# 13 The Dirac Equation

A two-component spinor

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

transforms under rotations as

$$\chi \rightarrow e^{-i\theta \mathbf{n}\cdot\mathbf{J}}\chi,$$

with the angular momentum operators, $J_i$ given by:

$$J_i = \frac{1}{2}\sigma_i,$$

where $\sigma$ are the Pauli matrices, $\mathbf{n}$ is the unit vector along the axis of rotation and $\theta$ is the angle of rotation.

For a relativistic description we must also describe Lorentz boosts generated by the operators $K_i$. Together $J_i$ and $K_i$ form the algebra (set of commutation relations)

$$\begin{aligned}
\left[K_i, K_j\right] &= -i\varepsilon_{ijk}J_k \\
\left[J_i, K_j\right] &= i\varepsilon_{ijk}K_k \\
\left[J_i, J_j\right] &= i\varepsilon_{ijk}J_k
\end{aligned}$$

For a spin-$\frac{1}{2}$ particle $K_i$ are represented as

$$K_i = \pm\frac{i}{2}\sigma_i,$$

giving us two inequivalent representations.

Starting with a spin-$\frac{1}{2}$ particle at rest, described by a spinor $\chi(0)$, we can boost to give two possible spinors

$$\chi_R(\mathbf{p}) = e^{\alpha/2\mathbf{n}\cdot\sigma}\chi(0) = (\cosh(\alpha/2) + \mathbf{n}\cdot\sigma\sinh(\alpha/2))\chi(0)$$

or

$$\chi_L(\mathbf{p}) = e^{-\alpha/2\mathbf{n}\cdot\sigma}\chi(0) = (\cosh(\alpha/2) - \mathbf{n}\cdot\sigma\sinh(\alpha/2))\chi(0)$$

where

$$\sinh(\alpha) = \frac{|\mathbf{p}|}{m}$$

and

$$\cosh(\alpha) = \frac{E_p}{m}$$

so that

$$\begin{aligned}
\chi_R(\mathbf{p}) &= \frac{(E_p + m + \sigma\cdot\mathbf{p})}{\sqrt{2m(E_p+m)}}\chi(0) \\
\chi_L(\mathbf{p}) &= \frac{(E_p + m - \sigma\cdot\mathbf{p})}{\sqrt{2m(E_p+m)}}\chi(0)
\end{aligned}$$

Under the parity operator the three-moment is reversed $\mathbf{p} \leftrightarrow -\mathbf{p}$ so that $\chi_L \leftrightarrow \chi_R$. Therefore if we require a Lorentz description of a spin-$\frac{1}{2}$ particles to be a proper representation of parity, we must include *both* $\chi_L$ and $\chi_R$ in one spinor (note that for massive particles the transformation $\mathbf{p} \leftrightarrow -\mathbf{p}$ can be achieved by a Lorentz boost). This we define a 4-component spinor

$$u_\alpha(p,\lambda) \sim \begin{pmatrix} \chi_R^\lambda \\ \chi_L^\lambda \end{pmatrix}, \quad (\alpha = 1\cdots 4).$$

Here $\lambda = \pm 1$ is the helicity of the particle.

Using

$$(E_p + m - \sigma\cdot\mathbf{p})\,(E_p + m + \sigma\cdot\mathbf{p}) \;=\; E_p^2 + m^2 - 2mE_p - |\mathbf{p}|^2 \;=\; 2m(E_p + m)$$

we can invert the expressions for $\chi_{L(R)}$ to get

$$\chi(0) \;=\; \frac{(E_p + m - \sigma\cdot\mathbf{p})}{\sqrt{2m(E_p + m)}}\chi_R(\mathbf{p})$$

and using

$$(E_p + m - \sigma\cdot\mathbf{p})^2 \;=\; E_p^2 + m^2 + p^2 + 2mE_p - 2(E_p + m)\sigma\cdot\mathbf{p} \;=\; 2\,(E_p + m)\,(E_p - \sigma\cdot\mathbf{p})$$

we also have

$$\chi_L(\mathbf{p}) \;=\; \frac{(E_p - \sigma\cdot\mathbf{p})}{m}\chi_R(\mathbf{p})$$

and

$$\chi_R(\mathbf{p}) \;=\; \frac{(E_p + \sigma\cdot\mathbf{p})}{m}\chi_L(\mathbf{p})$$

## 13.1 Dirac ($\gamma$)- Matrices

Define a set of four $4\times 4$ $\gamma$-matrices

$$\gamma^0 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i \;=\; \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

In terms of these matrices the above equations relating $\chi_L$ to $\chi_R$ may be expressed as

$$\left(\gamma^0 E_p - \gamma\cdot\mathbf{p})\right)_\alpha^\beta u_\beta(\mathbf{p},\lambda) \;=\; m u_\alpha(\mathbf{p},\lambda)$$

or in manifestly Lorentz invariant form

$$\left(\gamma^\mu p_\mu - m\right)_\alpha^\beta u_\beta(\mathbf{p},\lambda) \;=\; 0,$$

(Note that there is an implied $4\times 4$ unit matrix in front of the $m$ inside the parenthesis)

The $\gamma$-matrices obey the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} I,$$

where $I$ indicates the $4 \times 4$ unit matrix (we suppress this henceforth).

This is known as a "Clifford Algebra".

This set of anti-commutation relations implies that

$$\left(\gamma^\mu p_\mu + m\right)\left(\gamma^\nu p_\nu - m\right) = g^{\mu\nu} p_\mu p_\nu - m^2 = p^2 - m^2.$$

So that the equation obeys by the 4-component spinor $u_\alpha(\mathbf{p}, \lambda)$ describes a particle which is "on-shell" i.e. $p^2 = m^2$.

The Dirac equation for the wave-function of a relativistic moving spin-$\frac{1}{2}$ particle is obtained by making the replacing $p_\mu$ by the operator $i\partial_\mu$ giving

$$\left(i\gamma^\mu\partial_\mu - m\right)_\alpha^\beta \Psi_\beta(x) = 0,$$

which has solution

$$\Psi_\alpha(x) = e^{-ip\cdot x} u_\alpha(\mathbf{p}, \lambda)$$

with $p^2 = m^2$.

There is a minor problem in attempting to write the Hermitian conjugate of this equation since the matrix $\gamma^0$ is Hermitian whereas the space-like matrices, $\gamma^i$, are anti-Hermitian.

The Hermitian conjugate of the Dirac equation is

$$\Psi^\dagger(x)\left(-i\gamma^0 \overleftarrow{\partial}_0 - i\gamma^j \overleftarrow{\partial}_j - m\right) = 0,$$

which cannot be expressed in manifestly Lorentz invariant form. However, if we multiply on the right by $\gamma^0$ and then anti-commute $\gamma^0$ through the other $\gamma$-matrices so that

$$\gamma^i\gamma^0 = -\gamma^0\gamma^i,$$

we get

$$\Psi^\dagger(x)\gamma^0\left(-i\gamma^0 \overleftarrow{\partial}_0 + i\gamma^j \overleftarrow{\partial}_j - m\right) = 0.$$

Now we define $\overline{\Psi}$ to be

$$\overline{\Psi} = \Psi^\dagger\gamma^0$$

and we can write the conjugate equation in manifestly Lorentz invariant from as

$$\overline{\Psi}\left(i\gamma\cdot \overleftarrow{\partial} + m\right) = 0.$$

By multiplying on the left by $\gamma^0$ we can write the Dirac equation in a from similar to the Schrödinger equation namely:

$$i\frac{\partial\Psi}{\partial t} = \left(i\alpha^j\partial_j + m\gamma^0\right)\Psi$$

with the Hermitian matrices $\alpha^j$ given by

$$\alpha^j = \gamma^0\gamma^j \quad (j = 1\cdots 3).$$

We can then identify the Hamiltonian for a relativistic spin-$\frac{1}{2}$ as

$$H = i\alpha^j\partial_j + m\gamma^0.$$

The transformation of the four-component spinor $u_\alpha$ under general Lorentz transformations (rotations and boosts) can be treated by defining the antisymmetric tensor matrices

$$(\sigma^{\mu\nu})_\alpha^\beta \equiv \frac{i}{2}[\gamma^\mu,\gamma^\nu]_\alpha^\beta.$$

Using the commutation relations between Pauli matrices

$$\left[\sigma^i,\sigma^j\right] = 2i\varepsilon_{ijk}\sigma^k,$$

we have

$$\begin{aligned}\sigma^{ij} &= \varepsilon_{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma_k\end{pmatrix}\\ \sigma^{0j} &= \begin{pmatrix} i\sigma^j & 0\\ 0 & -i\sigma_j\end{pmatrix}\end{aligned}$$

Under a general Lorentz transformation

$$u \rightarrow e^{\frac{1}{2}i\omega_{\mu\nu}\sigma^{\mu\nu}}u.$$

For $\mu,\nu = i,j$ this is a rotation. If $\mu = 0$ then

$$u = \sqrt{2m}\begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix} \rightarrow \sqrt{2m}\begin{pmatrix} e^{-\frac{1}{2}\omega_{oj}\sigma^j}\chi_L\\ e^{+\frac{1}{2}\omega_{oj}\sigma^j}\chi_R\end{pmatrix},$$

as required for a boost in direction $j$ with velocity $v$, given by $\omega_{oj} = \tanh v$.

## 13.2 Negative Energy States

The solution to the Dirac equation is

$$\Psi_\alpha(x) = e^{-ip\cdot x}u_\alpha(\mathbf{p},\lambda), \quad (\lambda = \pm 1)$$

where the four-component spinor $u_\alpha$ obeys

$$(\gamma\cdot p - m)\,u_\alpha \;=\; 0,$$

gives us two allowed solutions. Since the $\gamma$ matrices are $4\times 4$ matrices, we expect two more solutions. These are

$$\Psi_\alpha(x) \;=\; e^{ip\cdot x} v_\alpha(\mathbf{p},\lambda),$$

where the spinor $v_\alpha$ obeys the equation

$$(\gamma\cdot p + m)\,v_\alpha \;=\; 0.$$

In terms of the left- and right- two-component spinors $\chi_{L(R)}(\mathbf{p},\lambda)$, $v_\alpha$ may be expressed as

$$v_\alpha(\mathbf{p},\lambda) \;=\; \sqrt{2m}\begin{pmatrix} \chi_R(\mathbf{p},\lambda) \\ -\chi_L(\mathbf{p},\lambda) \end{pmatrix}.$$

The spinors $u_\alpha$ and $v_\alpha$ transform into each other by the "charge-conjugation" operator $C$:

$$u \;=\; C\overline{v}^T,$$

where the superscript $T$ indicates the transpose ($u$, $v$ are considered to be column spinors, whereas $\overline{u}\,\overline{v}$ are row spinors).

This means that the equation for $u$ may be written

$$\gamma\cdot pC\overline{v}^T \;=\; mC\overline{v}^T$$

and the equation for $v$ may be written

$$-\overline{v}\gamma\cdot p = m\overline{v}$$

Taking the transpose of this gives

$$-\gamma^T\cdot p\,\overline{v}^T \;=\; m\overline{v}^T$$

and multiplying the equation for $u = C\overline{v}^T$ by $C^{-1}$ we have

$$\left(C^{-1}\gamma C\right)\cdot p\,\overline{v}^T = m\overline{v}^T.$$

These relations have to hold for *any* momentum , $p$, so that we must have

$$C^{-1}\gamma^\mu C \;=\; -\gamma^{\mu T}.$$

This is a general property of the charge conjugation matrix $C$. In the representation for the $\gamma$-matrices considered so far, namely

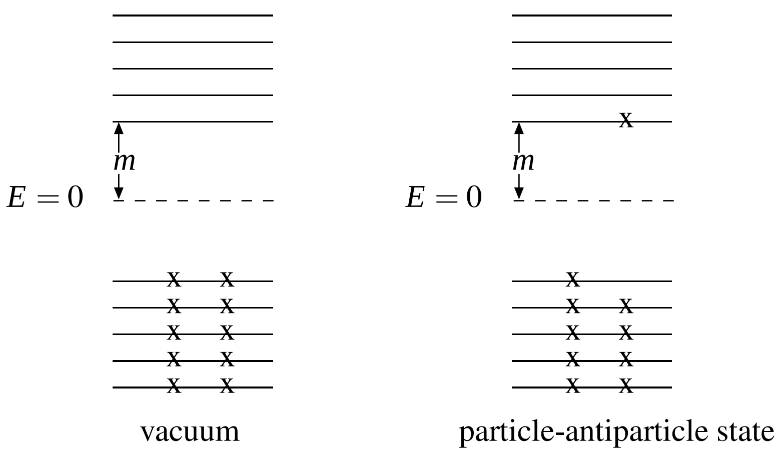
$$\gamma^0 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\;\; \gamma^j \;=\; \begin{pmatrix} 0 & -\sigma^j \\ \sigma^j & 0 \end{pmatrix},$$

we may write $C$ as

$$C = i\gamma^2\gamma^0,$$

but as we shall see there are other representations for the $\gamma$-matrices, for which this may be different.

These solutions with spinor $v$ have negative energy $-E_p$. The existence of negative energy solutions would normally cause difficulty since energies would then be unbounded from below and all particles would "fall" into the $E_p \to -\infty$ states. But for fermions this is *not* the case if all the negative energy states are filled. This was Dirac's picture of the vacuum, called the "sea". All negative states are filled (two particles in each energy-level with opposite helicities), and all of the positive energy states are empty




A one-particle state has one of the positive energy-levels filled and is prevented from "falling" into a negative energy state by the Pauli exclusion principle, since all these states are filled. A "hole" in a negative energy state is interpreted as an antiparticle. Thus if a particle in a negative energy state is promoted to a positive energy state, leaving a hole in a negative energy level, this is equivalent to the creation of a particle-antiparticle pair. Note that the gap between the lowest positive energy state and the highest negative energy state is $2m$, which is the minimum energy required to produce such a particle-antiparticle pair. If the positive energy particle falls back into the hole in the negative energy levels then this represents the annihilation of the particle and antiparticle, releasing energy in excess of $2m$.

## 13.3 Weyl and Dirac Representations

The matrices $\gamma^\mu$ are defined by their anti-commutation relations (the Clifford algebra). So far we have considered only one representation of these matrices and in this representation the $u$ and $v$

spinors are written

$$u(\mathbf{p},\lambda) = \frac{1}{\sqrt{2(E_p+m)}}\begin{pmatrix} (E_p+m+\sigma\cdot\mathbf{p})\chi(0) \\ (E_p+m-\sigma\cdot\mathbf{p})\chi(0)\end{pmatrix}$$

and

$$v(\mathbf{p},\lambda) = \frac{1}{\sqrt{2(E_p+m)}}\begin{pmatrix} (E_p+m+\sigma\cdot\mathbf{p})\chi(0) \\ -(E_p+m-\sigma\cdot\mathbf{p})\chi(0)\end{pmatrix}.$$

This is a useful representation (called the "Weyl representation" or "chiral representation") for the case massless particles for which $m=0$ and $E_p=|\mathbf{p}|\equiv E$. In this case the spinors simplify to

$$u = \sqrt{2E}\begin{pmatrix} \frac{(1+\sigma\cdot\hat{\mathbf{p}})}{2}\chi(0) \\ \frac{(1-\sigma\cdot\hat{\mathbf{p}})}{2}\chi(0)\end{pmatrix} = \sqrt{2E}\begin{pmatrix}\chi_R \\ \chi_L\end{pmatrix},$$

$\hat{\mathbf{p}}$ is the unit vector in the direction of momentum and we have

$$\sigma\cdot\hat{\mathbf{p}}\chi_R = \chi_R$$

$$\sigma\cdot\hat{\mathbf{p}}\chi_L = -\chi_L$$

$\chi_{R(L)}$ are eigenstates of helicity $\sigma\cdot\hat{\mathbf{p}}$ with eigenvalues $\pm 1$. Although the parity operator takes us from positive to negative helicity ($\mathbf{p}\leftrightarrow-\mathbf{p}$ under parity, but the spin operators $\sigma^i$ are unchanged as they are axial vectors), there can be *no* Lorentz transformation from one to the other in the case of massless particles. We cannot move into a frame in which the particle appears to be moving backwards, since the massless particle is moving with the speed of light.

Furthermore $u$ and $v$ both obey the equation

$$\not{p}u = 0,$$

$$\not{p}v = 0,$$

in the massless case.

Note that we have introduced the "slash" notation $\not{p}$ to mean $\gamma\cdot p$.

$\chi_L$ and $\chi_R$ are called "Weyl spinors" or "chiral spinors"

Since we now know that the neutrino has a mass there are *no* known massless spin-$\frac{1}{2}$ particles. Nevertheless for experiments carried out at sufficiently large energies the masses can usually be neglected and this Weyl representation is therefore useful for treating such high-energy processes.

For massive particles (or where the energies is too low for the particle masses to be neglected), it is more convenient to use the "Dirac representation", with $\gamma$-matrices related to the Weyl representation by a unitary transformation

$$\gamma^\mu_{Dirac} = S\gamma^\mu_{weyl}S^{-1},$$

where

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Clifford algebra is not affected by this transformation. In the Dirac representation we have

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix},$$

and the spinors $u$ and $v$ are given by

$$u_{Dirac} = Su_{Weyl} = \sqrt{(E_p+m)}\begin{pmatrix} \chi(0) \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi(0) \end{pmatrix}$$

$$v_{Dirac} = Sv_{Weyl} = \sqrt{(E_p+m)}\begin{pmatrix} \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi(0) \\ \chi(0) \end{pmatrix}.$$

In the rest-frame of the particle, the positive energy solutions are

$$u = \begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \sqrt{2m} \\ 0 \\ 0 \end{pmatrix}$$

and the negative energy solutions are

$$v = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2m} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2m} \end{pmatrix}.$$

In this representation

$$\not{p} \equiv \gamma\cdot p = \begin{pmatrix} E_p & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -E_p \end{pmatrix},$$

so that (using $(\sigma\cdot\mathbf{p})^2 = |\mathbf{p}|^2$)

$$\begin{aligned} \not{p}u &= \sqrt{E_p+m}\begin{pmatrix} \left(E_p - \frac{\sigma\cdot\mathbf{p}^2}{E_p+m}\right)\chi \\ \left(1 - \frac{E_p}{E_p+m}\right)\sigma\cdot\mathbf{p}\chi \end{pmatrix} = \frac{1}{\sqrt{E_p+m}}\begin{pmatrix} \left(E_p^2 - |\mathbf{p}|^2 + mE_p\right)\chi \\ m\sigma\cdot\mathbf{p}\chi \end{pmatrix} \\ &= m\sqrt{E_p+m}\begin{pmatrix} \chi \\ \frac{\sigma\cdot\mathbf{p}}{E_p+m}\chi \end{pmatrix} = mu \end{aligned}$$

Similarly it may be shown that $v$ obeys the equation

$$\not{p}v = -mv.$$

We now define a fifth $\gamma$-matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3.$$

In the Dirac representation we have

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whereas in the Weyl representation, we have

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that $\gamma^5$ anti-commutes with *all* the other $\gamma$-matrices.

$$\left\{\gamma^5, \gamma^\mu\right\} = 0$$

The helicity eigenstates can be projected out using the projection operators

$$\frac{(1 \pm \gamma^5)}{2}.$$

In the Weyl representation, these projection operators are simply

$$\frac{(1+\gamma^5)}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{(1-\gamma^5)}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Since $\gamma^5\gamma^5 = 1$ these are projection operators in the sense that

$$\frac{(1 \pm \gamma^5)}{2}\frac{(1 \pm \gamma^5)}{2} = \frac{(1 \pm \gamma^5)}{2}$$

and

$$\frac{(1+\gamma^5)}{2}\frac{(1-\gamma^5)}{2} = 0.$$

## 13.4 Normalisation of States

The spinors respect relativistic normalisation,

$$\begin{aligned} u^\dagger_\alpha(\mathbf{p},\lambda)u_\alpha(\mathbf{p},\lambda') &= 2E_p\delta_{\lambda\lambda'} \\ v^\dagger_\alpha(\mathbf{p},\lambda)v_\alpha(\mathbf{p},\lambda') &= 2E_p\delta_{\lambda\lambda'} \\ u^\dagger_\alpha(\mathbf{p},\lambda)v_\alpha(\mathbf{p},\lambda') &= 0 \end{aligned}$$

In terms of $\overline{u}$ and $\overline{v}$ these become

$$\begin{aligned} \overline{u}(\mathbf{p},\lambda)u(\mathbf{p},\lambda') &= 2m\delta_{\lambda\lambda'} \\ \overline{v}(\mathbf{p},\lambda)v(\mathbf{p},\lambda') &= -2m\delta_{\lambda\lambda'} \end{aligned}$$

## 13.5 Projection Operators

$$(P_+)_{\alpha\beta} \equiv \sum_{\lambda=\pm 1} \frac{u_\alpha(\mathbf{p},\lambda)\bar{u}_\beta(\mathbf{p},\lambda)}{2m}$$

This is a projection operator because

$$\begin{aligned}(P_+)_{\alpha\beta}(P_+)_{\beta\gamma} &= \sum_{\lambda,\lambda'=\pm 1} \frac{u_\alpha(\mathbf{p},\lambda)\bar{u}_\beta(\mathbf{p},\lambda)}{2m}\frac{u_\beta(\mathbf{p},\lambda')\bar{u}_\gamma(\mathbf{p},\lambda')}{2m} = \sum_{\lambda,\lambda'=\pm 1} \frac{u_\alpha(\mathbf{p},\lambda)\bar{u}_\gamma(\mathbf{p},\lambda')}{4m^2} 2m\delta_{\lambda\lambda'} \\ &= \sum_{\lambda=\pm 1} \frac{u_\alpha(\mathbf{p},\lambda)\bar{u}_\gamma(\mathbf{p},\lambda)}{2m}\end{aligned}$$

Moreover, from $(\not{p}-m)u=0$, we have

$$(\not{p}-m)_{\alpha\beta}(P_+)_{\beta\gamma} = 0.$$

We must be able to construct $P_+$ from the $\gamma$-matrices. This leads to the solution

$$P_+ = \frac{(\not{p}+m)}{2m}$$

Using

$$\not{p}\not{p} = p^2,$$

we see that

$$\frac{(\not{p}+m)}{2m}\frac{(\not{p}+m)}{2m} = \frac{p^2+m^2+2m\not{p}}{4m^2} = \frac{(\not{p}+m)}{2m}$$

Likewise it may be shown that

$$(P_-)_{\alpha\beta} \equiv \sum_{\lambda=\pm 1} \frac{v_\alpha(\mathbf{p},\lambda)\bar{v}_\beta(\mathbf{p},\lambda)}{2m} = \frac{(\not{p}-m)_{\alpha\beta}}{2m}$$

## 13.6 Dirac Equation in an Electromagnetic Field

The coupling of a spin-$\frac{1}{2}$ with electric charge $e$ to an electromagnetic field is achieved by the minimal coupling

$$\partial_\mu \rightarrow \partial_\mu + ieA_\mu.$$

so that the Dirac equation becomes

$$\left(i\gamma^\mu(\partial_\mu + ieA_\mu) - m\right)\Psi = 0$$

Write the spinor $\Psi$ in terms of two-component spinors as

$$\Psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$$

In terms of these component and in the Dirac representation for the $\gamma$-matrices, the Dirac equation becomes

$$\begin{aligned} i\dot{\chi} &= eA_0\chi + m\chi + i\sigma^j\partial_j\Phi - e\sigma\cdot\mathbf{A}\Phi \\ -i\dot{\Phi} &= -eA_0\chi + m\Phi - i\sigma^j\partial_j\chi + e\sigma\cdot\mathbf{A}\chi \end{aligned}$$

In the non-relativistic limit, we consider the positive energy solutions and define

$$\chi = e^{-imt}\tilde{\chi}$$

$$\Phi = e^{-imt}\tilde{\Phi},$$

where $\tilde{\chi}$, $\tilde{\Phi}$ are slowly varying functions of time. The Dirac equation becomes

$$\begin{aligned} i\frac{d\tilde{\chi}}{dt} &= eA_0\tilde{\chi} + i\sigma^j\partial_j\tilde{\Phi} - e\sigma\cdot\mathbf{A}\tilde{\phi} \\ -i\frac{d\tilde{\Phi}}{dt} &= -eA_0\tilde{\Phi} + 2m\tilde{\Phi} - i\sigma^j\partial_j\tilde{\chi} + e\sigma\cdot\mathbf{A}\tilde{\chi} \end{aligned}$$

If $A_0 \ll 2m$ the second of these equations is approximately solved by

$$\tilde{\Phi} = \frac{i\sigma^j\left(\partial_j + ieA_j\right)\tilde{\chi}}{2m},$$

and inserting this into the first equation gives

$$i\frac{d\tilde{\chi}}{dt} = eA_0\tilde{\chi} - \frac{1}{2m}\sigma^j\left(\partial_j + ieA_j\right)\sigma^k\left(\partial_k + ieA_k\right)\tilde{\chi}$$

Using

$$\sigma\cdot\mathbf{a}\sigma\cdot\mathbf{b} == \left(\frac{1}{2}\left\{\sigma^j,\sigma^k\right\} + \frac{1}{2}\left[\sigma^j,\sigma^k\right]\right)a_jb_k = \mathbf{a}\cdot\mathbf{b} + i\varepsilon_{ijk}a_ib_j\sigma_k$$

and

$$\left[\partial_j + ieA_j, \partial_k + ieA_k\right] = ie\left(\partial_jA_k - \partial_kA_j\right) = ieF_{jk} = -ie\varepsilon_{jkl}B^l$$

gives

$$\sigma^j\left(\partial_j + ieA_j\right)\sigma^k\left(\partial_k + ieA_k\right) = \left(\partial + ie\mathbf{A}\right)^2 - \sigma\cdot\mathbf{B}$$

so that the equation for $\tilde{\chi}$ becomes

$$i\frac{d\tilde{\chi}}{dt} = eA_0\tilde{\chi} - \frac{1}{2m}\left(\partial + ie\mathbf{A}\right)^2\tilde{\chi} + e\frac{\sigma\cdot\mathbf{B}}{2m}\tilde{\chi}.$$

The first two terms give the usual electromagnetic potential terms for a charged particle moving in an electromagnetic field.

The last term may be written as $\mu \cdot \mathbf{B}$, where $\mu$ is the magnetic moment associated with the spin of the particle.

$$\mu = g_s \mu_B \mathbf{S}$$

where $\mu_B$ is the "Bohr magneton"

$$\mu_B \equiv \frac{e}{2m}$$

$$S^j = \frac{1}{2}\sigma^j$$

This gives the value $g_s = 2$ for the gyromagnetic ratio $g_s$ associated with the spin of a fermion, in contrast with the value $g_l = 1$ for the contribution to the magnetic moment of a charged particle due to its orbital angular momentum