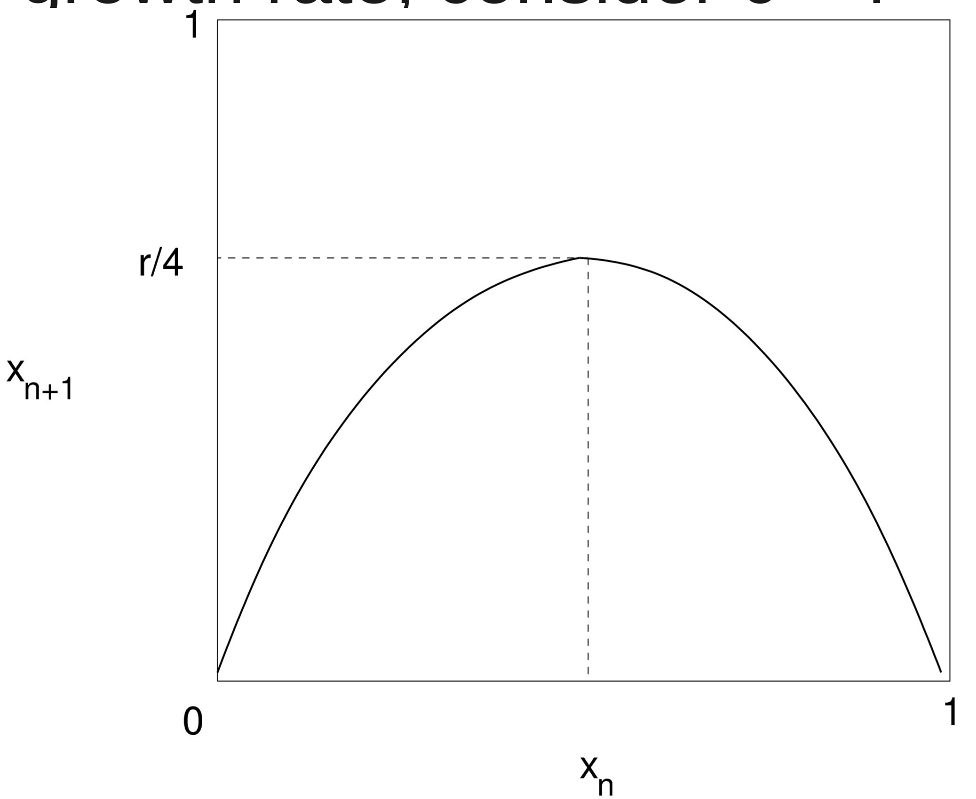
- In general, for non-linear iterated maps, even in 1d, far more exciting behaviour than seen for linear systems so far is possible ...
- To get some idea of this, let's have a look at the logistic map
- Easy to build a computer simulation to implement the following yourself

Logistic Map

- Remember the logistic map

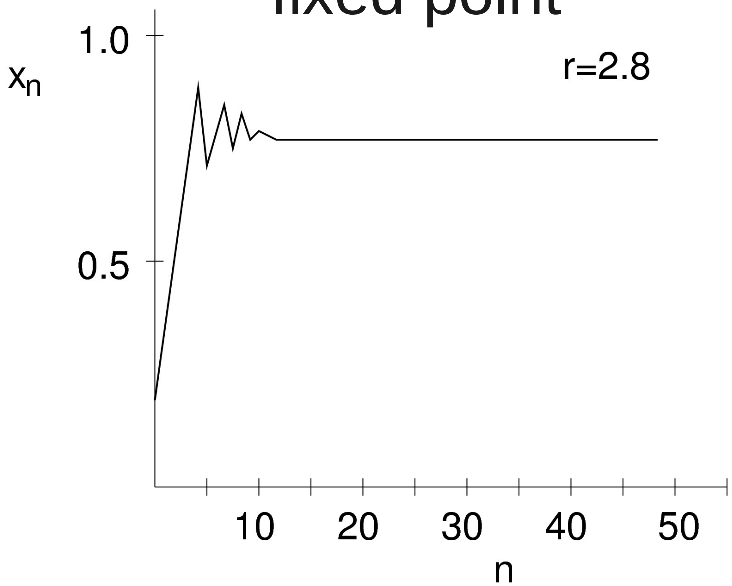
$$x_{n+1}=r\,x_n\left(1-x_n\right)$$

  - x_n ... population in nth generation
  - r ... growth rate, consider 0<=r<=4



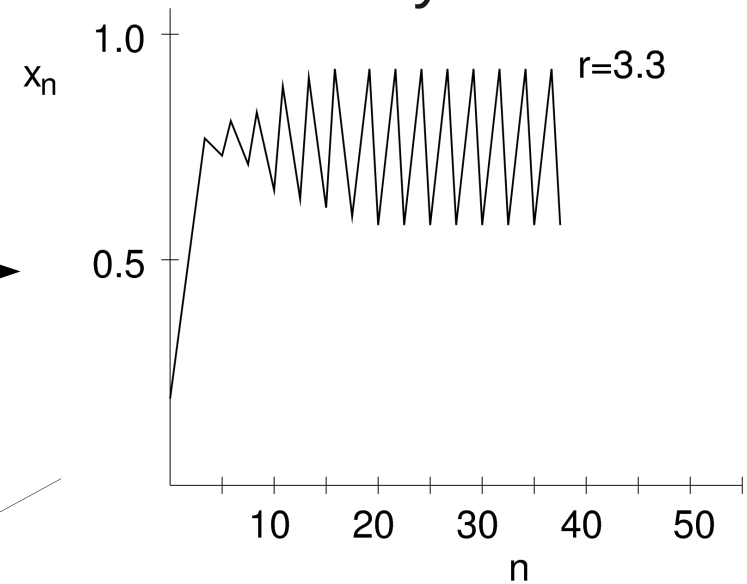
Let's just simulate for increasing r ...

fixed point

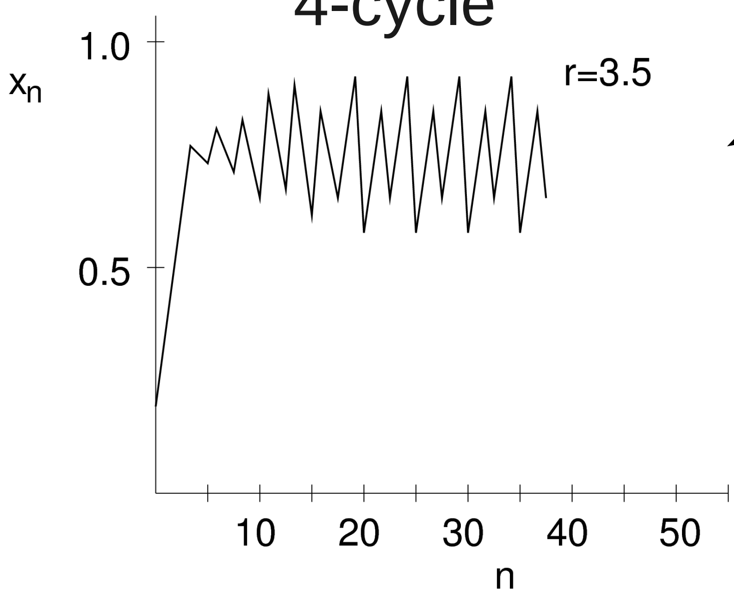


Period
Doubling

2-cycle



4-cycle



Period
Doubling

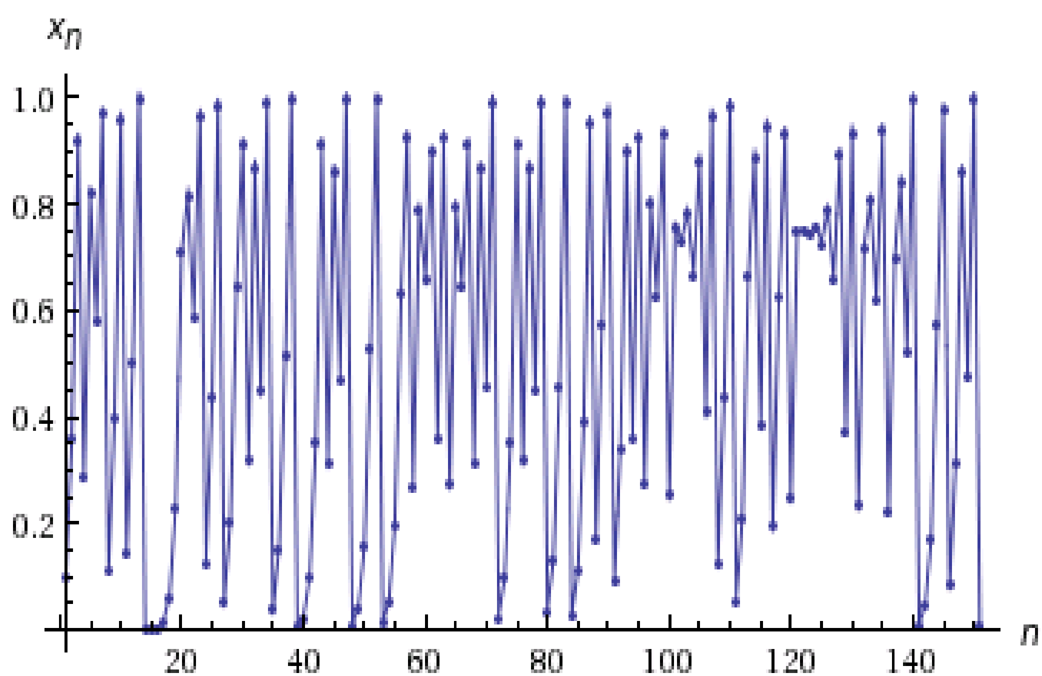
Further Period
Doublings to 8, 16, 32, ... cycles

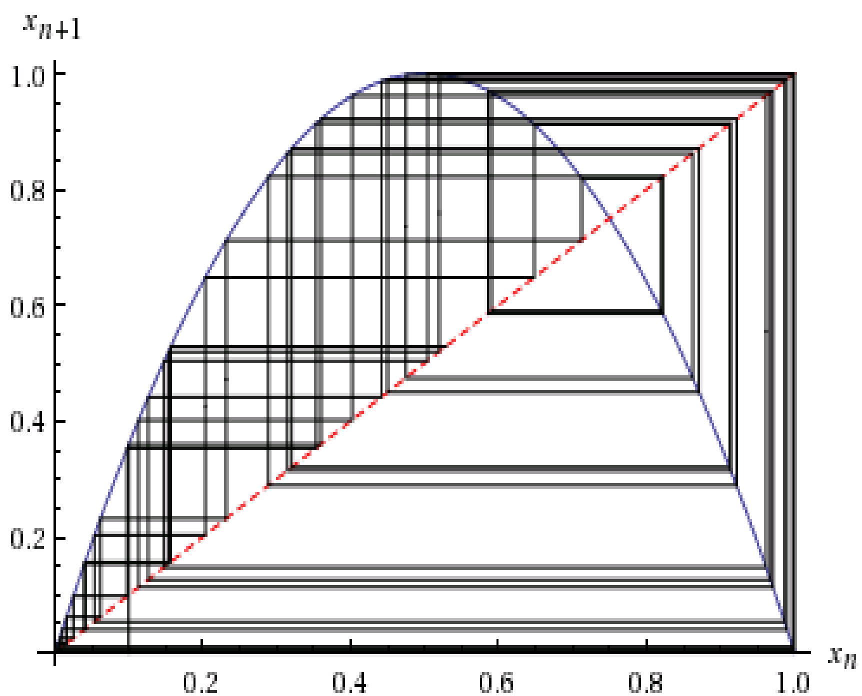
Period Doubling

- r_n ... value of r where 2^n-cycle is born
  - $r_1=3$ 2-cycle
  - $r_2=3.449...$ 4-cycle
  - $r_3=3.54409...$ 8-cycle
  - $r_4=3.5644...$ 16-cycle
  - $r_{inf}=3.569946...$ infinite cycle
- Distances between successive bifurcations become smaller and smaller ... geometric convergence
- What about $r>r_{inf}$?

Chaos ...

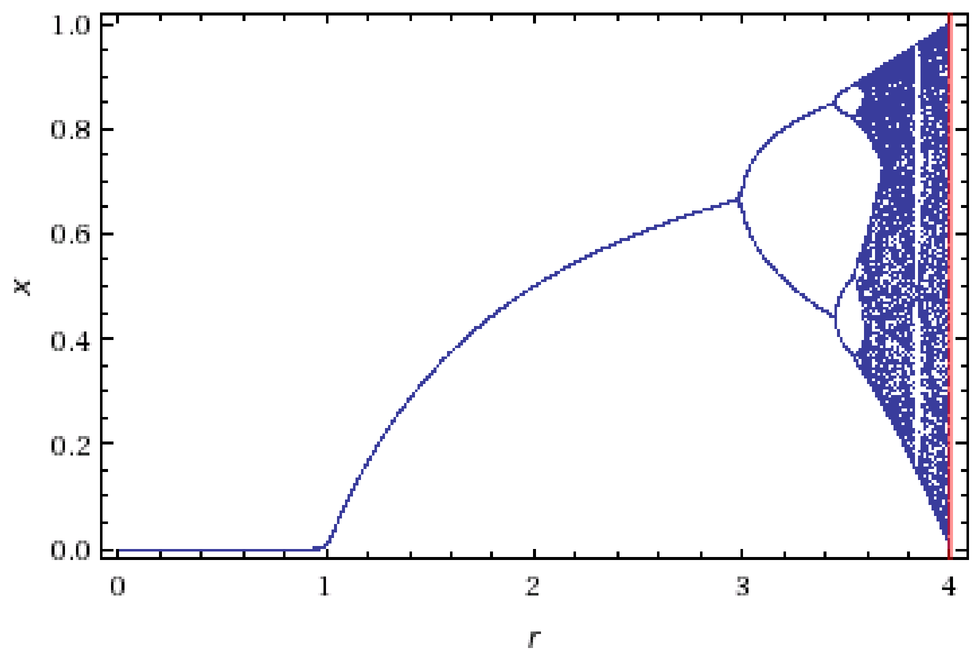
- For example r=3.9 – aperiodic irregular dynamics similar to what we have seen for continuous systems
- However ... not all $r > r_{inf}$ have chaotic behaviour!





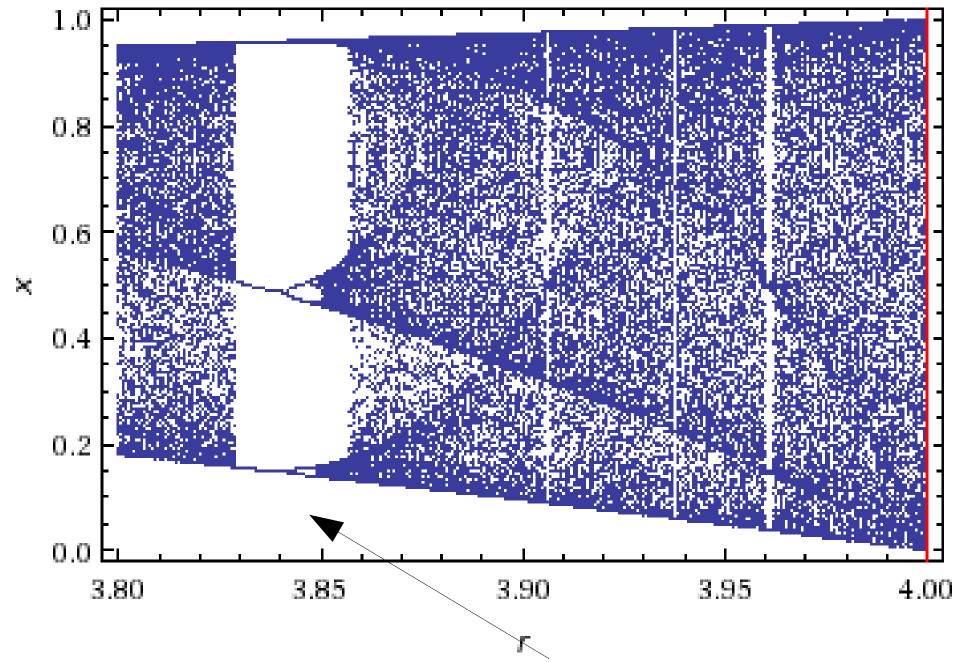
(lines successively connect the first 50 iterates and the dashed line $y = x$)

Bifurcation Diagram



(iterates 100 through 150 for each *r*)

Zoomed in:



(iterates 300 through 450 for each *r*)

stable 3-cycle

- For $r>r_{inf}$ diagram shows mixture of order and chaos, periodic windows separate chaotic regions
- Blow-up of parts appear similar to larger diagram ...
- This is still an exciting problem of study for complexity theory

Summary

- What is important to remember:
  - What is a map?
  - What is a linear map? How can we solve them?
  - Cobwebs
  - How to do equilibrium analysis for non-linear maps