3 Energy-Momentum Tensor

The energy-momentum tensor, \( T_{\mu\nu} \) is defined by

\[
T_{\mu\nu} = \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\nu \phi - g_{\mu\nu} L.
\]

We see immediately, using the definition of the canonical momentum, \( \pi(x) \), that \( T_{00} \) is the Hamiltonian density.

3.1 The Momentum Operator

The momentum operator for a system described by a Lagrangian density \( L \) is given by the \( \mu = 0 \) components of this tensor, integrated over space (and normal ordered so that the momentum of the vacuum is zero)

\[
P_\nu = \int d^3x : T_{0\nu} :.
\]

Now in terms of the expansion in creation and annihilation operators, we have

\[
\frac{\partial L}{\partial (\partial^\mu \phi)} = \partial_\mu \phi = -i \int \frac{d^3p}{(2\pi)^3 2E_p} p^\mu \left( a(p)e^{-i p \cdot x} - a^\dagger(p)e^{i p \cdot x} \right),
\]

so that the operator \( P_i, \ (i = 1 \cdots 3) \) is

\[
P_i = \int \frac{d^3p}{(2\pi)^3 2E_p} \frac{d^3p'}{(2\pi)^3 2E'_p} d^3x \left( E_p p'_i + E_p' p_i \right) a^\dagger(p') a(p) e^{i(p-p') \cdot x}
\]

\[
= \int \frac{d^3p}{(2\pi)^3 2E_p} \frac{d^3p'}{2E'_p} \delta^3(p-p') \left( E_p p'_i + E_p' p_i \right) a^\dagger(p') a(p) e^{i(E_p-E_p') \cdot t}
\]

\[
= \int \frac{d^3p}{(2\pi)^3 2E_p} p_i a^\dagger(p) a(p).
\]

This is the number of particles with momentum \( p \), multiplied by \( p_i \) and integrated over all possible momenta (using the Lorenz invariant integration measure) \(^\dagger\)

For space-like components, the momentum operator may be written as

\[
P_i = \int d^3x \pi(x) \partial_i \phi(x)
\]

which can be seen to obey the commutation relation

\[
[P_\mu, \phi(x)] = -i \partial_\mu \phi(x).
\]

\(^\dagger\)We have dropped terms quadratic in the creation or annihilation operator, which can be shown to vanish.
The momentum operator generates infinitesimal translations, and for finite transformations with parameter $a^\mu$ we have
\[ e^{iP_\mu a^\mu} \phi(x) e^{-iP_\mu a^\mu} = \phi(x + a). \]

Using the Euler-Lagrange equations it can be shown that the energy-momentum tensor is “conserved”, i.e. its divergence cancels:
\[
\begin{align*}
\partial^\mu T_{\mu \nu} &= \partial^\mu \frac{\partial L}{\partial(\partial^\nu \phi)} \partial_\nu \phi + \frac{\partial L}{\partial(\partial^\mu \phi)} \partial_\mu \phi - \partial_\nu \frac{\partial L}{\partial(\partial^\mu \phi)} \partial_\mu \phi \\
&= \frac{\partial L}{\partial \phi} \partial_\nu \phi + \frac{\partial L}{\partial(\partial^\mu \phi)} \partial_\mu \phi - \frac{\partial L}{\partial(\partial^\nu \phi)} \partial_\nu \phi - \frac{\partial L}{\partial(\partial^\mu \phi)} \partial_\mu \phi \\
&= 0,
\end{align*}
\]

where the Euler-Lagrange equations have been used in the first term in the last step. In components we may write the zero component of this conservation law as
\[
\frac{d}{dt} T_{0 \nu} - \frac{\partial}{\partial x_i} T_{i \nu} = 0
\]

Integrating over all space, the second term vanishes as it is the integral of a derivative (assumed to vanish at spatial infinity) and we are left with
\[
\frac{d}{dt} \int d^3 x T_{0 \nu} = \frac{d}{dt} P_\nu = 0,
\]
i.e, the total energy and momentum of the system are conserved (as expected).

### 3.2 The Angular Momentum Operator

In 3 dimensions
\[ \mathbf{L} = \mathbf{r} \times \mathbf{p} \]
where the component $L_i$ is given in terms of components of the energy-momentum tensor by
\[ L_i = \varepsilon_{ijk} \int d^3 x : x^j T^{0k} \]

We can generalise this to four dimensions (thereby including the generators of Lorentz boosts as well as rotations) by defining a 3-rank tensor
\[ \mathcal{M}^{\mu \nu \rho} = (x^{\mu} T^{\nu \rho} - x^{\nu} T^{\mu \rho}) \]
and hence an antisymmetric 2-rank tensor
\[ M^{\mu \nu} = \int d^3 x \mathcal{M}^{\mu \nu \rho}. \]
We can see that $M^{ij} = \varepsilon^{ijk}L_k$, $(i, j = 1 \cdots 3)$ are the usual angular momentum operators, whereas $M^0i$ generate Lorentz boosts.

By expanding the fields in terms of creation and annihilation operators (as we did for the momentum operators) and performing some algebra we can show that

$$M^{0i}|p\rangle = -pt|p\rangle - E_p \frac{\partial}{\partial p_i}|p\rangle$$

Now consider the infinitesimal operator

$$(1 - i\delta v_i M^0i)$$

acting on a one-particle momentum state $|p\rangle$, where $\delta v_i$ is an infinitesimal boost velocity in the direction $i$. $\delta v_i E_p = \delta p_i$, the change in momentum in the $i$-direction and $p_i \delta v_i = \delta E$, the change in energy. Thus we have

$$(1 - i\delta v_i M^0i) |p\rangle = e^{-i\delta E t}|p\rangle + \delta p_i \frac{\partial}{\partial p_i}|p\rangle = e^{-i\delta E t}|p + \delta p\rangle,$$

showing that the above operator is an infinitesimal boost in the direction $i$.

Likewise it may be shown that the operator

$$\left(1 - ie^{ijk}\delta \theta_i M_{jk}\right)$$

acting on $|p\rangle$ gives a state in which the momentum is rotated by a small angle $\delta \theta$ about the $i$ axis.

As before, the divergencelessness of the energy-momentum tensor leads to the conservation law

$$\partial_{\rho} M^{\mu \nu \rho},$$

which in turn (after integrating over all space) gives the conservation of angular momentum

$$\frac{d}{dt} M^{\mu \nu} = 0.$$