# MATH3084/MATH6162 Integral Transform Methods

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# Contents

# 1 Introduction

An integral transform relates an input function $f(t)$ to an output function $Tf(u)$ by the relation

$$Tf(u) = \int_{t_1}^{t_2} dtK(t,u)f(t) \tag{1}$$

where $K(t,u)$ is a function of two variables, known as the **kernel** of the transform. There are many useful choices of $K(t,u)$ and each choice defines a different integral transform. The functions $(f(t), Tf(u))$ are sometimes referred to as integral-transform pairs. The introduction of such a transform in a given problem may be advantageous if $Tf(u)$ can be determined more easily than $f(t)$; for example, $f(t)$ may satisfy a differential equation while $Tf(u)$ may satisfy an algebraic equation. Most useful kernels have an associated inverse kernel $K^{-1}(u,t)$ which gives the *inverse transform*

$$f(t) = \int_{u_1}^{u_2} duK^{-1}(u,t)Tf(u). \tag{2}$$

A symmetric kernel is one for which the kernel and inverse kernel coincide.

The most commonly used integral transforms are *infinite*, i.e. the range of integration is infinite, and the most popular integral transforms are Laplace transforms; Fourier transforms (also called complex Fourier transforms or exponential Fourier transforms); Fourier sine and cosine transforms and the Hankel transform. We will consider all these transforms - their properties and their applications - in this module. For example, the Laplace transform is defined as

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt, \tag{3}$$

i.e. the integral over $t$ extends over the positive real axis. In this case the kernel is

$$K(s,t) = e^{-st}. \tag{4}$$

Note that notations for integral transforms vary widely; there are no universal conventions for the labels of the arguments of the kernels, and notations for the transform itself also vary. Engineers typically use $(s,t)$ for the Laplace transform, with $t$ being interpreted as a time coordinate, and $(k,x)$ for the Fourier transform, with $x$ interpreted as a spatial coordinate. However one can equally denote the Laplace transform as

$$\bar{f}(k) = \int_0^\infty f(x)e^{-kx}dx, \tag{5}$$

where the function $f(x)$ is viewed as being dependent on a spatial coordinate $x$.

Many other infinite transforms have been studied and tabulated. For the transform to be useful, it is important that one can explicitly calculate the required integrals. For the Laplace and Fourier transforms the required integrals can generally be computed using complex variable theory methods and extensive tables of integral transform pairs also exist.

# 2 Complex variable theory

## 2.1 The fundamental theorem of algebra

Complex numbers arise as a natural extension of the number system:

- Within the *natural* numbers ($x \in \mathbb{N}$) one can solve $2 + x = 3$, but not $3 + x = 2$;

- Within the *integer* numbers ($x \in \mathbb{Z}$) one can solve $3 + x = 2$, but not $2x = 3$;
- Within the *rational* numbers ($x \in \mathbb{Q}$) one can solve $2x = 3$, but not $x^2 = 2$;
- Within the *real* numbers ($x \in \mathbb{R}$) one can solve $x^2 = 2$, but not $x^2 = -2$;
- Within the *complex* numbers ($x \in \mathbb{C}$) one can solve any polynomial equation.

The **fundamental theorem of algebra:** Every (nonconstant) polynomial with complex coefficients has at least one zero in $\mathbb{C}$.

**Corollary:** A polynomial of order $n$ has exactly $n$ zeros (roots) in $\mathbb{C}$. Roots may occur with multiplicity, i.e. the roots may be coincident.

## 2.2 Cartesian and polar representations

A complex number $z$ can be written in the form $z = x + iy$ where $x$ and $y$ are real numbers and $i^2 = -1$. We call $x$ the *real* part of $z$ and denote it by $\operatorname{Re} z$. Similarly, we call $y$ the *imaginary* part of $z$ and denote it by $\operatorname{Im} z$.

A complex number can also be written as $z = re^{i\theta} = r\cos\theta + ir\sin\theta$. Here $r = |z|$ is called the *modulus* or absolute value of $z$ and $\theta = \arg z$ is called the *argument* of $z$. Note that $\arg z$ is not uniquely defined: we can add any integer multiple of $2\pi$ to $\arg z$ without altering $z$.

The two representations

$$z = x + iy = re^{i\theta} \tag{6}$$

are linked by

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{7}$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}. \tag{8}$$

Since $z = re^{i\theta} = e^{\ln r}e^{i\theta} = e^{\ln r + i\theta}$, the complex natural logarithm function ln is given by

$$\ln z = \ln|z| + i\arg z. \tag{9}$$

**Argand diagrams** The relationship between Cartesian and polar representation of complex numbers becomes clear on an Argand diagram - see Figure 1. This represents the complex plane by a **Re**al and an **Im**aginary axes (horizontal and vertical respectively, referring to x and y).

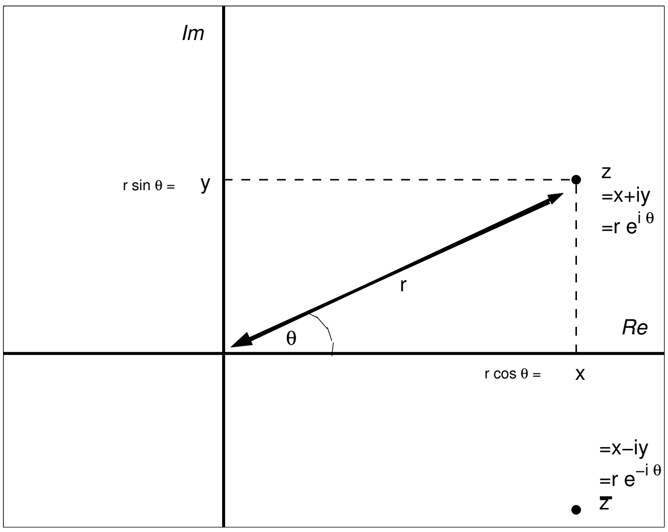



Figure 1: A complex number is represented by a point in the complex plane.

The **complex conjugate** of $z$ is defined to be

$$\bar{z} = x - iy = re^{-i\theta}. \tag{10}$$

From the definitions it follows that

$$|\bar{z}| = |z| \quad , \; \arg \bar{z} = -\arg z \quad \text{and} \quad z\bar{z} = r^2 \quad . \tag{11}$$

In the complex plane $\bar{z}$ is a reflection of $z$ in the x-axis.

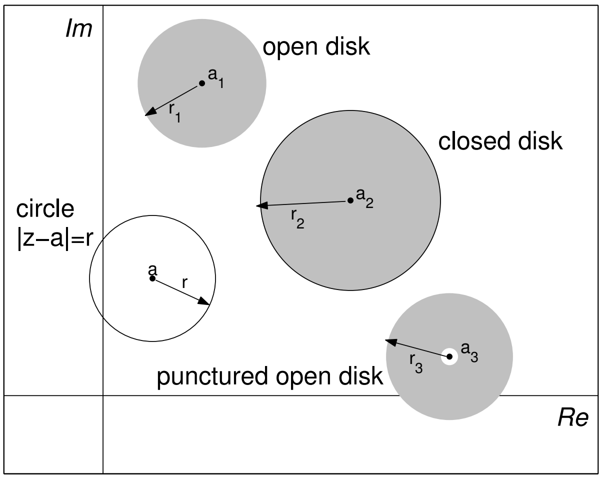



Figure 2: Examples of disks in the complex plane.

**Disks in the complex plane** A circle of radius $r$ with centre $a$ in the complex plane has the Eq.

$$|z - a| = r, \tag{12}$$

where $a$ can be complex and $r$ is real and positive. The inside of the circle

$$|z - a| < r, \tag{13}$$

is called an **open disk**. If we include the boundary it becomes a closed disk:

$$|z - a| \leq r. \tag{14}$$

If we remove the point $a$ from the disk it is called a **punctured disk -** such a disk is usually open:

$$0 < |z - a| < r \tag{15}$$

but can also be closed:

$$0 < |z - a| \leq r. \tag{16}$$

## 2.3 Holomorphic and analytic functions

**Complex functions** A mapping $f$ which maps a complex number $z$ to a *unique* complex number $w$, that is $w = f(z)$, is a *function.* We will later look at mappings such that $w$ is not unique; these *multifunctions* (see section 2.11) are not true functions but mappings. For example, ln is not a proper function of $z$, since $\arg z$ is not unique.

**The complex derivative** We say that $f$ is *differentiable* at $z$ if

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{17}$$

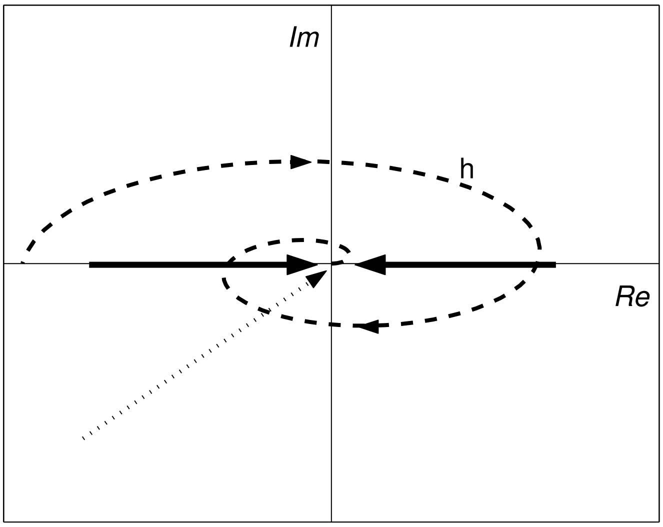



Figure 3: In this definition the limit must be independent of the direction in which the *complex* number $h$ approaches zero.

exists. When it exists, the limit is called the *derivative* of the function $f$ at $z$, and is denoted by $f'(z)$. To calculate the complex derivative of $f(z)$ we need to consider

$$h = \delta x + i\delta y. \tag{18}$$

Notice that for the limit in Eq. (17) to exit, it must be independent of the direction in which $h$ is taken to zero.

If the derivative of $f$ exists everywhere in a domain $D$, then $f$ is called **holomorphic** on $D$. A function which is holomorphic on $\mathbb{C}$ is called *entire*.

**Cauchy-Riemann equations** Let $f(z)$ be holomorphic in a certain domain, and let $z = x + iy$. It is often convenient to write

$$f(z) = u(x, y) + iv(x, y), \tag{19}$$

thereby splitting the complex function $f$ into real and imaginary parts just like $z$. Then using (18)

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) \; - \; u(x, y) - iv(x, y)}{\delta x + i\delta y.} \tag{20}$$

Take first $\delta y = 0$ (so, approach with $h$ to zero along the $x$ axis). Then $h = \delta x$, so that

$$\begin{aligned} \lim_{h\to 0} \tfrac{f(z+h)-f(z)}{h} &= \lim_{\delta x\to 0} \frac{u(x + \delta x, y) + iv(x + \delta x, y) - u(x, y) - iv(x, y)}{\delta x} \\ &= \lim_{\delta x\to 0} \left[\frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i\frac{v(x + \delta x) - v(x, y)}{\delta x}\right] \\ &= u_x(x, y) + iv_x(x, y). \end{aligned} \tag{21}$$

If, instead, we take $\delta x = 0$ (i.e., approach with $h$ to zero along the y axis), then $h = i\delta y$ and we obtain

$$\begin{aligned} \lim_{h\to 0} \tfrac{f(z+h)-f(z)}{h} &= \frac{u(x, y + \delta y) + iv(x, y + \delta y) - u(x, y) - iv(x, y)}{i\delta y} \\ &= -iu_y(x, y) + v_y(x, y). \end{aligned} \tag{22}$$

Now, if $f(z)$ is holomorphic, then $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = f'(z)$ does not depend on the direction, and the two expressions must be equal:

$$u_x + iv_x = f'(z) = v_y - iu_y \tag{23}$$

Taking the real and imaginary parts of $f'(z)$ and equating them we obtain the **Cauchy-Riemann equations**:

$$\boxed{u_x = v_y, \qquad u_y = -v_x.} \tag{24}$$

Now assume that $u(x,y)$ and $v(x,y)$ are twice differentiable, and that their first and second derivative are continuous. (In fact, as we shall see later, if $f(z)$ is holomorphic, $u$ and $v$ are automatically guaranteed to have continuous partial derivatives of all orders!) By taking another derivative of the Cauchy-Riemann equations, we obtain

$$u_{xx} = v_{yx} = -u_{yy}, \qquad\qquad v_{yy} = u_{xy} = -v_{xx}, \tag{25}$$

and hence

$$\boxed{u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0.} \tag{26}$$

We find that $u(x,y)$ and $v(x,y)$ both satisfy **Laplace's equation**. Such functions are called **harmonic.**

Conversely, it can be shown that if $u$ and $v$ obey the Cauchy-Riemann equations, then $f = u+iv$ is holomorphic. Hence:

| | | |
|---|---|---|
| $f$ is holomorphic | if and only if $\Leftrightarrow$ | $u$ and $v$ satisfy the Cauchy-Riemann equations (and have continuous first derivatives). |

**Example:** Show that $f(z) = z^2$ is holomorphic for all $z \in \mathbb{C}$.
$f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i2xy$ so

$$\begin{array}{ll} u = x^2 - y^2 & v = 2xy \\ u_x = 2x & v_y = 2x \\ u_y = -2y & v_x = 2y \end{array} \tag{27}$$

Hence $u_x = v_y$ and $v_x = -u_y$ $\Rightarrow$ The Cauchy-Riemann equations are satisfied
$\Rightarrow$ $f$ is holomorphic.

**Analytic functions** A function $f$ is analytic if it is *locally* representable by a power series (see section 2.4 below). It may be shown that the properties of analytic and holomorphic functions are equivalent:

| | | |
|---|---|---|
| $f$ is analytic | if and only if $\Leftrightarrow$ | $f$ is holomorphic. |

so that we can use the Cauchy-Riemann equations to test if a function is analytic.
**Example:** Determine whether $f(z) = \bar{z}$ is analytic.
$f(z) = \bar{z} = x - iy$, so

$$\begin{array}{ll} u = x & v = -y \\ u_x = 1 & v_y = -1 \\ u_y = 0 & v_x = 0 \end{array} \tag{28}$$

Hence $v_x = -u_y$ but $u_x \neq v_y$ $\Rightarrow$ The Cauchy-Riemann equations are not satisfied
$\Rightarrow$ $f$ is not holomorphic and not analytic.

More generally, functions of $\bar{z}$ are not holomorphic and not analytic.

## 2.4 Taylor series

Suppose that $f(z)$ is analytic in the open disk $D = \{z : |z-a| < r\}$. Then, one can show (although we do not prove this here) that the following holds:

1. $f(z)$ has derivatives of all orders, $f'$, $f''$, $f'''$, which are all analytic in $D$. (This is a striking result, that has no analogue in the world of real-valued functions. Think of the real function $f(x) = x^{3/2}$. It has a continuous first derivative, but its second derivative fails to exist at $x = 0$.)

2. At any point in $D$, $f(z)$ can be written as

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n \,(z-a)^n, \;\text{ where } a_n = \frac{f^{(n)}(a)}{n!} \tag{29}$$

and $f^{(n)}$ denotes the $n$-th derivative of $f$. This is called the **Taylor** series of $f(z)$ about $z = a$. The Taylor series converges (uniformly) anywhere in $D$.

3. The derivatives and integrals of $f$ can be found through *term-by-term* differentiation or integration of the Taylor series. For example,

$$f'(z) \;=\sum_{n=1}^{\infty} n\, a_n \,(z-a)^{n-1}. \tag{30}$$

**Radius of convergence** The region of convergence of the Taylor series is always a disk, so that there is some $R$ for which the series converges for all $|z-a| < R$, and diverges for all $|z-a| > R$. (If $f$ is analytic in the whole complex plane then $R$ becomes $\infty$.) $R$ is called the *Radius of Convergence* (RoC). It can be thought of as the largest possible $r$ for which (29) is true, or, equivalently, the distance to the nearest point (from $a$) where $f$ is no longer analytic.
**Example:** Find the Taylor Series of $f(z) = \frac{1}{1-z}$ about $z = 0$ and state its RoC.
Since $f$ is analytic at $z = 0$ (can you prove that?), we can choose to calculate its derivatives at this point by taking the limit along the real axis (as the direction does not matter). This is convenient, since we can then treat $f$ as a function of a real variable $x$, and calculate the derivatives of $f(x)$ in the usual manner: $f'(z) = (1-z)^{-2}$, $f''(z) = 2(1-z)^{-3}$, and so on. We find that the standard binomial expansion applies:

$$f(z) = (1-z)^{-1} = \sum_{n=0}^{\infty} z^n. \tag{31}$$

We know that this converges along the real axis for $|x| < 1$. Since the region of convergence must be a disk, we conclude that the expansion converges on the complex plane for $|z| < 1$. Hence, RoC=1.
**Example:** Find the Taylor Series of $f(z) = \frac{1}{1-z}$ about $z = 3$ and state its RoC.
Again, $f$ is analytic at $z = 3$, and we can use the binomial expansion (this time with respect to the variable $z - 3$) to obtain

$$f(z) = (1-z)^{-1} = (-2-(z-3))^{-1} = -\frac{1}{2}\left(1-\left(\frac{3-z}{2}\right)\right)^{-1} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{3-z}{2}\right)^{n} \tag{32}$$

which converges for $|\frac{3-z}{2}| < 1 \Rightarrow |3-z| < 2 \Rightarrow$ RoC=2.

These results become intuitive if we sketch them on an Argand diagram—see Figure 4.

**Binomial Expansion** The **Binomial series** often offers the quickest way in deriving power series:

$$\begin{aligned}(1+z)^k \;&=\; 1 + kz + \tfrac{k(k-1)}{2}z^2 + \tfrac{k(k-1)(k-2)}{3!}z^3 + \ldots + \tfrac{k(k-1)(k-2)\ldots(k-(n-1))}{n!}z^n + \ldots \\ &=\; 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots[k-(n-1)]}{n!}\, z^n \qquad \text{which converges for } |z| < 1.\end{aligned} \tag{33}$$

Many functions have well known power series, and hence RoC's. These are the same as for their real counterparts, with $x$ replaced by $z$. In all cases the RoC about any point $a$ is determined by the location of "problem points": these are the **singularities** of $f$ (see Section 2.6).

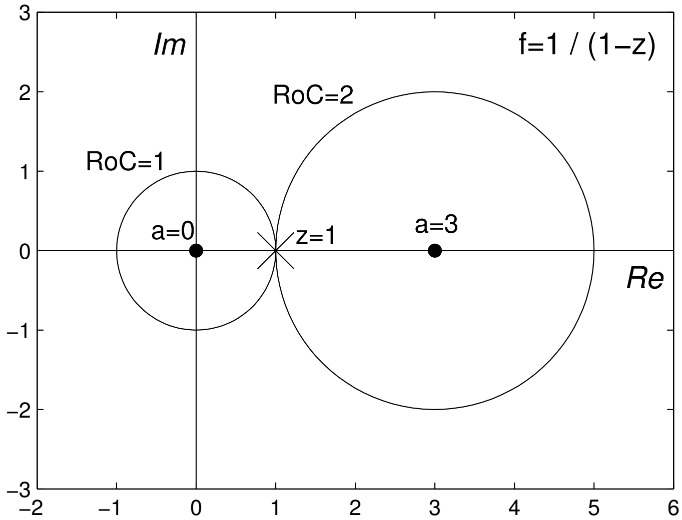



Figure 4: We might expect the function $f(z) = \frac{1}{1-z}$ not to be analytic at $z = 1$.

## 2.5 Laurent series

If $f(z)$ is analytic in an **annulus** (ring) $A = \{z : r < |z - a| < R\}$, then there exists a **Laurent series** such that for all $z$ in $A$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \tag{34}$$

where the coefficients $c_n$ are unique but *not* equal to $\left(f^{(n)}(a)/n!\right)$. The part $\sum_{n=-\infty}^{-1}$ with negative powers is called the **principal part** of the Laurent series. The Laurent series converges uniformly in $A$, and can be differentiated or integrated term-by term in the same way as Taylor series.

The smallest possible value of $r$ and the largest possible value of $R$ are again determined so that the annulus is squeezed between the nearest singularities (see figure 5).

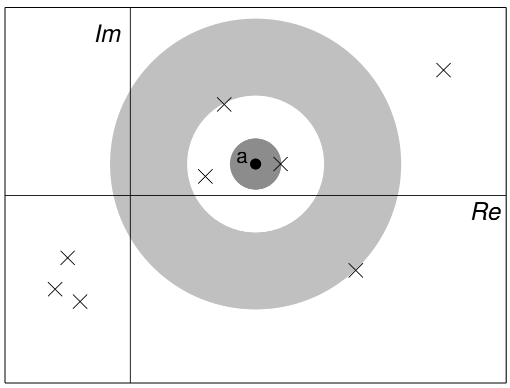



Figure 5: A Taylor series exists inside an analytic disk. A Laurent series exists inside an analytic annulus. Singularities are indicated by crosses.

If $z = a$ is itself a singularity then there is no Taylor series, since the disk $D = \{z : |z - a| < 0\}$ is empty. If $z = a$ is a singularity and $R$ is the distance from $a$ to the nearest singularity, then a Laurent series exists on the *punctured* open disk $0 < |z - a| < R$.

**Example:** Find the Laurent series of $f(z) = \frac{1}{1-z}$ about $z = 0$ and state its annulus of convergence. We found previously that $f$ had a Taylor series about $z = 0$ within $D = \{z : |z| < 1\}$, i.e., with an RoC of 1. This was due to a singularity at $z = 1$. We might expect a Laurent series of $f$ about $z = 0$ to be valid for $\{z : 1 < |z|\}$, if there are no other singularities of $f(z)$. Using the binomial expansion again we find

$$\begin{aligned} f(z) = & \ \frac{1}{1-z} = \frac{\frac{1}{z}}{\frac{1}{z}-1} = -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \\ = & \ -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n} = -\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} = -\sum_{m=1}^{\infty}\left(\frac{1}{z}\right)^{m} = -\sum_{n=-\infty}^{-1} z^{n}. \end{aligned} \tag{35}$$

The binomial expansion is valid for $\left|\frac{1}{z}\right| < 1$, i.e., the Laurent series is valid in the (semi-infinite) annulus $A = \{z : 1 < |z|\}$.
Note that this Laurent series has negative powers of $z$, unlike a Taylor series; in general, of course, the Laurent series has both positive and negative powers.

## 2.6 Singularities

A point $a$ is a **regular point** of $f(z)$ if $f$ is analytic at $a$. The point $a$ is a **singularity** of $f(z)$ if $a$ is a limit point of regular points which is not itself regular.

The point $a$ is an **isolated singularity** if $f(z)$ is analytic on the punctured disk $\{z : 0 < |z-a| < r\}$ for some $r > 0$.

If there exists no $r > 0$, however small, such that $f(z)$ is analytic on $\{z : 0 < |z-a| < r\}$, then $a$ is called a **non-isolated essential singularity**, but we will not be dealing with these in depth here.

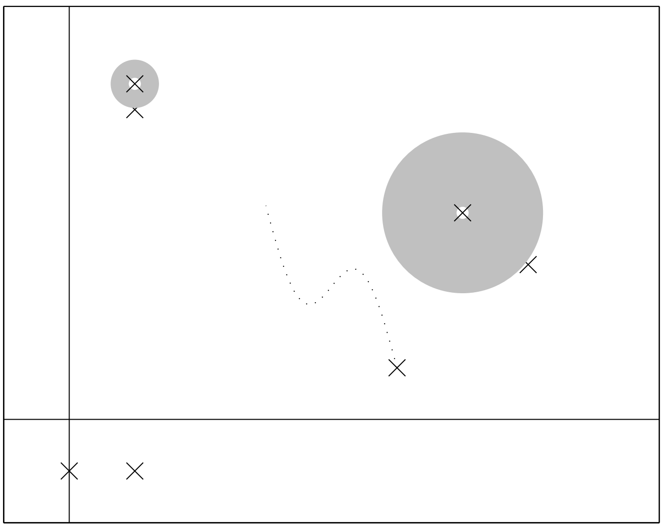


Figure 6: Isolated singularities.

There are three types of isolated singularities:

**Removable singularity** The point $a$ is a **removable singularity** if the principal part of the Laurent series about $a$ is zero.
**Example:** The Laurent series for $\frac{1}{z}\sin z$ about $z = 0$.

$$z^{-1}\sin z = z^{-1}\left(z - z^3/6 + \dots\right) \ = \ 1 - z^2/6 + \dots \tag{36}$$

So $\sum_{n=-\infty}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^n$ because $c_n = 0$ for all $n < 0$; the principal part of the Laurent series is zero, i.e., $= \sum_{n=-\infty}^{-1} c_n z^n = 0$.

As power series are unique this is equivalent to the Taylor series.

**Essential singularity** The point $a$ is an **isolated essential singularity** if the principal part of the Laurent series has an infinite number of terms.
**Example:** The Laurent series for $e^{1/z}$ about $z = 0$.

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{m=-\infty}^{0} \frac{1}{n!} z^m \tag{37}$$

The values of $n$ for which $c_{-n} \neq 0$ form an infinite sequence.

**Poles** The point $a$ is a **pole of order** $k$ if $c_{-k} \neq 0$ and $c_n = 0$ for all $n < -k$.
Since $k$ is lowest power of $(z - a)$ we have

$$f(z) = \sum_{n=-k}^{\infty} c_n (z-a)^n. \tag{38}$$

Another way of stating this is that

$$f(z)(z-a)^k \tag{39}$$

is an analytic function. Hence

$$f(z) = \frac{g(z)}{(z-a)^k}, \tag{40}$$

where $g(z)$ is an analytic function. It is often convenient to work out the Laurent series for $f(z)$ by using the standard Taylor series expansion for the analytic function $g(z)$.
**Example:** The Laurent series for $\frac{e^z}{z^2}$ about $z = 0$.

$$\begin{aligned} \tfrac{1}{z^2} e^z = &\; z^{-2}\left(1 + z + \tfrac{z^2}{2} + \tfrac{z^3}{6} + \dots\right) = z^{-2} + z^{-1} + \tfrac{1}{2} + \tfrac{1}{6} z + \dots \\ = &\; \textstyle\sum_{n=0}^{\infty} \frac{1}{n!} z^{n-2} \;=\; \sum_{m=-2}^{\infty} \frac{1}{(m+2)!} z^m. \end{aligned} \tag{41}$$

A pole of order 1 is called a *simple pole.* A pole of order 2 is called a *double pole.* Poles of order three and higher are usually referred to as such, although can be labeled triple etc.

## 2.7 Residues

The coefficient $c_{-1}$ is called the **residue of** $f$ **at** $a$ and is denoted by $\text{Res}(f : a)$.

Note that a removable singularity has no residue, since $c_{-1} = 0$ as part of $c_{-1} = c_{-2} = c_{-3} = \ldots = 0$. Both poles and essential singularities have residues since $c_{-1} \neq 0$ in general, although we can have a residue of zero in these cases (but recall that non-removable singularities would always contain at least one non-zero negative coefficient in their Laurent expansion).

It is important for us to calculate residues quickly as they are used a great deal in complex integration.

If $a$ is a pole of order $k$, then its residue is

$$\text{Res}(f : a) = \frac{1}{(k-1)!} \lim_{z \to a} \left(\frac{d}{dz}\right)^{(k-1)} (z-a)^k f(z) \qquad \text{(pole of order } k \text{ only)} \tag{42}$$

(Proof by calculation from the Laurent series.) For a simple pole ($k = 1$) this general formula reduces to

$$\text{Res}(f : a) = \lim_{z \to a} (z-a) f(z) \qquad \text{(simple pole only)}. \tag{43}$$

**A special case:** For functions $f$ of the form

$$f(z) = \frac{g(z)}{(z-a)^k} = (z-a)^{-k} g(z), \tag{44}$$

where $g$ is nonzero and analytic at $a$ (i.e. a pole of $f(z)$ of order $k$ at $a$) we have:

$$\text{Res}(f : a) = \frac{1}{(k-1)!} g^{(k-1)}(a), \qquad \text{(pole of order } k \text{ only)}. \tag{45}$$

**A special case for simple poles:** For a simple pole of the form

$$f(z) = \frac{g(z)}{(z-a)}, \tag{46}$$

where $g$ is nonzero and analytic at $a$ we have

$$\mathrm{Res}(f : a) = g(a) \qquad \text{(simple pole only)}. \tag{47}$$

**More generally, for simple poles:** If $f$ is given in the form

$$f(z) = \frac{g(z)}{h(z)} \tag{48}$$

where $g$ and $h$ are analytic at $a$, with $g(a) \neq 0$, $h(a) = 0$, and $h'(a) \neq 0$, then

$$\mathrm{Res}(f : a) = \frac{g(a)}{h'(a)} \qquad \text{(simple pole only)}. \tag{49}$$

(Proof by calculation from (47).)

For poles of higher order not of the form (44) it may be necessary to apply the general formula (42), or obtain the Laurent series directly (up to $c_{-1}$). In the case of essential singularities Eq. (42) no longer applies, and the only way to obtain the residue is by deriving the Laurent series (up to $c_{-1}$).

**Example:** Find the residues of all the real singularities of $f(z) = \frac{z^2}{(z-2)(z^2+1)}$.
The singularities of $f(z)$ are given by considering

$$(z-2)(z^2+1) \;=\; 0 \;=\; (z-2)(z+i)(z-i); \tag{50}$$

hence $f(z)$ has simple poles at $z = 2$, $z = i$ and $z = -i$.
For this question we need only consider $z = 2$.
We note that $f(z)$ only has one zero (root), which is at $z = 0$. Hence the singularity at $z = 2$ is not removable. The residue is

$$\mathrm{Res}(f : 2) = \lim_{z\to 2}(z-2)\left(\frac{z^2}{(z-2)(z^2+1)}\right) = \lim_{z\to 2}\frac{z^2}{(z^2+1)} = \frac{2^2}{2^2+1} = \frac{4}{5}. \tag{51}$$

**Example:** Find the residue $f(z) = \frac{1}{z^2(z+2)^3}$ at $z = -2$.
$z = -2$ is a pole of order three.

$$\mathrm{Res}(f : -2) = \lim_{z\to -2}\frac{1}{2!}\frac{d^2}{dz^2}\left((z+2)^3\frac{1}{z^2(z+2)^3}\right) = \lim_{z\to -2}\frac{1}{2}\frac{d}{dz}\left(\frac{-2}{z^3}\right) = \lim_{z\to -2}\frac{1}{2}\left(\frac{+6}{z^4}\right) = \frac{3}{16}. \tag{52}$$

**Example:** Find the residue of all singularities of $f(z) = \frac{\cos z + z^3}{\sin z}$.
Let $g(z) = \cos z + z^3$ and $h(z) = \sin z$.
$f$ has singularities whenever $h(z) = 0$, i.e. $z = n\pi,\ n \in \mathbb{Z}$. These are simple poles.
In addition, $h'(n\pi) = \cos(n\pi) \neq 0$ and $g(n\pi) \neq 0$, so we may use (49):

$$\mathrm{Res}(f : n\pi) = \frac{g(n\pi)}{h'(n\pi)} = \frac{\cos n\pi + (n\pi)^3}{\cos n\pi} = \frac{(-1)^n + n^3\pi^3}{(-1)^n} = 1 + (-1)^n n^3\pi^3. \tag{53}$$

And so, we have $\mathrm{Res}(f : 0) = 1,\ \mathrm{Res}(f : \pi) = 1 - \pi^3,\ \mathrm{Res}(f : 2\pi) = 1 + 8\pi^3, \ldots$
The Laurent series of $f(z)$ about the singular points $z = 0, \pi, \ldots$ have the form

$$f(z; 0) = \frac{1}{z} - \frac{z}{3} + z^2 - \frac{z^3}{45}\cdots \tag{54}$$

$$f(z;\pi) \;=\; \frac{1-\pi^3}{z-\pi} \;-\; 3\pi^2 \;-\; \left(\frac{\pi^3}{6} + 3\pi + \frac{1}{3}\right)(z-\pi) \;-\; \left(1 + \frac{\pi^2}{2}\right)(z-\pi)^2 \;+\; \ldots, \tag{55}$$

and so on. We can see, as a check, that $\mathrm{Res}(f)$ are given by the coefficients of $z^{-1}$ in these expansions. This is expected, as all the poles are simple.

## 2.8 Contour integration, Cauchy theorem and residue theorem

**Contours** We are used to integrating a *real* function along the *real* line: considering the value of $f(x)$ as $x$ ranges between $\alpha$ and $\beta$ say and finding the area underneath.

For integration over a complex variable $z$ we move in two dimensions across the complex plane. Because of this it is not only the **end points** that are important but also the **route** across the complex plane between them.

A **curve** $\gamma$ is the graph of a continuous function $z(t)$ from a real interval $a \leq t \leq b$ to the complex plane:

$$\gamma = \{z(t) : a \leq t \leq b\}. \tag{56}$$

We will call $z(t)$ together with the interval $[a, b]$ the **parameterised curve**.

If $z(a) = z(b)$ then $\gamma$ is **closed**.

If $z(t_1) = z(t_2)$ only if $t_1 = t_2$ for all $t \in (a, b)$ then $\gamma$ is **simple**, which means that it does not intersect itself. Note that this does not include end points.

A **contour**, also called a **path**, is a finite sequence of directed smooth curves. Contours can be open or closed; we will restrict to simple contours in what follows. When we integrate complex functions, we usually do so over contours, not just real intervals.

Since a closed path starts where it finishes we need to also define its orientation. By convention we take the integration in the **anticlockwise** direction.

(See Figure 7 on the next page.)

**Integration along a contour** The integral of a function $f(z)$ along a curve $\gamma$ is given by

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt, \tag{57}$$

where the curve is parameterised as $z(t)$ with $t$ real. The value of the integral does not depend on the choice of this parameter $t$ but only on the curve $\gamma$. (Proof by change of variable from $t$ to $\tau$, for an arbitrary monotonic function $t(\tau)$.) Note however that the direction of the integral is affected by the orientation of the contour–this is why we need a convention (anticlockwise). To go in the opposite direction would result in a change of sign.

**Example:** Integrate the function $(z - a)^n$ around a circle with radius $r$ and center $a$, i.e., find $I_n = \int_{C(a:r)} (z - a)^n\,dz$.

The integration contour is $\gamma = \{z : |z - a| = r\}$: This can be parameterized as

$$z(\theta) = a + re^{i\theta} : 0 \leq \theta < 2\pi. \tag{58}$$

(Whenever the contour is circular it is more common to label the contour by the parameter $\theta$, i.e. the polar angle, range than by $t$.) Since $z(\theta) = a + re^{i\theta} \Rightarrow z'(\theta) = ire^{i\theta}$, and so:

$$I_n = \int_0^{2\pi} \left(\left(a + re^{i\theta}\right) - a\right)^n ire^{i\theta} d\theta = ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta \tag{59}$$

$$= \begin{cases} \frac{r^{n+1}}{n+1} \left[e^{i(n+1)\theta}\right]_0^{2\pi} & = 0, \quad \text{for } n \neq -1 \\ i\,[\,\theta\,]_0^{2\pi} & = 2\pi i, \quad \text{for } n = -1. \end{cases} \tag{60}$$

This result is important for the Residue Theorem later on:

$$\int_{C(a:r)} (z - a)^n\,dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1 \end{cases} \quad \text{for any } r. \tag{61}$$

**Cauchy's Theorem** Let $f(z)$ be a function which is holomorphic in a region $D$ containing the contour $\gamma$. Then

$$\boxed{\int_\gamma f(z)\,dz = 0.} \tag{62}$$

This is the single most important theorem in complex variable theory. It follows directly from the Cauchy-Riemann equations, and it can be proved in a similar way to the proof of Stokes' theorem in vector calculus.

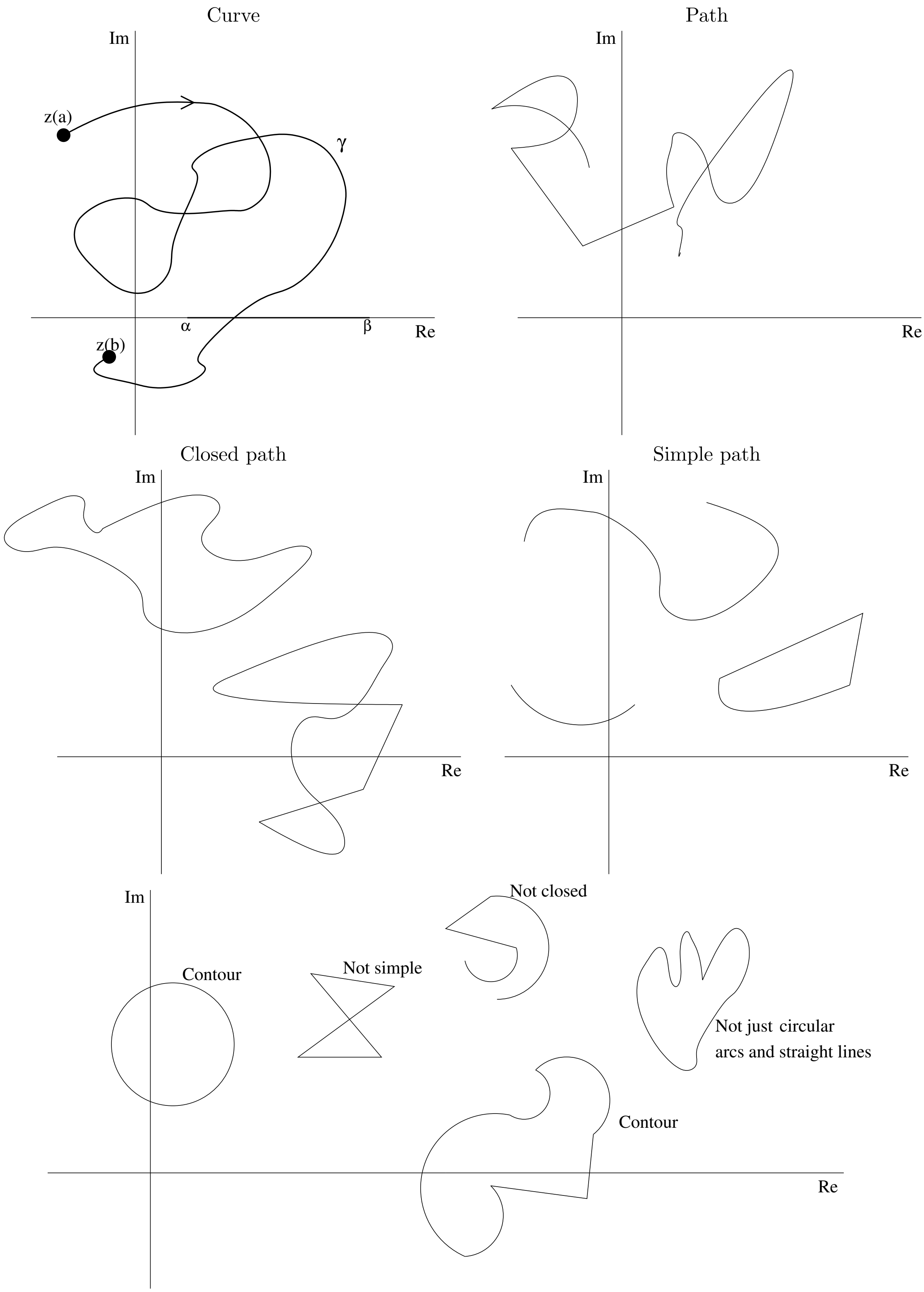



Figure 7: Curves in the complex plane.

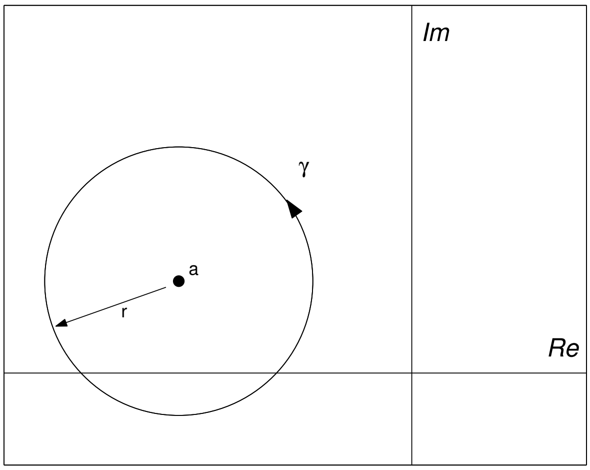



Figure 8: The circle $C(a:r) = \{z : |z-a| = r\} = \{z(\theta) = a + re^{i\theta} : 0 \le \theta < 2\pi\}$.

**Cauchy's Residue Theorem** Cauchy's residue theorem tells us how to integrate functions with isolated singularities over closed contours.

Let $f$ be holomorphic inside and on a contour $\gamma$ except for a finite number of poles $\{a_k\}$ inside (not *on*) $\gamma$. Then

$$\int_\gamma f\,dz = 2\pi i \sum_{\substack{\text{singularities } a_k \\ \text{of } f(z) \\ \text{inside } \gamma}} \mathrm{Res}(f:a_k). \tag{63}$$

**Proof outline:** Cauchy's residue theorem follows from Cauchy's theorem and (61)—see figure 9. Each isolated singularity $a_k$ can be covered by a disk $D_k$ with boundary $C(a_k : r_k)$, with a sufficiently small radius $r_k$ so that no disk overlaps another. This implies that $a_k$ is the only singularity inside $D_k$.

Inside this disk $f(z)$ has Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a_k)^n, \tag{64}$$

so that

$$\begin{aligned}
\int_{C(a_k:r_k)} f(z)\,dz &= \int_{C(a_k:r_k)} \sum_{n=-\infty}^{\infty} c_n (z-a_k)^n\,dz \\
&= \sum_{n=-\infty}^{\infty} c_n \int_{C(a_k:r_k)} (z-a_k)^n\,dz \\
&= \sum_{n=-\infty}^{\infty} c_n \left\{ \begin{array}{rl} 2\pi i, & n=-1 \\ 0, & n \ne -1 \end{array} \right. \\
&= 2\pi i\, c_{-1} \;=\; 2\pi i \,\mathrm{Res}(f:a_k).
\end{aligned} \tag{65}$$

Remember that by definition the contour $C(a_k, r_k)$ is orientated anticlockwise.

Now consider the contour $\gamma^*$ consisting of going around $\gamma$ anticlockwise, down to each disk and around each disk *clockwise*. There are no singularities inside $\gamma^*$ so that

$$\int_{\gamma^*} f(z)\,dz \;=\; 0. \tag{66}$$

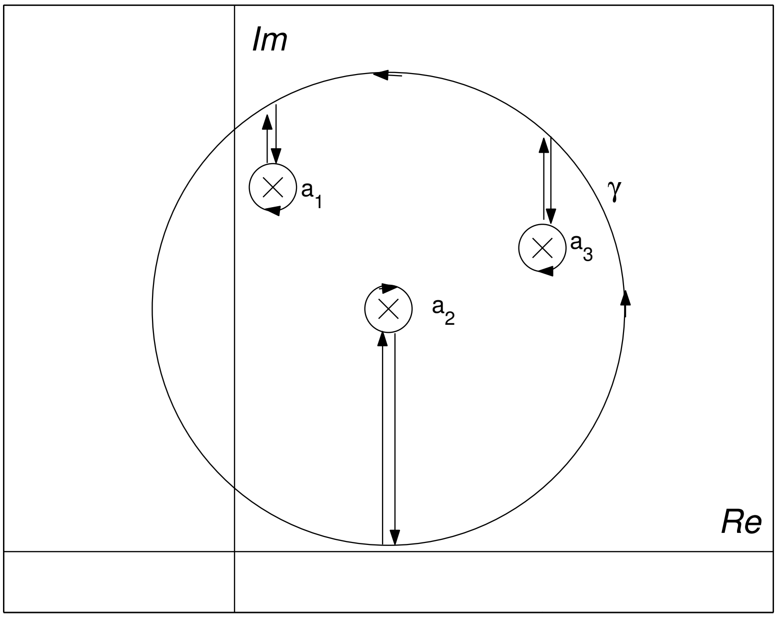



Figure 9: An example to show why the residue theorem holds true.

Here the up and down contours cancel each other out, since they are the same integral but in opposite directions, so that $\gamma^*$ only consists of:

$$\gamma \text{ going anticlockwise, with all the circles } C(a_k, r_k) \text{ going clockwise.}$$

Hence

$$0 = \int_{\gamma^*} f(z)\,dz \;=\; +\int_{\gamma} f(z)\,dz \;-\; \sum_k \int_{C(a_k,r_k)} f(z)\,dz \tag{67}$$

So

$$\int_{\gamma} f(z)\,dz \;=\; \sum_k 2\pi i \;\mathrm{Res}(f : a_k). \tag{68}$$

The **deformation theorem** states that the integral around a closed contour is unchanged as the contour is deformed, provided that the singularities contained within the contour remain the same. One can therefore deform the integration contour without changing the value of the integral providing that the deformation does not cross any singularity.

## 2.9 Cauchy integral formula, $NP$ integral, argument principle

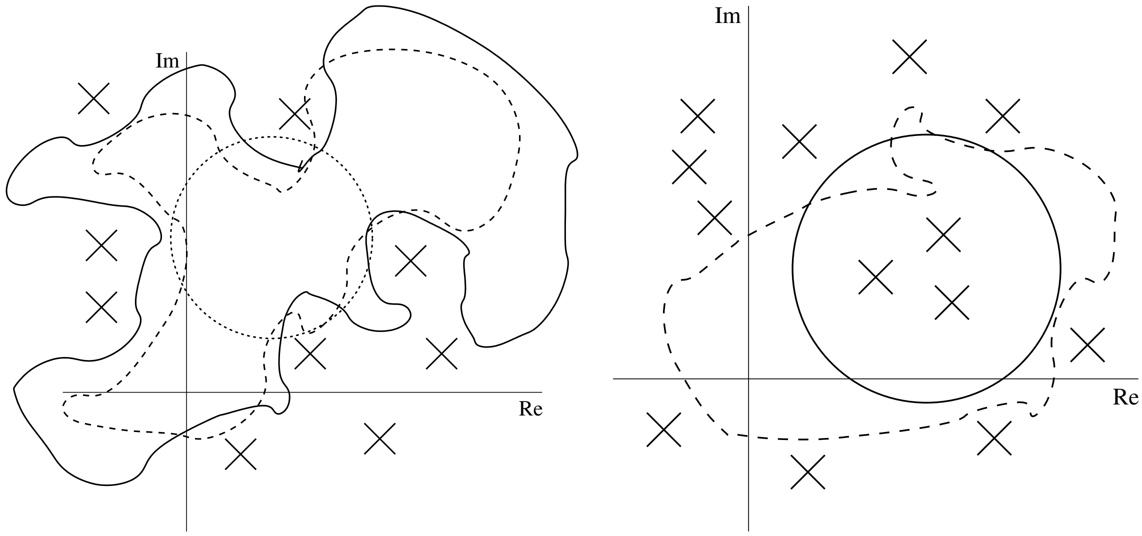
A nice consequence of the residue theorem is the **Cauchy integral formula**: Let $g(z)$ be analytic on and inside the simple closed curve $\gamma$. Then

$$\frac{1}{2\pi i}\int_{\gamma} \frac{g(z)}{z-a}\,dz = g(a) \tag{69}$$

This is proved by applying the residue theorem to the function $g(z)/(z-a)$, which by assumption has a simple pole at $z = a$.

**Example:** Evaluate the integral $f(z) = \frac{1}{(z-1)^2(z-2)}$ about the following contours:

*(i)* $C(0 : \frac{1}{2})$; *(ii)* $C(0 : 3)$; *(iii)* $C(1 : \frac{1}{2})$; *(iv)* $C(2 : 3)$.




**Cauchy's Theorem** is independent of the contour, provided it remains inside an analytic region.

**The Residue Theorem** is independent of the contour provided the singularities inside remain unchanged.

Figure 10: The Deformation Theorem.

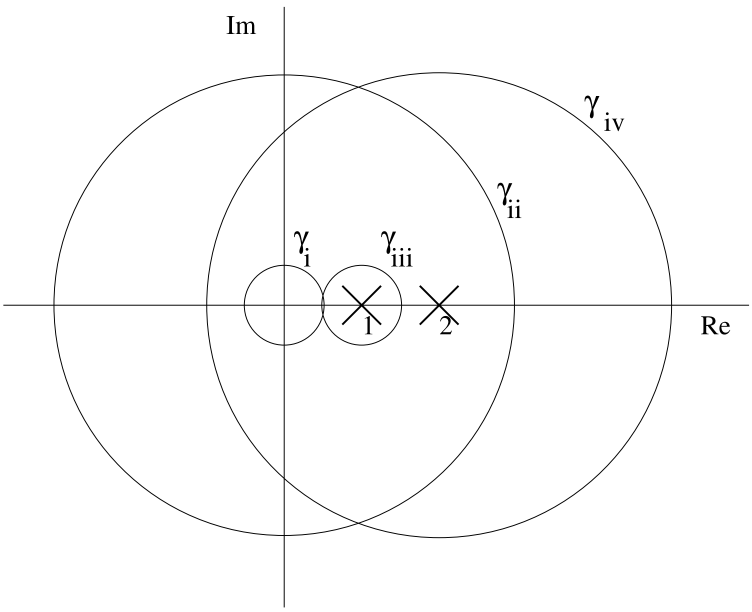



Figure 11: Sketch diagram for $f(z) = \frac{1}{(z-1)^2(z-2)}$.

Singularities: The function $f$ has a pole of order 1 at $z = 2$ and pole of order 2 at $z = 1$.

$$\operatorname{Res}(f:2) = \lim_{z\to 2}(z-2)f(z) \;= \lim_{z\to 2}\frac{(z-2)}{(z-2)(z-1)^2} \;=\; \frac{1}{(z-1)^2}\bigg|_{z=2} \;=\; (2-1)^{-2} = 1. \quad (70)$$

$$\operatorname{Res}(f:1) = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2 f(z) \;= \lim_{z\to 1}\frac{d}{dz}(z-2)^{-1} \;=\; -(z-2)^{-2}\big|_{z=1} \;=-1. \quad (71)$$

Alternatively we could look at the Laurent Series of $f$ about each singularity to find the coefficient of $\frac{1}{(z-a)}$: e.g, for $z=1$

$$\begin{aligned} f(z) &= \tfrac{1}{(z-1)^2(z-2)} = \tfrac{1}{(z-1)^2}\tfrac{1}{-1+(z-1)} = \tfrac{-1}{(z-1)^2}\left(1-(z-1)\right)^{-1} \\ &= \tfrac{-1}{(z-1)^2}\left(1+(z-1)+(z-1)^2+\ldots\right) = -\tfrac{1}{(z-1)^2} - \tfrac{1}{(z-1)} - 1 - \tfrac{1}{(z-1)} - \ldots \Rightarrow c_{-1} = -1. \end{aligned} \tag{72}$$

*(i)* $f(z)$ has no singularities inside $C(0:\frac{1}{2}) \Rightarrow \int_{C(0:\frac{1}{2})} f(z)dz = 0.$

*(ii)* $f(z)$ has the singularities $z=1$ and $z=2$ inside $C(0:3) \Rightarrow$

$$\int_{C(0:3)} f(z)dz = 2\pi i\left(\mathrm{Res}(f:1)+\mathrm{Res}(f:2)\right) = 2\pi i\,(1-1) = 0. \tag{73}$$

*(iii)* $f(z)$ has the singularity $z=1$ inside $C(1:\frac{1}{2}) \Rightarrow$

$$\int_{C(1:\frac{1}{2})} f(z)dz = 2\pi i\mathrm{Res}(f:1) \;=\; -2\pi i. \tag{74}$$

*(iv)* $f(z)$ has the singularities $z=1$ and $z=2$ inside $C(2:3) \Rightarrow$

$$\int_{C(2:3)} f(z)dz \;=\; 2\pi i\left(\mathrm{Res}(f:1)+\mathrm{Res}(f:2)\right) \;=\; 0 \tag{75}$$

**Example:** Find the integral of $\frac{1}{1-z^4}$ about each of the contours $\gamma_0, \gamma_1, \gamma_2, \gamma_4, \gamma_5$ in figure 12.

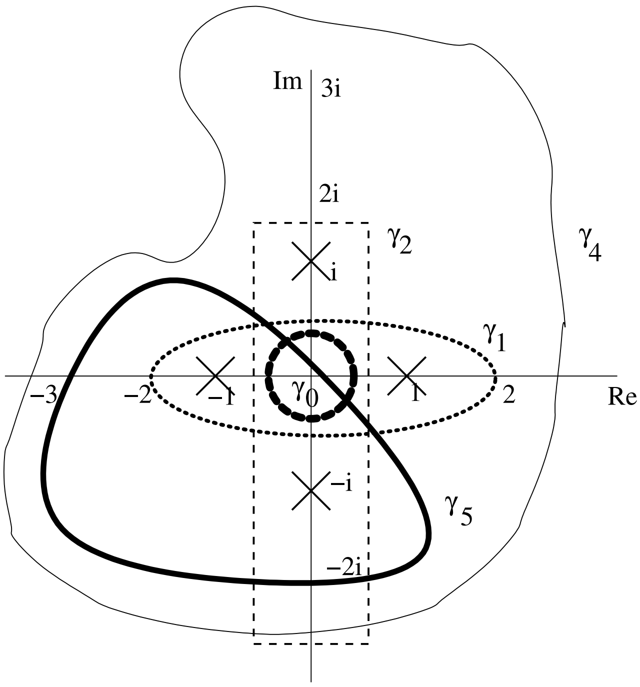



Figure 12: Sketch diagram for $f(z) = \frac{1}{1-z^4}$.

$f(z) = \frac{1}{1-z^4}$ has singularities when $z^4 = 1$: it has four simple poles at $z = 1, -1, i, -i$. This may be shown by considering

$$z^4 \;=\; 1 \;=\; e^{i0} \;=\; e^{i(0+2\pi k)}, \;\; k \in \mathbb{Z}. \tag{76}$$

This gives the roots as:

$$z_k \;=\; \left(e^{i2\pi k}\right)^{\frac{1}{4}} \;=\; e^{i2\pi k/4} = \; e^{i\pi k/2}, \; k = 0,1,2,3. \tag{77}$$

Using additional values of $k$ simply duplicates these roots; a fourth order polynomial has exactly four roots.

Hence the roots are $e^{i\pi 0/2} = e^0 = 1$, $e^{i\pi/2} = i$, $e^{i\pi} = -1$, $e^{i\pi 3/2} = e^{-i\pi/2} = -i$.
Using the "$g/h'$" rule we then get

$$\text{Res}(f : z_k) = \text{Res}\left(\frac{1}{1-z^4} : z_k\right) = \left.\frac{1}{-4z^3}\right|_{z=z_k} = -\frac{1}{4}\,z_k^{-3} = -\frac{z_k}{4z_k^4} = -\frac{z_k}{4}. \tag{78}$$

Explicitly, $\text{Res}(f:1) = -\frac{1}{4}$, $\text{Res}(f:-1) = \frac{1}{4}$, $\text{Res}(f:i) = -\frac{i}{4}$, $\text{Res}(f:-i) = \frac{i}{4}$ so that

$$\begin{aligned}
\int_{\gamma_0} f(z)\,dz &= 0;\\
\int_{\gamma_1} f(z)\,dz &= 2\pi i\left(\tfrac{-1}{4} + \tfrac{1}{4}\right) = 0;\\
\int_{\gamma_2} f(z)\,dz &= 2\pi i\left(\tfrac{-i}{4} + \tfrac{i}{4}\right) = 0;\\
\int_{\gamma_4} f(z)\,dz &= 2\pi i\left(\tfrac{-1}{4} + \tfrac{1}{4} + \tfrac{-i}{4} + \tfrac{i}{4}\right) = 0;\\
\int_{\gamma_5} f(z)\,dz &= 2\pi i\left(\tfrac{1}{4} + \tfrac{i}{4}\right) = \tfrac{\pi}{2}\left(-1+i\right).
\end{aligned} \tag{79}$$

**Example:** Evaluate the integral of $f(z) = \frac{1}{z^2(1+\sin z)}$ about $C(0:1)$; to find any residues first derive the Laurent Series.
The function has simple poles when $\sin z = -1$, but these are not inside $C(0:1)$.
$f(z)$ has a double pole at $z = 0$. Expanding about $z = 0$:

$$1 + \sin z = 1 + z - \frac{z^3}{3!} + \ldots \;\Rightarrow\; \frac{1}{1+\sin z} = 1 - z + \ldots \;\Rightarrow\; \frac{1}{z^2}\frac{1}{1+\sin z} = \frac{1}{z^2} - \frac{1}{z} + \ldots \tag{80}$$

$$\Rightarrow\; c_{-1} = -1 \;\Rightarrow\; \int_{C(0:1)} f(z)dz = 2\pi i \text{Res}(f:0) = -2\pi i. \tag{81}$$

**Principle of the argument** Consider a function $f(z)$ as $z$ travels along a contour $\gamma$. The curve traced by $f(z)$ is called the image of $\gamma$ under $f$ and is denoted by $f(\gamma)$.

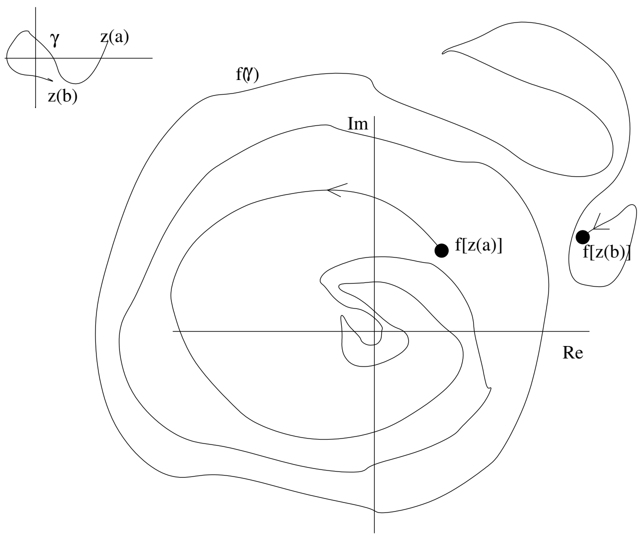



Figure 13: The path $f(\gamma)$ traced by $f(z)$ as $z$ follows the contour $\gamma$.

We define the **Winding Number** W as the number of times $f(z)$ winds around the origin (zero) as $z$ goes along $\gamma$.
If N is the number of zeros of $f(z)$ inside $\gamma$, counted by their multiplicity
and P is the number of poles of $f(z)$ inside $\gamma$, counted by their multiplicity then

$$W = N - P. \tag{82}$$

This is the Principle of the Argument (or **Argument Principle**) and is used in Nyquist theory: see section 7.

**Proof outline:** Consider $f$ near a zero $a$ of order $n$. Then $f(z) = (z-a)^n g(z)$ where $g(z)$ is analytic (and nonzero) at $a$. We have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \left(\frac{n}{z-a}+\frac{g'(z)}{g(z)}\right)dz = \frac{1}{2\pi i}(2\pi i\, n + 0) = n. \text{ (since } g'/g \text{ is analytic)} \tag{83}$$

Therefore, each zero of $f$ contributes its multiplicity to the integral.

Similarly, consider $f(z)$ near a pole $b$ of order $p$. Then $f(z) = (z-b)^{-p}h(z)$ where $h(z)$ is analytic (and nonzero) at $b$. We have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \left(\frac{-p}{z-b}+\frac{h'(z)}{h(z)}\right)dz = \frac{1}{2\pi i}(-2\pi i\, p + 0) = -p. \text{ (since } h \text{ is analytic)} \tag{84}$$

Therefore, each pole of $f$ contributes its multiplicity to the integral, with a minus sign.

The integral $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$ is sometimes referred to as *the zero and pole counting integral.*

On the other hand, we note that $\frac{f'(z)}{f(z)} = \frac{d}{dz}(\ln f(z))$ so that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{d}{dz}(\ln f)\,dz \tag{85}$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{d}{dz}[\ln|f(z)|]d\gamma + \frac{1}{2\pi i}\int_\gamma \frac{d}{dz}[i\arg(f(z))]d\gamma = 0 + \frac{1}{2\pi i} i\, 2\pi W = W \tag{86}$$

Every time $f(z)$ goes round zero once, $\ln|f(z)|$, the real part of $\ln f(z)$, remains unchanged while its imaginary part, the argument of $f(z)$, increases by $2\pi$.

### 2.10 $ML$ estimates

We will often need to show that the integral along a curve vanishes as the curve is taken out to infinity. Consider a function $f(z)$ which is bounded so that it satisfies

$$|f(z)| \le M \tag{87}$$

for some real number $M$.

Let the length of the curve $\gamma$ which we are integrating along be given by $L$; technically this is defined as

$$L = \int_\gamma |dz| = \int_\gamma \left|\frac{dz}{d\theta}\right| d\theta \tag{88}$$

and it is particularly simple to work out when our contours are combinations of straight lines and circular arcs.

Then an upper bound on the absolute value of the integral is

$$\left|\int_\gamma f\,dz\right| \le M\,L. \tag{89}$$

If $ML \to 0$ in some limit then so does the absolute value of the integral. Sometimes this estimate is not sharp enough, that is, the integral vanishes but $\lim ML$ does not. Then we need a better estimate than an $ML$ estimate

**Example:** The integral of $\frac{1}{z^2+a^2}$ $(a \neq 0)$ about $C(0:R) = \{z : |z| = R\}$.

$\gamma = C(0:R)$ is a circle with radius $R$ and so has length $2\pi R$.

The largest value of $|f(z)| = \frac{1}{|z^2+a^2|}$ when $|z| = R \gg |a|$ is given by $\frac{1}{R^2-|a|^2}$ (see Figure 14, and the triangle inequalities below). Hence

$$\left|\int_\gamma f(z)\,dz\right| \le \frac{2\pi R}{R^2-|a|^2}. \tag{90}$$

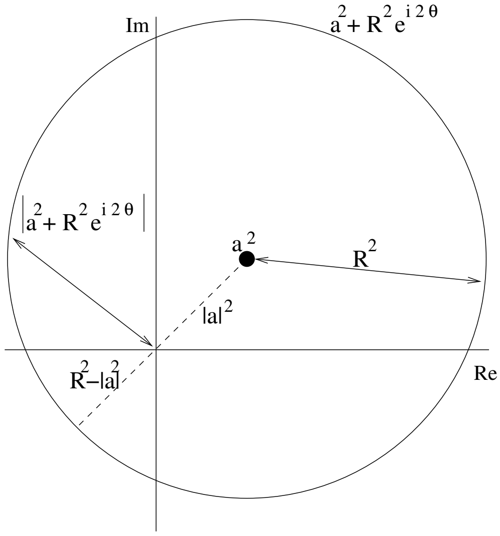



Figure 14: The smallest value of $|z^2 + a^2| = |(Re^{i\theta})^2 + a^2|$ is given by $R^2 - |a|^2$.

If we let the circle grow to cover the entire complex plane, $R \to \infty$, then

$$\frac{2\pi R}{R^2 - |a|^2} \to 0, \quad \text{so that} \quad \lim_{R\to\infty} \int_{\gamma_R} f(z)\, dz = 0. \tag{91}$$

If we let the circle shrink down to a point at the origin, $R \to 0$, then

$$\frac{2\pi R}{R^2 - |a|^2} \to 0, \quad \text{so that} \quad \lim_{R\to 0} \int_{\gamma_R} f(z)\, dz = 0. \tag{92}$$

(Both results are expected from the Residue Theorem: $f(z)$ has two simple poles at $\pm ia$, with corresponding residues $\pm 1/(2ia)$. For $R \gg a$ the contour $C(0 : R)$ contains both poles, and since their residues are equal and opposite in sign, the integral vanishes. For $R \ll a$ the contour does not contain any singularity and the integral vanishes by Cauchy's Theorem.)

To work out the maximum modulus of the integrand it is often useful to apply the **triangle inequalities**. For any complex numbers $z_1$ and $z_2$,

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{93}$$

while for any complex numbers $z_3$ and $z_4$

$$|z_3 - z_4| \ge ||z_3| - |z_4||. \tag{94}$$

(In the latter equality the overall modulus takes the absolute value, as $|z_3| - |z_4|$ can be negative.)

Consider now

$$\frac{(z_1 + z_2)}{(z_3 + z_4)}. \tag{95}$$

The modulus of this expression is maximised when the modulus of the numerator is maximised and the modulus of the denominator is minimised. Therefore, applying the triangle inequalities, we obtain

$$\left|\frac{(z_1 + z_2)}{(z_3 + z_4)}\right| = \frac{|z_1 + z_2|}{|z_3 + z_4|} \le \frac{|z_1| + |z_2|}{||z_3| - |z_4||}, \tag{96}$$

where we used $|-z_4| = |z_4|$.

For example, if

$$f(z) = \frac{1}{z^4 + \alpha}, \tag{97}$$

with $\alpha$ an arbitrary complex number

$$|f(z)| \le \frac{1}{||z^4| - |\alpha||}. \tag{98}$$

**Small circular arcs** Imagine that the function $f(z)$ which you are considering has a simple pole $a$ sitting *on* the contour you would like to integrate around. One way around this is to circumnavigate the point $a$ with a small circular arc—see Figure 15. (In practice we are usually interested in semi-circular arcs or circles.)

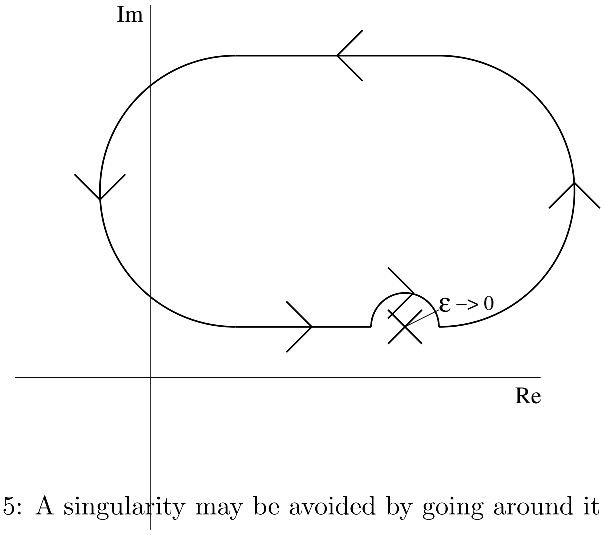



Figure 15: A singularity may be avoided by going around it in an arc.

Our arc with radius $\epsilon$ is given by $\gamma_\epsilon = \{z(\theta) = a + \epsilon e^{i\theta} : \theta_1 < \theta < \theta_2\}$ and should be orientated anticlockwise. Hence

$$\int_{\gamma_\epsilon} f(z)\,dz \ = \int_{\theta_1}^{\theta_2} f\left(a + \epsilon e^{i\theta}\right)\, i\epsilon e^{i\theta}\, d\theta. \tag{99}$$

Because $f(z)$ has a simple pole at $a$, we can write

$$f(z) = g(z) + \frac{\mathrm{Res}(f:a)}{z-a} \tag{100}$$

where $g(z)$ is analytic. Then

$$\begin{aligned}
\int_{\gamma_\epsilon} f(z)\,dz &= \int_{\theta_1}^{\theta_2} \left[g(a+\epsilon e^{i\theta}) + \frac{\mathrm{Res}(f:a)}{a+\epsilon e^{i\theta} - a}\right] i\epsilon e^{i\theta} d\theta \\
&= \int_{\theta_1}^{\theta_2} g(a+\epsilon e^{i\theta})\, i\epsilon e^{i\theta} d\theta + \mathrm{Res}(f:a) \int_{\theta_1}^{\theta_2} i d\theta \qquad (101) \\
&= \epsilon \int_{\theta_1}^{\theta_2} g(a+\epsilon e^{i\theta})\, i e^{i\theta} d\theta + i\,\mathrm{Res}(f:a)\ (\theta_2 - \theta_1)\,.
\end{aligned}$$

Now as $\epsilon \to 0$ the first term $\to 0$, and we obtain

$$\lim_{\epsilon\to 0} \int_{\gamma_\epsilon} f(z)\,dz \ = i\ (\theta_2 - \theta_1)\ \mathrm{Res}(f:a). \tag{102}$$

If $\gamma_\epsilon = C(0:\epsilon)$, given by $\theta_1 = 0$, $\theta_2 = 2\pi$, then this reduces to the Residue Theorem.

If we circumnavigate the singularity with a semicircular arc

$$\gamma_\epsilon = \{z(\theta) = a + \epsilon e^{i\theta} : 0 < \theta < \pi\} \tag{103}$$

then the integral becomes

$$\lim_{\epsilon\to 0} \int_{\gamma_\epsilon} f(z)\,dz \ = i\pi\,\mathrm{Res}(f:a). \tag{104}$$

## 2.11 Multifunctions

The archetype for all other **multi-valued functions** is the multi-valued function ln $z$, and so we consider it as a prototypical example. Recall that

$$\begin{aligned} \ln z &= \ln r &+& \quad i\theta \\ &= \ln |z| &+& \quad i \arg(z) \end{aligned} \tag{105}$$

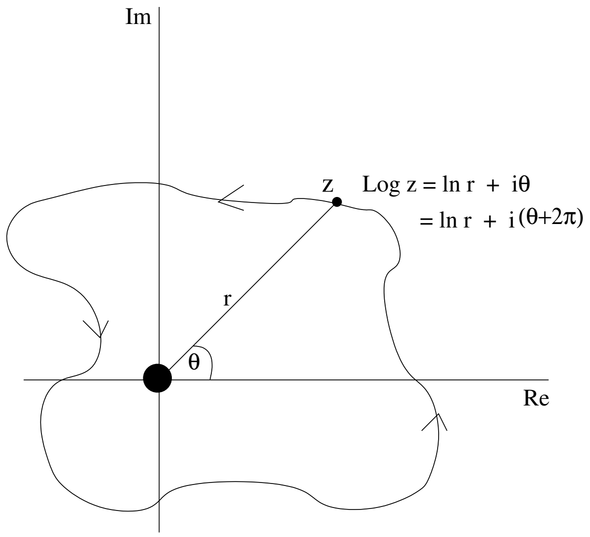



Figure 16: ln $z$ is a multifunction.

If we move $z$ around the origin on a closed curve in an anticlockwise direction, coming back to its starting point, then the value of $z$ is unchanged. Re ln $z = \ln |z|$ also goes back to its starting value, but Im ln $z = \arg z$ has increased by $2\pi$. Therefore ln $z$ is **multi-valued**—it is not a proper function.

**Branch points** Note that ln $z$ is not defined at zero, because $\ln |z|$ is not. The change in Im ln $z$ depends only on how often one goes around $z = 0$, not on the shape of the curve. The point $z = 0$ is called a **branch point** of ln $z$.

The function $z^a$ can be written as

$$z^a = e^{a \ln z} = e^{a(\ln |z| + i \arg z)} = |z|^a e^{ia \arg z}. \tag{106}$$

As $z$ goes around zero, $z^a$ keeps its modulus, but changes its argument by $2a\pi$. Therefore $z = 0$ is a branch point of $z^a$ unless $a$ is an integer. If $a = 1/n$, where $n$ is an integer, then $z^a$ goes back to what it was after one has gone around zero $n$ times. $z^{1/n}$ is called **$n$-valued**; one also says it has $n$ **branches**. For irrational $a$, $z^a$ has infinitely many branches; so does ln $z$.

For example, consider $z^{1/2}$. This can be written as

$$z^{1/2} = |z|^{1/2} e^{\frac{i}{2} \arg z} = r^{1/2} e^{\frac{i}{2}\theta}, \tag{107}$$

where in the latter equality we have switched to polar coordinates. As $\theta \to \theta + 2\pi$

$$z^{1/2} \to r^{1/2} e^{\frac{i}{2}\theta + i\pi} = -r^{1/2} e^{\frac{i}{2}\theta} \tag{108}$$

i.e. the multifunction picks up a minus sign. This multifunction has two branches, since it returns to its original value after winding twice around zero.

One can make more complicated multi-valued functions from the multifunctions given above. For example:
$z^{m/n}$ is $n$-valued; $\sin \sqrt{z}$ is double-valued; $\cos \sqrt{z}$ is single valued (when $z$ goes once around the origin, $\sqrt{z} \to -\sqrt{z}$, but $\cos z$ is even.)

**Branch cuts** To make $f(z)$ single valued we can introduce **branch cuts** into the complex plane. It connects a branch point either to infinity, or to another branch point.

As long as the path $z$ takes (i.e. the contour it follows) does not cross a branch cut, the function $f(z)$ is single valued. If we cross the branch cut in any way, the value of the function jumps. The position of the branch cut is arbitrary, as long as it ends at the branch point (if we think of infinity as one branch point).

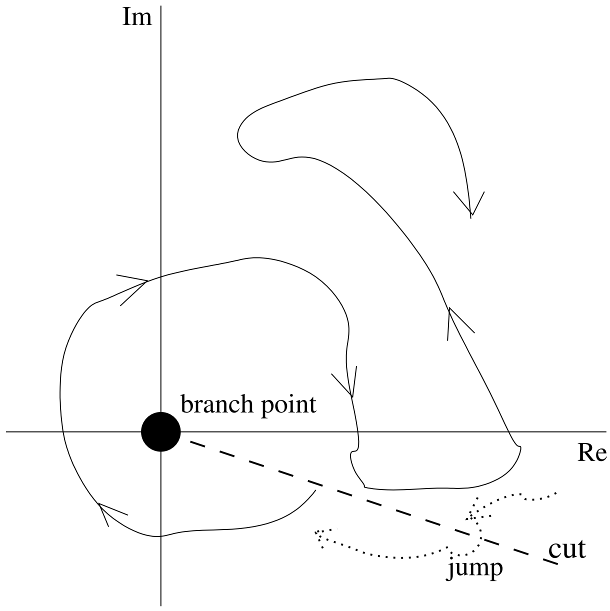



Figure 17: Multifunctions only jump if a branch cut is crossed.

Going to infinity in any direction can be thought of as going to a single "point at infinity". This point is a branch point of, for example, $\ln z$ in the precise sense that zero is a branch point of $\ln \frac{1}{z}$. If we turn infinity into a point, then any branch cut links two branch points. It is sometimes useful to make use of the **extended complex plane**, defined as $\mathbb{C} \cup \infty$.

$\ln z$ has a branch cut going from zero to infinity. One can go to "infinity" in any direction, for example to $\infty$, to $-i\infty$, or to $(3-i)\infty$.

**Example:** $f(z) = \sqrt{z(z-1)}$

$f(z)$ has the two branch points 0 and 1. $f(z)$ is double-valued around either of them.

There exist two possible types of branch cuts: One linking 0 to 1, and the other linking both to infinity (see figure 18). Which one we choose will depend on what problem we are considering.

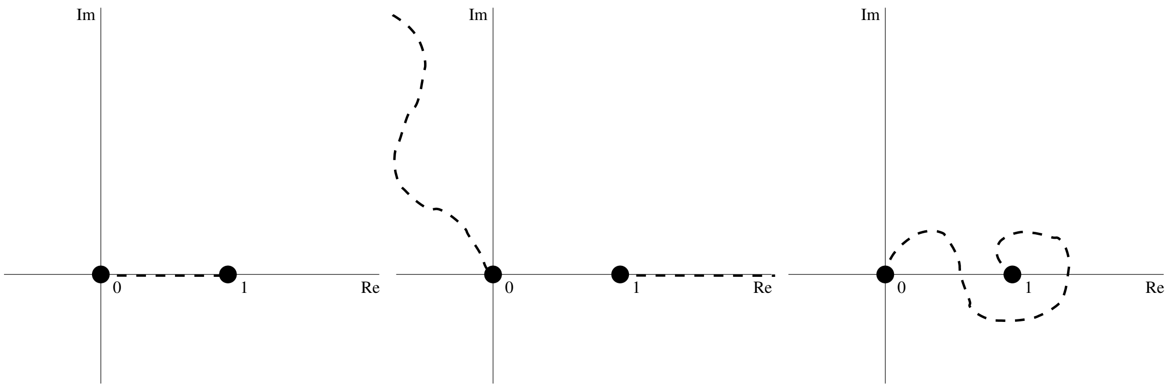



Figure 18: Possible branch cuts (the dashed line) for $f(z) = \sqrt{z(z-1)}$.

Branch cuts define the **domain** of a function. For $\ln z$, the introduction of a branch cuts means we restrict the values that $\theta$ can take. For example:

a branch cut from 0 to $+\infty$ gives the restriction $0 \le \theta < 2\pi$;
a branch cut from 0 to $-\infty$ gives the restriction $-\pi < \theta \le \pi$.

**Contour integration of multifunctions** When integrating multi-valued functions we *must not cross a branch cut*, since then the integral is not uniquely defined. The same is true if we evaluate a contour integral using the residue theorem. We can *deform* the branch cut continuously to avoid the contour that we want (remember its path is arbitrary, only the end points are important). For example, the branch cut of $\ln z$ is sometimes usefully put on the negative real axis.

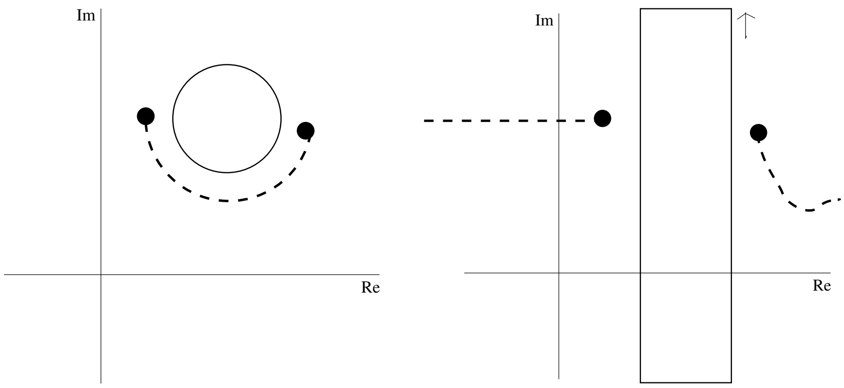



Figure 19: Contours must not cross branch cuts.

**Example:** Integrate $f(z) = \frac{z^{\frac{1}{2}}}{4+z^2}$ over an infinite semicircular contour in the upper half plane. Hence evaluate the real integral $I = \int_0^\infty \frac{x^{1/2}}{4+x^2}dx$

$z^{\frac{1}{2}} = \left(re^{i(\theta+2\pi n)}\right)^{\frac{1}{2}} = r^{\frac{1}{2}}\, e^{i(\frac{\theta}{2}+\pi n)}$ (where $n \in \mathbb{Z}$) gives two distinct values, namely $r^{\frac{1}{2}}\, e^{i\frac{\theta}{2}}$ and $r^{\frac{1}{2}}\, e^{i(\frac{\theta}{2}+\pi)} = -r^{1/2}e^{i\theta/2}$, $\Rightarrow f(z)$ is a multifunction.

$f(z)$ has a branch point at $z = 0$.

$f(z)$ has simple poles at $z = 2i$ and $z = -2i$.

On $\gamma_R$, $z$ may be parameterized by $z(\theta) = Re^{i\theta}$ $(0 \le \theta < \pi)$, so $dz = iRe^{i\theta}d\theta$.

$$\int_{\gamma_R} f(z)\, dz = \int_0^\pi \frac{\sqrt{Re^{i\theta}}}{4+\left(Re^{i\theta}\right)^2} iRe^{i\theta}\, d\theta. \tag{109}$$

Using $ML$ estimates:

$$L \;=\; \pi R, \qquad |f(z)| \;=\; \frac{\left|\sqrt{Re^{i\theta}}\right|}{|4+R^2e^{i2\theta}|} \;<\; \frac{R^{\frac{1}{2}}}{R^2-4} \;=\; M. \tag{110}$$

Letting the semicircle cover the entire upper half of the complex plane:

$$\left|\int_{\gamma_R} f(z)\, dz\right| \;<\; \frac{\pi R^{\frac{3}{2}}}{R^2-4} \;\to\; 0,\; R \;\to\; +\infty. \tag{111}$$

Hence

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\, dz = 0. \tag{112}$$

Similarly, letting $\gamma_\epsilon$ shrink down to a point we have

$$\left|\int_{\gamma_\epsilon} f(z)\, dz\; \right| < \; \frac{\pi\epsilon^{\frac{3}{2}}}{-\epsilon^2+4} \;\to\; 0,\; \epsilon \;\to\; 0. \tag{113}$$

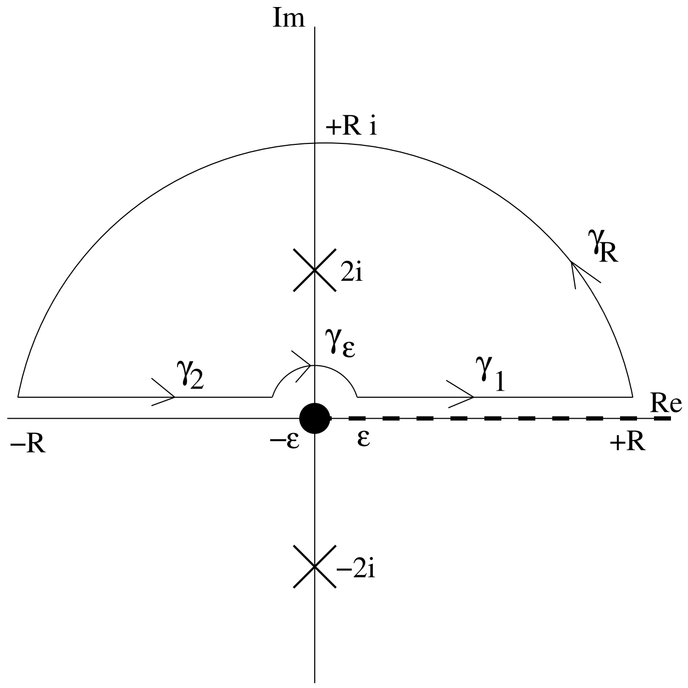



Figure 20: A semicircular contour.

Hence

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} f(z)\,dz = 0. \tag{114}$$

The line $\gamma_1$ may be parameterized by $z = x$, for $\epsilon < x < R$. Hence

$$\int_{\gamma_1} f(z)\,dz \;=\; \int_\epsilon^R \frac{x^{\frac{1}{2}}}{4+x^2}dx. \tag{115}$$

As $\epsilon \to 0$ and $R \to \infty$ this tends to the integral

$$I \;=\; \int_0^\infty \frac{x^{\frac{1}{2}}}{4+x^2}dx. \tag{116}$$

The line $\gamma_2$ may be parameterized by $z = x$, for $-R < x < -\epsilon$. We must be careful now as $x$ is negative:

$$\int_{\gamma_2} f(z)\,dz \;=\; \int_{-R}^{-\epsilon} \frac{x^{\frac{1}{2}}}{4+x^2}dx \;=\; -\int_{-\epsilon}^{-R} \frac{x^{\frac{1}{2}}}{4+x^2}dx \;=\; -\int_\epsilon^R \frac{(-X)^{\frac{1}{2}}}{4+(-X)^2} - dX \;=\; i\int_\epsilon^R \frac{X^{\frac{1}{2}}}{4+X^2}dX. \tag{117}$$

Here in the last two equalities we used the redefinition $X = -x$ with $X$ positive. As $\epsilon \to 0$ and $R \to \infty$ the last equality tends to the integral

$$i\int_0^\infty \frac{X^{\frac{1}{2}}}{4+X^2}dX \;=\; iI, \tag{118}$$

where $I$ was the original integral of interest. (Remember that you can always relabel an integration variable!)

If we consider the entire contour $\gamma$ to be the union of $\gamma_R$, $\gamma_2$, $\gamma_\epsilon$, $\gamma_1$ in the combined limit $\epsilon \to 0$, $R \to \infty$ then the integral along $\gamma$ becomes

$$\begin{aligned}\int_\gamma &= \lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_{\gamma_R} + \int_{\gamma_\epsilon} + \int_{\gamma_1} + \int_{\gamma_2}\right) \\ &= \quad 0 \quad +0 \quad +I \quad +iI \quad = (1+i)I.\end{aligned} \tag{119}$$

On the other hand, we have the Residue Theorem, which tells us that:

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{\substack{\text{singularities } a_k\\ \text{of } f(z)\\ \text{inside } \gamma}} \text{Res}(f:a_k) \;=\; 2\pi i \text{Res}(f:2i)$$
$$= 2\pi i\, \frac{z^{\frac{1}{2}}}{2z}\Bigg|_{z=2i} \;=\; \pi i\, z^{-\frac{1}{2}}\Big|_{z=2e^{i\frac{\pi}{2}}} \;=\; \pi i 2^{-\frac{1}{2}}\, e^{-i\frac{\pi}{4}} \;=\; \pi i \frac{1}{\sqrt{2}}\frac{1-i}{\sqrt{2}} = \frac{\pi}{2}(i+1). \tag{120}$$

To obtain these formulae, note that the residue in this case is most conveniently calculated with what we called earlier the $g/h'$ formula. The function of interest is

$$f(z) = \frac{z^{\frac{1}{2}}}{z^2+4} \tag{121}$$

To calculate the residue, we identify $g(z) = z^{\frac{1}{2}}$ and $h(z) = (z^2+4)$. Then $h'(z) = 2z$ giving the residue used above. We have to be careful when we take the square root: the value of a square root is most easily calculated by writing the complex number in polar form. In the case at hand, we remember the standard relation that $i = \exp(i\pi/2)$.

Equating (119) and (120), we finally obtain

$$I \;=\; \int_0^\infty \frac{x^{\frac{1}{2}}}{4+x^2}dx \;=\; \frac{\pi}{2}. \tag{122}$$

Note that because $I$ is a real integral our result must also be real.

Compare this to the result from real integration:

$$\begin{aligned}\int_0^\infty \frac{x^{\frac{1}{2}}}{4+x^2}dx \;&=\; \left[\frac{1}{4}\ln\left(\frac{2-2\sqrt{x}+x}{2+2\sqrt{x}+x}\right) + \frac{1}{2}\arctan\left(\sqrt{x}-1\right) + \frac{1}{2}\arctan\left(\sqrt{x}+1\right)\right]_0^\infty \\ &=\; \left(\frac{1}{4}\ln 1 \;+\; \frac{1}{2}\frac{\pi}{2} \;+\; \frac{1}{2}\frac{\pi}{2}\right) \;-\; \left(\frac{1}{4}\ln 1 \;+\; \frac{1}{2}\left(-\frac{\pi}{4}\right) \;+\; \frac{1}{2}\left(\frac{\pi}{4}\right)\right) \\ &=\; \frac{\pi}{2} \;-\; 0.\end{aligned} \tag{123}$$

# 3 Evaluation of real integrals by contour integration

Given some real integral $I = \displaystyle\int_a^b F(x)dx$ we can attempt to evaluate it by the following:

**I**. Choose a suitable closed contour $\gamma$ which will yield the real limits $a$ and $b$ (often 0 or $\pm\infty$).

**II**. Choose a suitable complex function $f(z)$ such that it is equal to the desired real function $F(x)$ on the relevant part of the contour.

**III**. Demonstrate, for example by an $ML$ estimate, that the integral of the remaining contour is zero, or evaluate it explicitly.

**IV**. Evaluate the integral on the closed contour by residues and equate.

Definite integrals over the real line can be evaluated by closing the integration "at infinity", for example with a semicircle whose radius goes to $\infty$.

The best approach to problems is to remember which integrals can be calculated, and with what trick, and then to derive the details from scratch. The following is a guide to the most common examples but is by no means exhaustive.

## 3.1 Semicircle contour: for integrals over the whole real line

Examples of functions can be integrated with a semicircle contour (see figure 21) include

$$I \;=\; \int_{-\infty}^\infty \frac{P(x)}{Q(x)}dx \tag{124}$$

$$\text{and} \int_{-\infty}^{\infty} \frac{e^{ikx}P(x)}{Q(x)}dx, \tag{125}$$

where $P$ and $Q$ are polynomials, and $Q(x)$ has no zeroes on the real line. $P/Q$ is called a **rational function**.

The order of the polynomial $Q$ must be at least the order of $P$ *plus two*, in order for the large semicircle contribution to vanish by an $ML$ estimate.

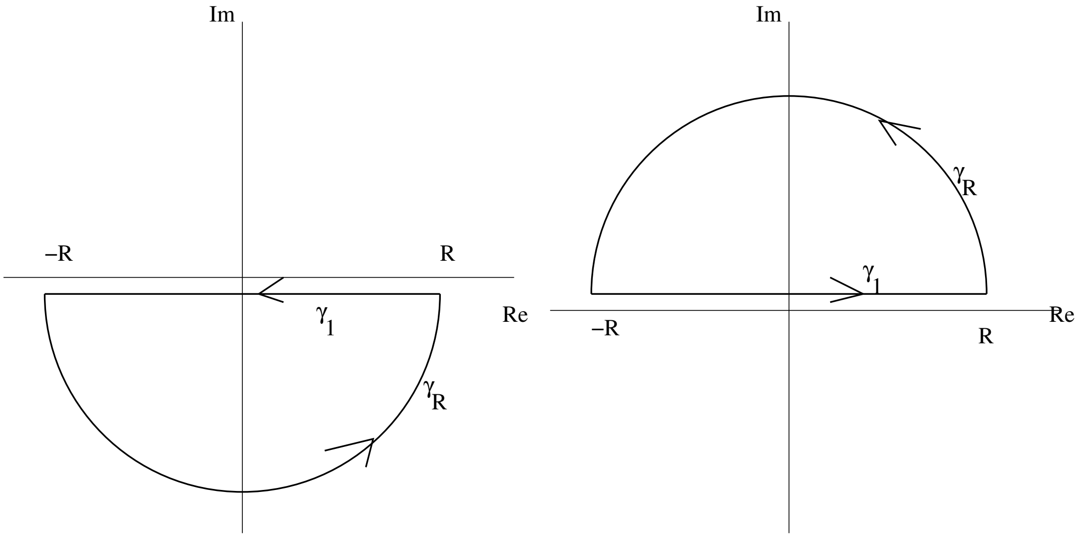



Figure 21: Closing a contour on the real line with a semicircular arc. The left diagram is appropriate for $k < 0$ and the right diagram is appropriate for $k > 0$. Note that we transverse the real line in different directions.

Whether we close the contour above or below depends on the value of the integrand on the semicircle, i.e., where $z = Re^{i\theta}$, as $R \to \infty$.

For integrals of the form (124) we can close the contour above or below.

If $k > 0$ in (125) then we must close the contour above because $e^{ikx}$ is exponentially small in the upper half plane, but exponentially large in the lower half plane. Conversely, if it is $k < 0$ we must close in the lower half.

If $P$ and $Q$ are real polynomials, we can take the real and imaginary parts of this integral to obtain $\cos kx$ and $\sin kx$ in the integrand instead of $e^{ikx}$. For example

$$\int_{-\infty}^{\infty} \frac{\cos kx\, dx}{1+x^2} = \mathrm{Re}\Big(\int_{-\infty}^{\infty} \frac{e^{ikx}\, dx}{1+x^2}\Big). \tag{126}$$

**Example:** Evaluate $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$.

**I** $F(x) = \frac{1}{(1+x^2)^2}$ is of the form (124) so we choose a semicircular contour, which we may close above or below.

**II** The function $f(z) = \frac{1}{(1+z^2)^2}$ is equal to $F(x)$ on $z = x$.

$f(z)$ has singularities at $z = \pm i$ , which do not lie on the contour provided $R > 1$.

**III** Consider the contour $\gamma$ in two parts:

$$\int_{\gamma_1} F(z)\, dz = \int_{-R}^{+R} \frac{1}{(x^2+1)^2} dx \;\to\; I,\; R \to \infty. \tag{127}$$

$$\begin{aligned} \Big|\int_{\gamma_R} F(z)\, dz\Big| &= \Big|\int_0^{\pi} \frac{1}{((Re^{i\theta})^2+1)^2} Re^{i\theta} d\theta\Big| < \int_0^{\pi} d\theta \max_{0\le\theta\le\pi} \left\{ \frac{|R||e^{i\theta}|}{|R^2 e^{i2\theta}+1|^2} \right\} \\ &\le \frac{\pi R}{(R^2-1)^2} \;\to\; 0,\; R \to \infty. \end{aligned} \tag{128}$$

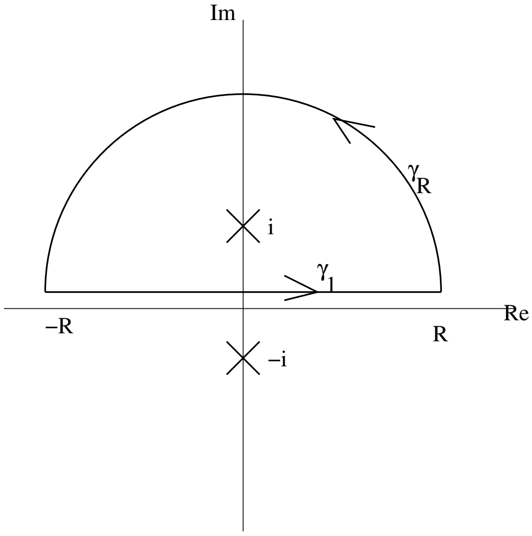



Figure 22: the contour closed above.

Putting these together gives

$$I \;=\; \lim_{R\to\infty} \int_\gamma F(z)\, dz. \tag{129}$$

**IV** $f(z)$ has a double pole at $z = +i$ and a double pole at $z = -i$.
However, only $+i$ is inside $\gamma$.

We need to evaluate the residues of all the singularities *inside* $\gamma$.

$$\mathrm{Res}(f(z) : i) = \lim_{z\to i} \frac{d}{dz}\left((z-i)^2 f(z)\right) = \lim_{z\to i} \frac{d}{dz}\,(z+i)^{-2} \;=\; -2(z+i)^{-3}\big|_{z=i} \;=\; -2(2i)^{-3} \;=\; -\frac{i}{4}. \tag{130}$$

Hence

$$\int_{\gamma(R>1)} F(z)\, dz \;=\; 2\pi i \left(-\frac{i}{4}\right) \;=\; \frac{\pi}{2}. \tag{131}$$

Taking the limit $R \to \infty$ gives $I = \frac{\pi}{2}$, which is real as required.

Although we may conclude, by the deformation theorem, that

$$\int_\gamma F(z)\, dz = \frac{\pi}{2} \quad \text{for all } R \text{ satisfying } R > 1, \tag{132}$$

it is important to remember that $\int_{\gamma_1} = I$ and $\int_{\gamma_R} = 0$ only in the limit $R \to \infty$. For all $1 < R < \infty$ this does not hold; instead we only have

$$\int_{\gamma_1} + \int_{\gamma_R} = \int_\gamma = \frac{\pi}{2}. \tag{133}$$

## 3.2 Fourier transforms

The Fourier transforms of a function may often be evaluated by considering contour integrals which include the real line. The same applies to finding the inverse of a Fourier transform.
**Example:** Find $f(x)$ given $\hat{f}(k) = 2a/(a^2 + k^2)$, where $a > 0$.

$$f(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + k^2}\, e^{ikx}\, dk. \tag{134}$$

This is an integral on the real line of a form we are familiar with: we will use a semicircular contour.

To determine whether to close the contour above or below, we need to find if the exponential goes to zero (rather than blow up) as the imaginary part of $k$ goes to plus or minus zero. (Here we have an *inverse* FT, and so this exponential is $e^{ikx}$, but in a FT it would be $e^{-ikx}$, so be careful.) Since the sign of the complex exponential is determined by the sign of $x$ (remember we are integrating with respect to $k$) , we need to consider $x > 0$ and $x < 0$ separately.

Let us first take $x > 0$ so that we close above (see figure 23).

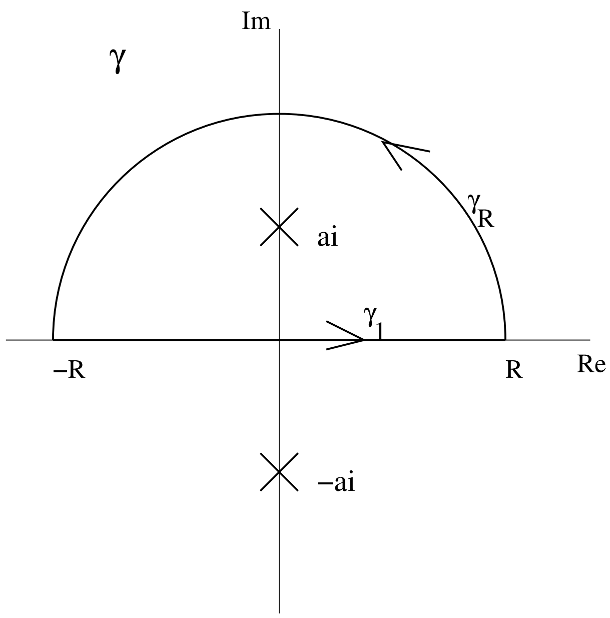



Figure 23: The integral contour for $F(k)$ for $x > 0$.

The integrand given by $F(k) = \frac{2a}{a^2+k^2}e^{ikx}$ has poles when $k^2 = -a^2$ i.e., $k = \pm ai$.

On $\gamma_1$, $k \in \mathbb{R}$ and $-R < k < R$, so

$$\int_{\gamma_1} F(k)dk \;=\; \int_{-R}^{+R} \frac{2a}{a^2+k^2}e^{ikx}dk \;\to\; 2\pi f(x), R \to \infty. \tag{135}$$

On $\gamma_R$, $k = Re^{i\theta}$ for $0 < \theta < \pi$:

$$\left|\int_{\gamma_R} F(k)dk\right| \;<\; \pi R\,\frac{2a}{R^2-a^2}\left|e^{ikx}\right| \;\to 0, R \to \infty, \tag{136}$$

because $|e^{ikx}| = |e^{ix\,\mathrm{Re}(k)}|\,|e^{-x\,\mathrm{Im}(k)}| = e^{-x\,\mathrm{Im}(k)}$ and $\mathrm{Im}(k) = \mathrm{Im}(Re^{i\theta}) = R\sin\theta > 0$ for $\theta \in (0, \pi)$, so that $e^{-x\,R\sin\theta} \to 0, R \to \infty$ for $x > 0$.

This is why we had to specify the sign of $x$ and (for $x > 0$) close in the upper half plane.

In the limit $R \to \infty$ we have:

$$f(x) \;=\frac{1}{2\pi}\int_{\gamma} F(k)\,dk \;=\frac{1}{2\pi}2\pi i \sum_{\substack{\text{singularities } a_k\\ \text{of } F(k)\\ \text{inside } \gamma}} \mathrm{Res}(F : a_k) \tag{137}$$

$$\begin{aligned}\text{for } x > 0,\;\; f(x) &= i\,\mathrm{Res}(\tfrac{2a}{a^2+k^2}e^{ikx} : ai)\\ &= i\left(\tfrac{2a}{2ai}\,e^{i^2ax}\right) \;=\; e^{-ax}.\end{aligned} \tag{138}$$

Remember, this result is only valid for $x > 0$. For $f(x)$ when $x < 0$ we may either repeat the calculations or use symmetry to note that

$$\begin{aligned}\text{for } x < 0,\;\; -f(x) &= i\,\mathrm{Res}(\tfrac{2a}{a^2+k^2}e^{ikx} : -ai)\\ &= i\left(\tfrac{2a}{-2ai}\,e^{-i^2ax}\right) \;=\; -e^{ax}.\end{aligned} \tag{139}$$

For $x < 0$ we close the contour below so that it now contains the singularity $-ia$ and not $+ia$. Here we have minus $f(x)$ on the left hand side because going around semicircular contour closed below in an anticlockwise direction means we integrate $F(k)$ from $+R$ to $-R$.

$$\text{Hence} \quad f(x) \; = \; \begin{cases} e^{-ax}, & x > 0, \\ e^{+ax}, & x < 0, \end{cases} \quad \Rightarrow \; f(x) \; = \; e^{-a|x|}. \tag{140}$$

Because Fourier Transforms are unique we have thereby shown that

$$\mathcal{F}\left[e^{-a|x|}\right] \; = \; \frac{2a}{a^2 + k^2}, \quad (a > 0). \tag{141}$$

Note that this is only valid for $a > 0$ since otherwise the function $f(x)$ would not have the property (198).

## 3.3 Box contour

When the integrand $F(z)$ has an infinite number of poles spaced equally along the imaginary line, we may wish to avoid having to sum over an infinite number of residues. In such cases $F(z)$ often has the property

$$F(z + ia) = CF(z) \tag{142}$$

for a real constant $a$ and a complex constant $C$. In this case we can close the contour from $\infty + ia$ to $-\infty + ia$ (see figure 24), including only one of the poles. The integral along the section from $\infty + ia$ to $-\infty + ia$ would equal $(-C)$ times the integral along the real axis (from $-\infty$ to $\infty$).

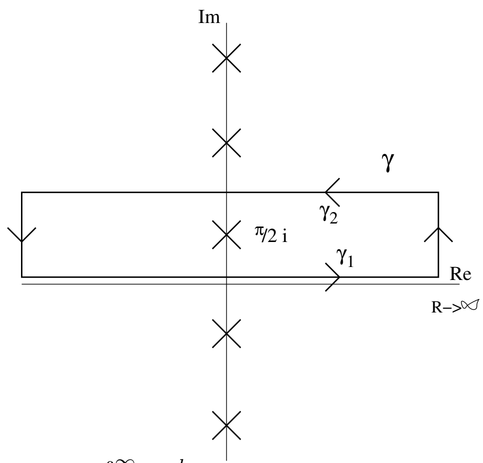



Figure 24: The box contour for $I = \int_{-\infty}^{\infty} \frac{dx}{\cosh x}$, which has poles at $z = (2n+1)\pi i/2$ for all integers $n$. Here we choose $a = \pi$.

Consider the integral

$$I = \int_\gamma \frac{dz}{\cosh z} \tag{143}$$

over the contour shown in figure 24. The function $\frac{1}{\cosh z}$ has poles along the imaginary axis as shown; these poles follow from the zeroes of $\cosh z$. Letting

$$\cosh z = \cosh(x + iy) = \cosh x \cos(y) + i \sinh x \sin(y) \tag{144}$$

we note that the real part vanishes for $y = (2n + 1)\pi/2$, with $n$ an integer or zero, while the imaginary part vanishes either at $y = n\pi$ or $x = 0$. However at $y = n\pi$ the real part does not vanish, so the locations at which both real and imaginary parts vanish are $x = 0$, $y = (2n+1)\pi/2$.

One can use $ML$ estimates to show that the contributions from the vertical parts of the rectangle vanish as the length of the rectangle is taken to be large. Clearly along $\gamma_1$ we obtain

$$\int_{\gamma_1} \frac{dz}{\cosh z} = \int_{-R}^{R} \frac{dx}{\cosh x}. \tag{145}$$

Along $\gamma_2$, $z = x + i\pi$ and hence

$$\cosh z = \cosh(x + i\pi) = \cosh x \cos(\pi) = -\cosh x \tag{146}$$

Thus

$$\int_{\gamma_2} \frac{dz}{\cosh z} = \int_R^{-R} \frac{dx}{(-\cosh x)} = \int_{-R}^{R} \frac{dx}{\cosh x}. \tag{147}$$

(Note that the integration on this part of the contour is in the direction of decreasing $x$.) Therefore

$$\int_\gamma \frac{dz}{\cosh z} = 2 \operatorname{Lim}_{R\to\infty} \int_{-R}^{R} \frac{dx}{\cosh x} = 2I. \tag{148}$$

Now we can compute the integral over the closed contour using the residue theorem; since the residue of the pole is $-i$

$$\int_\gamma \frac{dz}{\cosh z} = 2\pi i(-i) = 2\pi \tag{149}$$

and therefore $I = \pi$.

There are many more functions that can be integrated over the real line. In each case it is important to verify that it is possible to obtain both the estimates (for those parts of the contour not on the line, such as the semicircle with radius $R$) and residues (for singularities inside the contour), since otherwise we will not obtain the (real) value of the integral we require.

## 3.4 Integrals over the half line: using symmetry

If the integral is from 0 to $\infty$, we can trivially extend it to the whole real line only if the integrand is even. For example

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} dx \;\; \text{but} \;\; \int_0^\infty x e^{-x^2} dx \neq \frac{1}{2}\int_{-\infty}^{\infty} x e^{-x^2} dx \; (=0). \tag{150}$$

## 3.5 Integrals over the half line: using a keyhole contour with multifunctions

Although multifunctions complicate matters, this can be used to our advantage.

For a *(single valued)* function $f(z)$ we have, for $z = re^{i\theta}$ :

$$f(re^{i\theta}) \;=\; f\left(re^{i(\theta+2\pi)}\right). \tag{151}$$

This is not so for the multifunction ln $z$ for which

$$\ln\left(re^{i\theta}\right) = \ln r + i\theta, \quad \ln\left(re^{i(\theta+2\pi)}\right) = \ln r + i\left(\theta + 2\pi\right). \tag{152}$$

Consider two contour sections $\gamma_1$ and $\gamma_2$, just above and just below the positive real axis (see figure 25), that is, on either side of a branch cut for ln $z$. Then

$$\text{on } \gamma_1,\; z \approx x = xe^{i0}, \text{ while on } \gamma_2,\; z \approx xe^{i2\pi}. \tag{153}$$

This means that

$$\int_{\gamma_1} f(z)\,\ln z\,dz \;=\; \int_0^\infty f(x)\,\ln x\,dx, \quad \int_{\gamma_2} f(z)\,\ln z\,dz \;=\; \int_\infty^0 f(x)\,(\ln x + 2\pi i)\;dx. \tag{154}$$

Combining these together we have

$$\int_{\gamma_1+\gamma_2} f(z)\,\ln z\,dz \;=\; \int_0^\infty \left[f(x)\,\ln x \,-\, f(x)\,(\ln x + 2\pi i)\right]\,dx \;=\; \int_0^\infty -2\pi i\, f(x)\,dx. \tag{155}$$

We can therefore evaluate real integrals of the form

$$\int_0^\infty f(x)dx \tag{156}$$

(where $f(z)$ is a single-valued function) using

$$\int_0^\infty f(x)dx = -\frac{1}{2\pi i}\int_\gamma f(z)\,\ln\,z\,dz \tag{157}$$

where $\gamma$ is a "keyhole" contour—see figure 25. One must check, though, that both the large and small circle give vanishing contributions as $\epsilon \to 0$ and $R \to \infty$, and so it is best to derive this trick from scratch rather than memorize it. Effectively, Eq. (157) allows us to calculate real integrals $\int_0^\infty$ (or $\int_{-\infty}^0$) using the Residue Theorem—very convenient indeed!

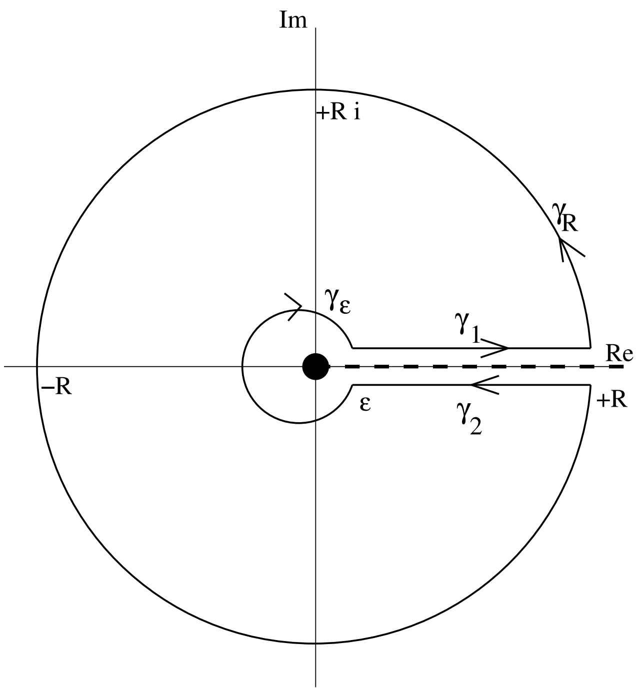



Figure 25: The keyhole contour for a branch cut $[0, -\infty)$: to cover the complex plane we let $R \to \infty$ and $\epsilon \to 0$.

The keyhole contour (but without the $\ln z$ trick) also works for

$$\int_0^\infty x^a f(x)\,dx \tag{158}$$

where $a$ is *not* an integer and where $f(z)$ is single-valued. This is because $x^a$ is a multifunction.

**Example:** Evaluate $I = \displaystyle\int_0^\infty \frac{dx}{1+x^4}$.

We will try the keyhole contour with

$$f(z) = \frac{\ln\,z}{1+z^4}. \tag{159}$$

$f(z)$ has a branch point at $z = 0$, so we use the branch cut $z = x$ $(x > 0)$.

$f(z)$ has simple poles when $z^4 = -1$: these are given by $a_k = e^{i(2k+1)\frac{\pi}{4}}$ for $k = 0, 1, 2, 3$. Explicitly:

$$a_0 = e^{i\frac{\pi}{4}} = \frac{+1+i}{\sqrt{2}}\ ,\ a_1 = e^{i\frac{3\pi}{4}} = \frac{-1+i}{\sqrt{2}}\ ,\ a_2 = e^{i\frac{5\pi}{4}} = \frac{-1-i}{\sqrt{2}}\ ,\ a_3 = e^{i\frac{7\pi}{4}} = \frac{+1-i}{\sqrt{2}}. \tag{160}$$

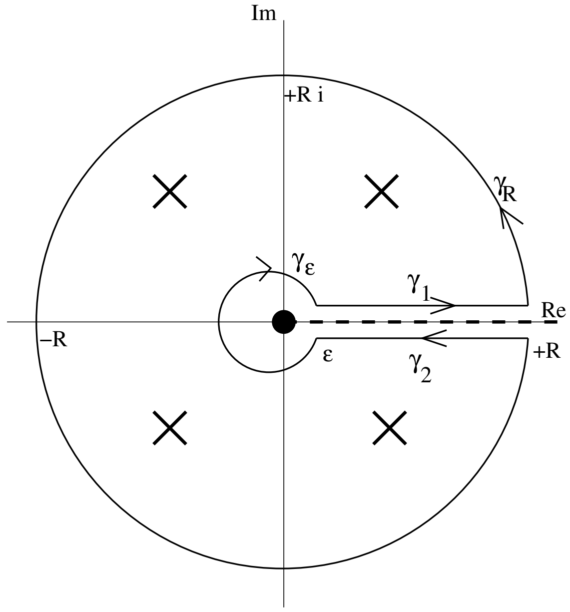



Figure 26: By sketching our contour we note that we require $\epsilon$<1 and $R$> 1 for all four singularities to be inside the contour.

Next draw a keyhole contour $\gamma$ as in figure 26, and consider its four pieces:

$$\int_\gamma f(z)\,dz \;=\; \int_{\gamma_1} f(z)\,dz + \int_{\gamma_R} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_\epsilon} f(z)\,dz$$

$$= \int_0^\infty \frac{\ln x}{1+x^4}\,dx + \int_0^{2\pi} \frac{\ln R + i\theta}{1+R^4e^{i4\theta}} iRe^{i\theta}\,d\theta + \int_\infty^0 \frac{\ln x + 2\pi i}{1+x^4}\,dx + \int_{2\pi}^0 \frac{\ln\epsilon + i\theta}{1+\epsilon^4 e^{i4\theta}} i\epsilon e^{i\theta}\,d\theta. \quad (161)$$

Now

$$\left|\int_0^{2\pi} \frac{\ln R + i\theta}{1+R^4e^{i4\theta}} iRe^{i\theta}\,d\theta\right| \;<\; (2\pi R)\left(\frac{\ln R + 2\pi}{R^4-1}\right) \;\to 0,\ R\to\infty, \quad (162)$$

and similarly

$$\left|-\int_0^{2\pi} \frac{\ln\epsilon + i\theta}{1+\epsilon^4e^{i4\theta}} i\epsilon e^{i\theta}\,d\theta\right| \;<\; \frac{2\pi\epsilon\,(\ln\epsilon + 2\pi)}{\epsilon^4-1} \;\to 0,\ \epsilon\to 0. \quad (163)$$

Hence

$$\int_\gamma f(z)\,dz \;=\; \int_{\gamma_1+\gamma_2} \frac{\ln\, z}{1+z^4}\,dz \;=\; -2\pi i I \quad (164)$$

[using Eq. 157]. But we also have

$$\int_\gamma f(z)\,dz \;= 2\pi i \sum_{\substack{\text{singularities } a_k \\ \text{inside } \gamma}} \mathrm{Res}(f : a_k). \quad (165)$$

Combining (164) and (165) gives

$$I = -\sum_{k=0}^{3} \mathrm{Res}(f(z) : a_k). \quad (166)$$

Evaluating all four residues:

$$\begin{aligned}
\text{Res}(f(z):a_k) =& \quad \text{Res}\left(\frac{\ln z}{1+z^4} : e^{i\frac{(2k+1)\pi}{4}}\right) \\
=& \quad \frac{\ln\left(e^{i\frac{(2k+1)\pi}{4}}\right)}{4\left(e^{i\frac{(2k+1)\pi}{4}}\right)^3} \qquad \text{using “}g/h^{'}\text{ rule”} \\
=& \quad \frac{i\frac{(2k+1)\pi}{4}}{4\left(e^{i\frac{(2k+1)\pi}{4}}\right)^4} e^{i\frac{(2k+1)\pi}{4}} \\
=& \quad \frac{i\frac{(2k+1)\pi}{4}}{-4} e^{i\frac{(2k+1)\pi}{4}} \qquad \text{(using } a_k^4+1=0\text{)} \\
=& \quad -i\frac{(2k+1)\pi}{16} e^{i\frac{(2k+1)\pi}{4}} \\
=& \quad i\frac{(2k+1)\pi}{16} e^{i\frac{(2k+5)\pi}{4}} \qquad \text{(using } -1=e^{i\pi}\text{)}
\end{aligned} \tag{167}$$

Finally, substituting these into (166):

$$\begin{aligned}
-I =& \quad \text{Res}\,(f:a_0) \quad + \quad \text{Res}\,(f:a_1) \quad + \quad \text{Res}\,(f:a_2) \quad + \quad \text{Res}\,(f:a_3) \\
=& \quad i\tfrac{1\pi}{16}e^{i\frac{5\pi}{4}} \quad + \quad i\tfrac{3\pi}{16}e^{i\frac{7\pi}{4}} \quad + \quad i\tfrac{5\pi}{16}e^{i\frac{9\pi}{4}} \quad + \quad i\tfrac{7\pi}{16}e^{i\frac{11\pi}{4}} \\
=& \quad i\tfrac{1\pi}{16}\,(-1-i)\,/\sqrt{2} + i\tfrac{3\pi}{16}\,(+1-i)\,/\sqrt{2} + i\tfrac{5\pi}{16}\,(+1+i)\,/\sqrt{2} + i\tfrac{7\pi}{16}\,(-1+i)\,/\sqrt{2} \\
=& \quad i\pi\,\left[(-1+3+5-7)+i(-1-3+5+7)\right]\,/16\sqrt{2} \\
=& \quad i\pi\,[8i]\;/\,16\sqrt{2} \\
I =& \quad \tfrac{\pi}{2\sqrt{2}} \qquad \text{(which is } \textit{real} \text{ as required).}
\end{aligned} \tag{168}$$

## 3.6 Segment contour

In some cases it may be useful to consider a “pie” segment contour, as in figure 27. This works for example for an integral of the type

$$\int_0^\infty f(x^n)dx, \tag{169}$$

where $n$ is an integer and $f(z)$ is single-valued. In this case we can choose the contour such that the integrals along the two straight sections of the “pie” are the same, up to an overall factor. For large $n$, this will be simpler than the keyhole contour because fewer residues need to be calculated.

Consider, for example, the integral

$$\int_0^\infty \frac{dx}{1+x^7}. \tag{170}$$

The singularities of $\frac{1}{1+z^7}$ are plotted in figure 27. A segment contour can be drawn to include only a single residue.

**Example:** Evaluate (once again) $I = \displaystyle\int_0^\infty \frac{1}{1+x^4}dx$.

Consider $f(z) = \frac{1}{1+z^4}$, which has four singularities, one in each quadrant.

We will use a segment contour covering only the first quadrant—see figure 28.

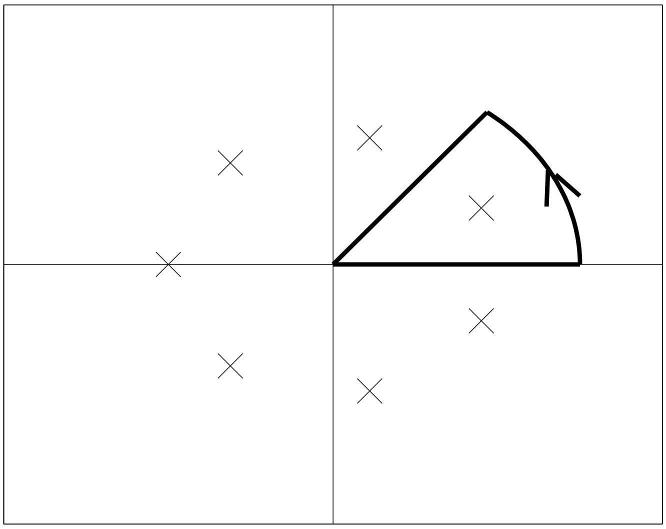


Figure 27: The segment contour for $\int_0^\infty \frac{dx}{1+x^7}$.

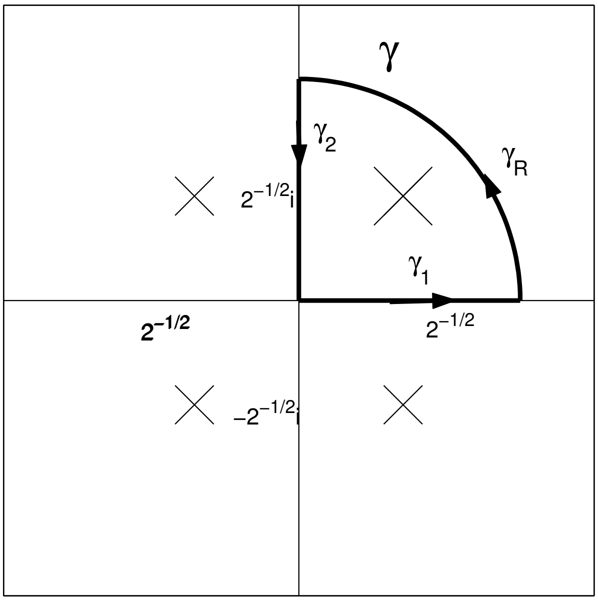



Figure 28: The segment contour for $\int_0^\infty \frac{dx}{1+x^4}$.

On $\gamma_1$, $z = x$, $0 < x < R$:

$$\int_{\gamma_1} f(z)\,dz = \int_0^R f(x)\,dx \to I,\ R \to \infty. \tag{171}$$

On $\gamma_2$, $z = iy$, $0 < y < R$:

$$\int_{\gamma_2} f(z)\,dz = \int_R^0 f(iy)\,idy = -\int_0^R \frac{1}{1+(iy)^4}\,idy = -i\int_0^R \frac{1}{1+y^4}\,dy = \to -iI,\ R \to \infty. \tag{172}$$

On $\gamma_R$, $z = Re^{i\theta},\ 0 < \theta < \frac{\pi}{2}$ :

$$\left|\int_{\gamma_R} f(z)\,dz\right| \;=\; \left|\int_0^{\frac{\pi}{2}} \frac{1}{1+R^4e^{i4\theta}}\,iRe^{i\theta}\,d\theta\right| \;<\; \frac{\pi}{2}R\frac{1}{R^4-1} \;\to 0,\ R \to \infty. \tag{173}$$

The only singularity of $f(z)$ inside the contour $\gamma$ is $a_0 = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$.

$$\text{Res}(f : a_0) = \frac{1}{4a_0^3} = \frac{a_0}{4a_0^4} = -\frac{a_0}{4} = -\frac{1+i}{4\sqrt{2}} \tag{174}$$

Note that we cannot use the residue calculated in (167) as this was for a different function, namely $\frac{1}{1+z^4} \ln z$. Although we are considering the same real integral we are using a different contour with a different integrand.

Using the Residue Theorem we now have

$$I(1-i) \;=\; \int_\gamma f(z)\,dz \;=\; 2\pi i \text{Res}(f(z) : a_0) = \; 2\pi i \frac{-1-i}{4\sqrt{2}} = \frac{\pi(-i+1)}{2\sqrt{2}} \tag{175}$$

giving $I = \frac{\pi}{2\sqrt{2}}$ as before.

## 3.7 Keyhole segment contour

With integrals of the form

$$\int_0^\infty x^a f(x^n)\,dx \tag{176}$$

where $n$ is an integer, $a$ is real or complex and not an integer, and $f(z)$ is single-valued we could of course use the keyhole contour, but it is easier to use only a keyhole segment—see figure 29 for the example

$$\int_0^\infty \frac{x^{1/2}}{1+x^6}\,dx. \tag{177}$$

As usual one can use $ML$ estimates to show that the circular arcs do not contribute to the integral.

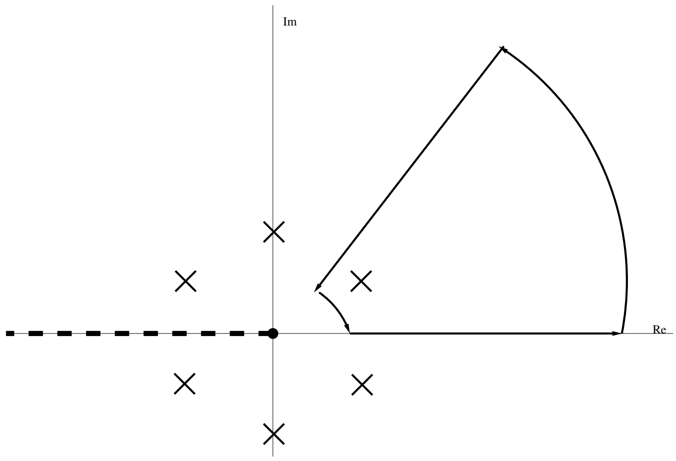



Figure 29: The keyhole segment contour for $\int_0^\infty \frac{x^{1/2}\,dx}{1+x^6}$. The dashed line is the branch cut.

Therefore the integral over the closed contour is equal to the contributions from the two straight lines:

$$\int_\gamma \frac{z^{1/2}}{1+z^6}\,dz = \int_\epsilon^R \frac{x^{1/2}}{1+x^6}\,dx + \int_R^\epsilon \frac{(xe^{i\alpha})^{1/2}}{1+(xe^{i\alpha})^6} d(xe^{i\alpha}), \tag{178}$$

where $\alpha$ is the angle of the segment. If we choose $6\alpha = 2\pi$, namely $\alpha = \pi/3$, then the second term becomes

$$e^{i\pi/2} \int_R^\epsilon \frac{x^{1/2}}{1+x^6} = -i \int_\epsilon^R \frac{x^{1/2}}{1+x^6} \tag{179}$$

and is therefore proportional to the integral which we wish to compute. Computing the contour integral with the residue theorem as usual, we find that

$$\int_\gamma \frac{z^{1/2}}{1+z^6}\,dz = \frac{\pi}{3\sqrt{2}}(1-i) \tag{180}$$

and therefore

$$\int_0^\infty \frac{x^{1/2}}{1+x^6} = \frac{\pi}{3\sqrt{2}}. \tag{181}$$

A semicircle is, of course, just a special case of a segment, and a **keyhole semicircle** contour may sometimes be appropriate. In all of the above examples one can do well with just the semicircle and keyhole contours; however, segment/box contours may be easier and more efficient, as they enclose fewer singularities.

## 3.8 Integrals over periodic functions

We can integrate periodic functions over one full period, which is simplified by a change of variable so that the period is $2\pi$. To calculate functions of the form

$$\int_0^{2\pi} f(\cos\theta,\ \sin\theta,\ \frac{\tan\frac{\theta}{2}}{\sin\theta+\cos 3\theta},\ \ldots)\ d\theta \tag{182}$$

(recall tan has period $\pi$, not $2\pi$) we use as a contour the **unit circle**:

$$C(0:1)\ =\ \{z(\theta)=e^{i\theta}\ :\ 0\le\theta<2\pi\ \}. \tag{183}$$

Remember that we need only consider singularities inside the unit disk.

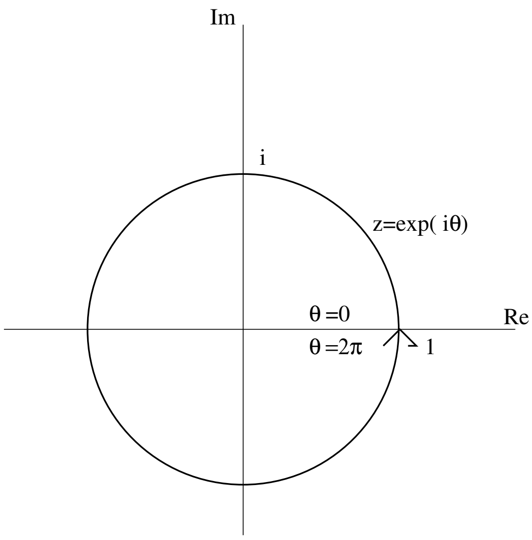



Figure 30: The unit circle contour for periodic functions: $z=e^{i\theta}\Rightarrow dz=ie^{i\theta}$.

It follows from

$$z=e^{i\theta}=\cos\theta+i\sin\theta,\quad \frac{1}{z}=e^{-i\theta}=\cos\theta-i\sin\theta, \tag{184}$$

that

$$\cos\theta=\frac{1}{2}(z+z^{-1}),\quad \sin\theta=\frac{1}{2i}(z-z^{-1}), \tag{185}$$

and similarly $\cos 2\theta=\frac{1}{2}(z^2+z^{-2})$, etc.

If $f$ is a rational function of $\sin\theta$ and $\cos\theta$, then it is also a rational function of $z$.

**Example:** Evaluate $I=\displaystyle\int_0^{2\pi}\frac{d\theta}{5-4\cos\theta}$.

The integrand is a periodic function so we use the contour $\gamma=C(0:1)$.

The choice of $f(z)$ is given by transforming $\theta\to z$:

$$\cos\theta\ =\ \frac{1}{2}\left(z+\frac{1}{z}\right),\ \ d\theta\ =\ \frac{dz}{iz} \tag{186}$$

$$\frac{d\theta}{5-4\cos\theta} = \frac{1}{5-\frac{4}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz} \;=\; \frac{-1}{2i}\frac{dz}{z^2-\frac{5}{2}z+1} \quad \text{so use } f(z) \;=\; \frac{1}{z^2-\frac{5}{2}z+1}. \tag{187}$$

Then:

$$\int_\gamma f(z)\,dz \;=\; \int_{C(0:1)} \frac{dz}{z^2-\frac{5}{2}z+1} \;=\; -2i\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta} \;=\; -2iI. \tag{188}$$

Evaluate $\int_\gamma f(z)\,dz$ using the Residue Theorem.

$$z^2-\frac{5}{2}z+1 \;=\; (z-2)\left(z-\frac{1}{2}\right) \tag{189}$$

so $f(z)$ has a simple pole at $z=2$ and a simple pole at $z=\frac{1}{2}$.

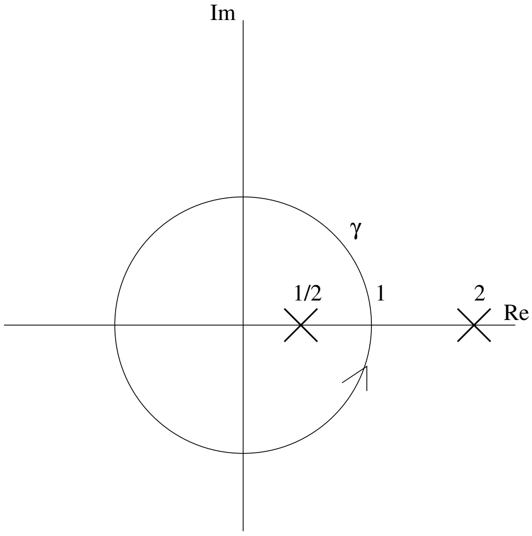



Figure 31: The unit circle contour for $f(z) = \frac{1}{z^2-\frac{5}{2}z+1}$.

$$\begin{aligned}
\int_\gamma f(z)\,dz = &\; 2\pi i \operatorname{Res}\left(f : \tfrac{1}{2}\right) \qquad \text{(since only } \tfrac{1}{2} \text{ inside } \gamma\text{)} \\
= &\; 2\pi i \lim_{z\to\frac{1}{2}} (z-\tfrac{1}{2}) f(z) \\
&= 2\pi i \; \left.\frac{1}{z-2}\right|_{z=\frac{1}{2}} \\
&= 2\pi i \; \frac{1}{-\frac{3}{2}} \\
&= -\frac{4\pi i}{3}
\end{aligned} \tag{190}$$

Equating we have $-2iI \;=\; \frac{4\pi}{3}i$ which gives $I = \; +\frac{2\pi}{3}$.

# 4 Fourier transforms

## 4.1 Derivation of the Fourier transform from the complex Fourier series

You have already covered the complex Fourier series in Year 2. If $f(x)$ is a periodic function with period $L$, that is $f(x+L) = f(x)$, then we can write it as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x}, \qquad k_n := \frac{2\pi n}{L}, \tag{191}$$

where the complex Fourier coefficients $c_n$ are given by

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-ik_nx}\,dx. \tag{192}$$

Note the integral could be taken over any other full period, for example from 0 to $L$. If $f(x)$ is real, then $c_n = c_{-n}^*$, where the star denotes the complex conjugate.

[You can verify (192) directly by changing $n$ to $m$ and then substituting the expression (191) for $f(x)$. After integrating, you get $c_m = c_n$, as required.]

From the complex Fourier series we can obtain the Fourier transform as a limit. Define

$$\hat{f}(k_n) := c_n L, \qquad \Delta k := k_{n+1} - k_n = \frac{2\pi}{L}. \tag{193}$$

With this notation, (191) and (192) become

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{f}(k_n)e^{ik_nx}\,\Delta k, \tag{194}$$

$$\hat{f}(k_n) = \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)e^{-ik_nx}\,dx. \tag{195}$$

So far, the function $\hat{f}(k)$ is defined only at the discrete points $k_n$. Now as we take the limit $L \to \infty$, we have $\Delta k \to 0$ and we can think of it as $dk$ under an integral, that is $\sum \Delta k \to \int dk$. At the same time, $\hat{f}(k)$ is then defined on the continuous real line. We have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dk, \tag{196}$$

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx. \tag{197}$$

(Note that in MATH3084/6162 I have a different convention from the one in MATH3083/6163.) When $f(x)$ is real, then $\hat{f}(k) = \hat{f}(-k)^*$ (the complex conjugate).

## 4.2 Complex Fourier transforms

**Fourier transforms**, also known as complex Fourier transforms, are used for real functions $f(x)$ with $x \in \mathbb{R}$; the parameter $x$ is usually viewed as a spatial coordinate.

If $f(x)$ is a piecewise continuously differentiable function for $-\infty < x < \infty$ and if it is integrable, meaning that

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty, \tag{198}$$

then the complex Fourier transform of $f(x)$ is defined to be

$$\mathcal{F}[f(x)] = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx. \tag{199}$$

By the Riemann-Lebesgue lemma, if $f(x)$ is integrable then $\hat{f}(k)$ vanishes at infinity.

If $\hat{f}(k)$ is also integrable inverse of this transform is given by

$$\mathcal{F}^{-1}\left[\hat{f}(k)\right] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dk. \tag{200}$$

By the Riemann-Lebesgue lemma, if $f(x)$ is integrable then $\hat{f}(k)$ vanishes at infinity.

The Fourier transform is useful for PDE problems on an infinite domain with fall-off boundary conditions: if we represent a function $f(x)$ as an inverse FT as in (200), for an integrable $\hat{f}(k)$, we know that $f(x) \to 0$ as $|x| \to \infty$.

The Fourier transform, on the other hand, is useful for problems associated with infinite domains.

**Properties** Fourier transforms have the following basic properties:

$$\mathcal{F}\left[af(x)+bg(x)\right] = a\hat{f}(k)+b\hat{g}(k) \tag{201}$$

$$\mathcal{F}\left[f(x-a)\right] = e^{-ika}\hat{f}(k) \tag{202}$$

$$\mathcal{F}\left[f(ax)\right] = \frac{1}{|a|}\hat{f}(k/a) \tag{203}$$

$$\mathcal{F}\left[xf(x)\right] = i\frac{d}{dk}\hat{f}(k) \tag{204}$$

$$\mathcal{F}\left[f^{(m)}(x)\right] = (ik)^m\hat{f}(k) \tag{205}$$

$$\mathcal{F}\left[f*g\right] := \mathcal{F}\left[\int_{-\infty}^{\infty} f(y)g(x-y)dy\right] = \hat{f}(k)\hat{g}(k) \tag{206}$$

Most of these properties place conditions upon the function $f(x)$. These conditions are usually satisfied if $f(x)$ and its derivatives are integrable and differentiable.

The Plancherel theorem (sometimes also called the Parseval theorem) says that if $f(x)$ is square-integrable, then so is $\hat{f}(k)$, and the two obey

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2\,dk. \tag{207}$$

## 4.3 Proofs

The shifting theorem (202) can be proved by a change of variable:

$$\mathcal{F}\left[f(x-a)\right] = \int_{-\infty}^{\infty} f(x-a)e^{-ikx}dx = \int_{-\infty}^{\infty} f(X)e^{-ik(X+a)}dX = e^{-ika}\hat{f}(k), \tag{208}$$

where we set $X = x-a$. For the product theorem (203) we again change variable:

$$\mathcal{F}\left[f(ax)\right] = \int_{-\infty}^{\infty} f(ax)e^{-ikx}dx = \int_{-\infty}^{\infty} f(X)e^{-ikX/a}\frac{dX}{a} = \frac{1}{a}\hat{f}(k/a) \tag{209}$$

where now $X = ax$ and we assume $a$ is positive. (If $a$ is negative we obtain a modulus sign as given in (203).)

Now consider (204)

$$\mathcal{F}\left[xf(x)\right] = \int_{-\infty}^{\infty} xf(x)e^{-ikx}dx = i\frac{d}{dk}\left[\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\right] = i\frac{d\hat{f}(k)}{dk}, \tag{210}$$

where we have assumed that the integration and differentiation commute.

The transform of derivatives (205) is proved using integration by parts:

$$\mathcal{F}\left[f^{(m)}(x)\right] = \int_{-\infty}^{\infty} f^{(m)}(x)e^{-ikx}\,dx = \left[f^{(m-1)}(x)e^{-ikx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(-ik)f^{(m-1)}(x)e^{-ikx}\,dx, \tag{211}$$

and the boundary term vanishes provided that $f^{(m-1)}(x)$ vanishes as $x \to \pm\infty$. Integrating by parts repeatedly we thence obtain

$$\mathcal{F}\left[f^{(m)}(x)\right] = (ik)^m \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx. \tag{212}$$

The convolution theorem is proven as for the Laplace convolution theorem, see Sec. 5.1 below.

## 4.4 Fourier transform pairs

One can easily show that if the Fourier transform of the (abstract) function $f$ is the (abstract) function $g$, then the Fourier transform of $g$ is $f$, just multiplied by $2\pi$ and with its argument reversed. As a formula

$$\mathcal{F}f(x) = g(k) \qquad \Leftrightarrow \mathcal{F}g(x) = 2\pi f(-k). \tag{213}$$

This is another way of saying that the Fourier transform is its own inverse, up to a factor of $2\pi$ and changing one sign.

For even or odd functions, the Fourier transform is its own inverse, up to a factor of $2\pi$ and a sign. It can be shown that

$$\begin{aligned} \textit{Even}\text{: if } f(x) &= \phantom{-}f(-x) \text{ then } \mathcal{F}\left[\mathcal{F}[f(x)]\right] = \mathcal{F}\left[\hat{f}(k)\right] = \phantom{-}2\pi f(x) \\ \textit{Odd}\text{: if } f(x) &= -f(-x) \text{ then } \mathcal{F}\left[\mathcal{F}[f(x)]\right] = \mathcal{F}\left[\hat{f}(k)\right] = -2\pi f(x) \end{aligned} \tag{214}$$

Fourier transforming an even function twice brings it back to $2\pi$ times that function,
Fourier transforming an odd function twice brings it back to $-2\pi$ times that function.

## 4.5 The Dirac delta function

The Dirac delta function has the formal properties

$$\delta(x-y) = 0 \qquad x \neq y \tag{215}$$

and

$$\int_{-\infty}^{\infty} f(y)\delta(x-y)dy = f(x) \tag{216}$$

which for $f(y) = 1$ implies

$$\int_{\infty}^{\infty} \delta(x-y)dy = 1. \tag{217}$$

(Strictly speaking, the Dirac delta function is not a function: it is a what is called a distribution, also called a generalised function. However it can be defined as the limit of a function and was introduced by Cauchy and Poisson in this way.)

From the definition of the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk \tag{218}$$

we deduce that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)e^{-iky}dy\right]e^{ikx}dk \tag{219}$$

and hence

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{ik(x-y)}dkdy \tag{220}$$

if we assume we can exchange the orders of integration. Then if

$$\int_{-\infty}^{\infty} e^{ik(x-y)}dk = 2\pi\delta(x-y) \tag{221}$$

we obtain

$$f(x) = \int_{-\infty}^{\infty} f(y)\delta(x-y)dy, \tag{222}$$

as required. We can rewrite the identity as

$$\int_{-\infty}^{\infty} e^{ikx}dk = 2\pi\delta(x), \tag{223}$$

implying that the Fourier transform of a constant function is proportional to the Dirac delta function.

## 4.6 The $\delta$-function as a limit of sharply peaked functions

Consider any integrable function $h(x)$ whose integral over the real line is one, that is

$$\int_{-\infty}^{\infty} h(x)\,dx = \int_{-\infty}^{\infty} h(x)e^{-i\cdot 0\cdot x}\,dx = \hat{h}(0) = 1. \tag{224}$$

Then define, for all $\epsilon > 0$, the functions

$$h_\epsilon(x) := \frac{1}{\epsilon}h\left(\frac{x}{\epsilon}\right). \tag{225}$$

One can easily show that

$$\hat{h}_\epsilon(k) = \hat{h}(\epsilon k). \tag{226}$$

In particular, this means that all $\hat{h}_\epsilon(x)$ still have integral one. The classic example of such a family of functions is

$$f_\epsilon(x) := \frac{e^{-\left(\frac{x}{2\epsilon}\right)^2}}{2\epsilon\sqrt{\pi}}, \tag{227}$$

which has Fourier transform

$$\hat{f}_\epsilon(k) = e^{-(\epsilon k)^2}. \tag{228}$$

Note the Fourier transform of a narrow Gaussian is a wide Gaussian, and vice versa. The Gaussian has the interesting property that it is, up to normalisation and changes of argument, its own Fourier transform.

We now have that for any *smooth* integrable function (test function) $\phi(x)$,

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} h_\epsilon(x)\phi(x)\,dx = \phi_0. \tag{229}$$

In this sense, we can informally write that

$$\lim_{\epsilon\to 0} h_\epsilon(x) = \delta(x) \tag{230}$$

and so we have that

$$\hat{\delta}(k) = \lim_{\epsilon\to 0}\hat{h}_\epsilon(k) = \hat{h}(0) = 1, \tag{231}$$

that is, the Fourier transform of the $\delta$-function is the unit function. This statement is very non-rigorous, because the $\delta$-"function" is not really a function, and the unit function is not integrable and so does not really have an (inverse) Fourier transform.

Recall that our formula for the inverse Fourier transform holds precisely if the Fourier transform of the unit function is the $\delta$-function, as we showed above in Sec. 4.5. What was non-rigorous about the argument there is our assumption that we can interchange the order of integration.

## 4.7 Fourier transforms in two or more dimensions

We may extend the Fourier transform pair to a function of two variables to obtain

$$\hat{f}(k_1,k_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(k_1x+k_2y)]dxdy \tag{232}$$

and correspondingly

$$f(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(k_1,k_2)\exp[i(k_1x+k_2y)]dk_1dk_2. \tag{233}$$

More generally, letting $\mathbf{x}$ denote a vector with $n$ Cartesian components and $\mathbf{k}$ similarly denote an $n$-dimensional vector

$$\hat{f}(\mathbf{k}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,dx_1\cdots dx_n \tag{234}$$

with

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\hat{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,dk_1\cdots dk_n. \tag{235}$$

## 4.8 Application of Fourier transforms to partial differential equations

**Partial Differential Equations** (PDEs) concern functions of more than one variable.

In many practical examples we need to consider a function $u(x,t)$ where $x \in (a,b)$ and $t \geq 0$, are time and space (in one dimension). For example,

$$u_t + u^3 = u_{xx} - \sin 8x, \tag{236}$$

where subscripts denote partial derivatives (e.g., $\frac{\partial u}{\partial x} = u_x$). To solve such equations we require both **boundary condition(s)** (BCs) and **initial condition(s)** (ICs):

The BCs $u(a,t)$ and $u(b,t)$ tell us what is happening at the spatial boundary, for all values of time.

The ICs $u(x,0)$ and $u_t(x,0)$ tell us what is happening initially, at all points in our spatial domain. (See figure 32.)

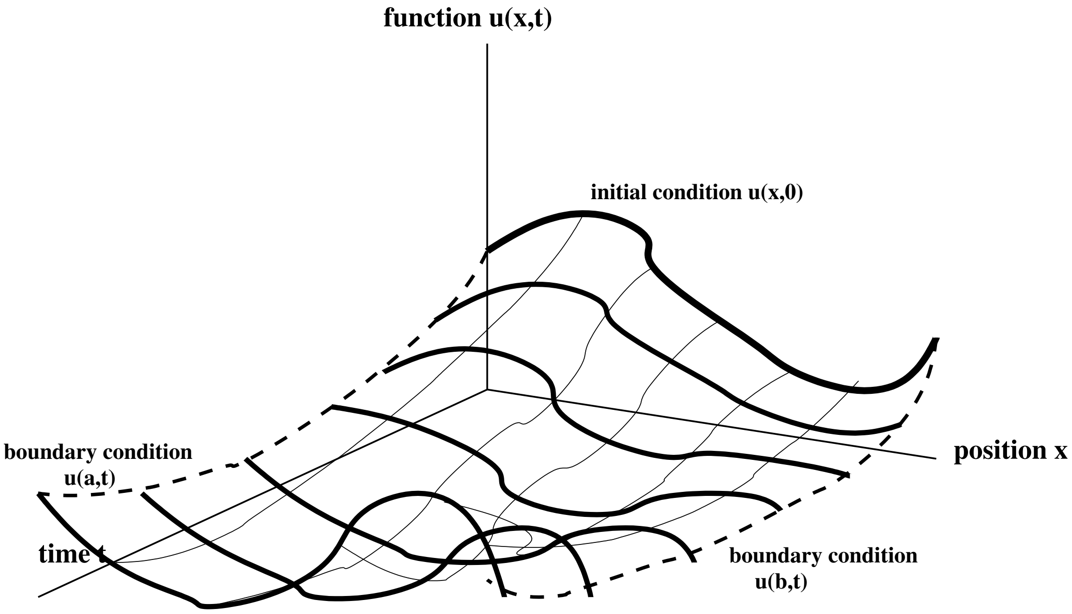



Figure 32: An example of function $u$ in time and one dimensional space.

For a function $f(x,t)$ where $x \in (-\infty, \infty)$ and $t \in [0, \infty)$,

$$\text{if } f(x,t) \to 0 \text{ as } x \to \pm\infty \text{ then} \tag{237}$$

the Fourier transform $\hat{f}(k,t)$ of $f$ with respect to $x$ has the following property:

$$\mathcal{F}\left[f_x(x,t)\right] = ik\hat{f}(k,t). \tag{238}$$

This results can be extended for higher order derivatives, with each higher order derivative in $x$ requiring an additional condition, e.g.,

$$\text{if } f(x,t) \to 0 \text{ and } f_x(x,t) \to 0 \text{ as } x \to \pm\infty, \text{ then} \tag{239}$$

$$\mathcal{F}\left[f_{xx}(x,t)\right] = -k^2\hat{f}(k,t). \tag{240}$$

Time derivatives are unaffected by integration with respect to space and so:

$$\mathcal{F}\left[f_t(x,t)\right] = \hat{f}_t(k,t), \qquad \mathcal{F}\left[f_{tt}(x,t)\right] = \hat{f}_{tt}(k,t). \tag{241}$$

In many cases a second order PDE in $f(x,t)$ can be transformed to a second order ODE in $\hat{f}(k,t)$, which can usually be solved more easily than the original PDE. The latter is an ODE in the sense that only $t$ derivatives but no $x$ or $k$ derivatives remain.

**Example:** Solve $u_t = u_x - u$ with BC $u \to 0$ as $|x| \to \infty$ and IC $u(x,0) = 1/(1+x^2)$; you may use the result

$$\mathcal{F}\left[1/(1+x^2)\right] = \pi e^{-|k|}. \tag{242}$$

**I** Transform Eq. using the fall-off boundary condition:
$u_t = u_x - u \;\Rightarrow\; \hat{u}_t = ik\,\hat{u} - \hat{u} \;\Rightarrow\; \frac{d\hat{u}}{dt} = (ik-1)\,\hat{u}$.
Transform initial condition: $u(x,0) = 1/(1+x^2) \;\Rightarrow\; \hat{u}(k,0) = \pi e^{-|k|}$ using (242).

**II** Obtain homogeneous solutions for transform equation: $\hat{u}(k,t) = A(k)\,e^{(ik-1)t}$.
Substituting initial condition into solution: $\pi e^{-|k|} = A(k)\,e^0 \Rightarrow A(k) = \pi e^{-|k|} \Rightarrow$

$$\hat{u}(k,t) = \pi e^{-|k|}\,e^{(ik-1)t}. \tag{243}$$

**III** Invert $\hat{u}(k,t)$ to get $u(x,t)$.

$$\begin{aligned} u(x,t) &= \pi e^{-t}\mathcal{F}^{-1}\left(e^{-|k|}e^{itk}\right) \\ &= \pi e^{-t}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|}e^{itk}e^{ikx}\,dk \end{aligned} \tag{244}$$

The modulus sign in the integrand suggests we should consider $k > 0$ and $k < 0$ separately:

$$\begin{aligned} 2e^t\,u(x,t) &= \int_0^\infty e^{-k}e^{i(t+x)k}\,dk + \int_{-\infty}^0 e^k e^{i(t+x)k}\,dk \\ &= \int_0^\infty e^{(-1+i(t+x))k}\,dk + \int_{-\infty}^0 e^{(+1+i(t+x))k}\,dk \\ &= \left[\frac{e^{(-1+i(t+x))k}}{-1+i(t+x)}\right]_0^\infty + \left[\frac{e^{(1+i(t+x))k}}{1+i(t+x)}\right]_{-\infty}^0 \\ &= \left(0 - \tfrac{1}{-1+i(t+x)}\right) + \left(\tfrac{1}{1+i(t+x)} - 0\right) \\ &= \frac{-(1+i(t+x)) + (-1+i(t+x))}{(1+i(t+x))\,(-1+i(t+x))} \\ &= \frac{-2}{i^2(t+x)^2-1} = \frac{2}{1+(t+x)^2}. \end{aligned} \tag{245}$$

Hence $u(x,t) = e^{-t}/(1+(t+x)^2)$.

**Example:** Invert $\pi e^{-|k|}\,e^{(ik-1)t}$ by inspection.

$$\begin{aligned} \pi e^{-|k|} &= \mathcal{F}\left(\tfrac{1}{1+x^2}\right) && \text{using (242)} \\ \pi e^{-|k|}\,e^{itk} &= \mathcal{F}\left(\tfrac{1}{1+(x+t)^2}\right) && \text{using (202)} \\ \pi e^{-|k|}\,e^{itk}\,e^{-t} &= \mathcal{F}\left(\tfrac{e^{-t}}{1+(x+t)^2}\right) && e^{-t} \text{ independent of } k \end{aligned} \tag{246}$$

**Example:** Solve the wave equation $\frac{1}{c^2}f_{tt} = f_{xx}$ with BCs $f, f_x \to 0$ as $|x| \to \infty$ and ICs $f(x,0) = g(x)$, $f_t(x,0) = 0$.

Transform equation and ICs:

$$\frac{d^2\hat{f}}{dt^2} + c^2k^2\hat{f} = 0, \tag{247}$$

$$\hat{f}(k,0) = \hat{g}(k) \quad \text{and} \quad \hat{f}_t(k,0) = 0. \tag{248}$$

Solve and impose ICs:

$$\hat{f} = A(k)\cos ckt + B(k)\sin ckt \tag{249}$$

$$-ckA\sin 0 + ckB\cos 0 = 0 \;\Rightarrow\; B(k) = 0, \text{ and then } \hat{g}(k) = A(k)\cos 0 \Rightarrow A(k) = \hat{g}(k). \tag{250}$$

So:

$$\hat{f}(k,t) \;=\; \hat{g}(k)\,\cos(ckt). \tag{251}$$

Now $\cos(ckt) = \frac{1}{2}\left(e^{ickt} + e^{-ickt}\right)$, so we may write

$$\hat{f}(k,t) \;=\frac{1}{2}\left(\hat{g}(k)\,e^{ictk} \;+\; \hat{g}(k)\,e^{-ictk}\,\right). \tag{252}$$

Using (202) we then get

$$f(x,t) \;=\frac{1}{2}\left(g(x+ct) \;+\; g(x-ct)\,\right). \tag{253}$$

In the examples above, Fourier transforms were applied to partial differential equations, with the transforms being taken along infinite spatial directions.

## 4.9 Fourier sine and cosine transforms

For functions $f(x)$ which are only defined on the half line $0 \le x < \infty$ we may define the **cosine transform**,

$$\mathcal{F}_c\left[f(x)\right] = \hat{f}_c(k) = \int_0^\infty f(x)\;\cos kx\,dx, \tag{254}$$

and the **sine transform**,

$$\mathcal{F}_s\left[f(x)\right] = \hat{f}_s(k) = \int_0^\infty f(x)\;\sin kx\,dx. \tag{255}$$

These transforms are unique, so that either one contains all the information necessary to reconstruct $f(x)$ when inverted.

It can easily be shown from the inversion formula for the complex Fourier transform that

$$f(x) \quad=\quad \frac{2}{\pi}\int_0^\infty \hat{f}_c(k)\cos kx\,dk \quad=\quad \frac{2}{\pi}\int_0^\infty \hat{f}_s(k)\sin kx\,dk. \tag{256}$$

Both the sine and cosine transforms are their own inverse, up to a factor of $\frac{\pi}{2}$.

**Example:** Find $\mathcal{F}_s\left[\frac{x}{1+x^4}\right]$.

We let $f(x) = \frac{x}{1+x^4}$ and note that $f(x)$ is an odd function, so that (letting $X = -x$)

$$\hat{f}_s(k) = \int_0^\infty \frac{x}{1+x^4}\sin kx\,dx = \int_0^{-\infty} \frac{-X}{1+X^4}(-\sin kX)\;-dX = \int_{-\infty}^0 \frac{X}{1+X^4}(\sin kX)\,dX. \tag{257}$$

Combining gives

$$\hat{f}_s(k) = \frac{1}{2}\int_{-\infty}^\infty \frac{x}{1+x^4}\sin kx\,dx. \tag{258}$$

Using $e^{ikx} = \cos kx + i\sin kx$ gives

$$\hat{f}_s(k) = \frac{1}{2}\;\mathrm{Im}\left(\int_{-\infty}^\infty \frac{x}{1+x^4}\,e^{ikx}\,dx\right). \tag{259}$$

We can solve this using a semicircular contour as before, closed in the upper half plane for $k > 0$ and in the lower half plane for $k < 0$.

On $\gamma_R$ $z = Re^{i\theta}$, $\theta \in (0,\pi)$ so

$$\left|\int_{\gamma_R} F(z)\,dz\right| \;\le\; (\pi\,R)\left(\frac{R}{R^4-1}\right) \;\to\; 0,\; R \;\to\; \infty. \tag{260}$$

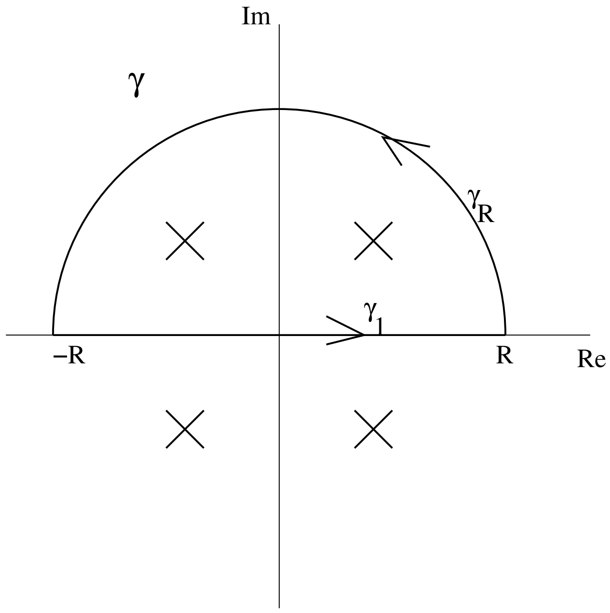



Figure 33: The integral contour for $F(z) = \frac{z}{1+z^4}\, e^{ikz}$ for $k > 0$.

On $\gamma_1$ $z = x$, $x \in (-R, R)$ so

$$\int_{\gamma_1} F(z)\, dz \;=\; \int_{-R}^{+R} \frac{x}{1+x^4} e^{ikx}\, dx \;\rightarrow\; \int_{-\infty}^{\infty} \frac{x}{1+x^4} e^{ikx}\, dx,\; R \;\rightarrow\; \infty. \tag{261}$$

Applying the Residue Theorem we have

$$\hat{f}_s(k) = \frac{1}{2}\,\mathrm{Im}\Big[2\pi i \sum_{\substack{\text{singularities } a_n \\ \text{of } F(z) \\ \text{inside } \gamma}} \mathrm{Res}(F : a_n)\Big] \;= +\pi\ \mathrm{Re}\Big[\sum_{\substack{\text{singularities } a_n \\ \text{of } F(z) \\ \text{inside } \gamma}} \mathrm{Res}(F : a_n)\Big]. \tag{262}$$

The singularities of $F(z)$ are given by $1 + z^4 = 0$:

$$a_n \;=\; e^{i\frac{(2n+1)\pi}{4}}, \qquad\qquad n = 0, 1, 2, 3. \tag{263}$$

These are all simple poles, but only

$$a_0 \;=\; e^{i\frac{\pi}{4}} \;=\; \frac{1+i}{\sqrt{2}} \qquad\text{and}\qquad a_1 \;=\; e^{i\frac{3\pi}{4}} \;=\; \frac{-1+i}{\sqrt{2}} \tag{264}$$

are inside $\gamma$.

$$\Rightarrow \mathrm{Res}\,(F(z) : a_n) \;=\; \frac{a_n\, e^{ika_n}}{4a_n^3} \;=\; \frac{a_n^2\, e^{ika_n}}{4a_n^4} \;=\; -\frac{a_n^2}{4} e^{ika_n}. \tag{265}$$

Hence

$$\begin{aligned} \hat{f}_s(k) &= \pi\mathrm{Re}\Big[\ \mathrm{Res}(F(z) : a_0) \;+\; \mathrm{Res}(F(z) : a_1)\ \Big] \\ &= -\tfrac{\pi}{4}\mathrm{Re}\Big[\ e^{i\frac{\pi}{2}} e^{i\frac{k}{\sqrt{2}} - \frac{k}{\sqrt{2}}} \;+\; e^{i\frac{3\pi}{2}} e^{-i\frac{k}{\sqrt{2}} - \frac{k}{\sqrt{2}}}\ \Big] \\ &= -\tfrac{\pi}{4} e^{-\frac{k}{\sqrt{2}}}\,\mathrm{Re}\Big[\ i e^{i\frac{k}{\sqrt{2}}} \;-\; i e^{-i\frac{k}{\sqrt{2}}}\ \Big] \\ &= \frac{\pi}{2} e^{-\frac{k}{\sqrt{2}}} \sin\frac{k}{\sqrt{2}} \qquad\qquad (k > 0) \end{aligned} \tag{266}$$

For $k<0$ we similarly obtain

$$\hat{f}_s(k) = -\frac{\pi}{2} e^{\frac{k}{\sqrt{2}}} \sin\frac{-k}{\sqrt{2}} \qquad (k<0) \tag{267}$$

Hence,

$$\hat{f}_s(k) = \frac{\pi}{2} e^{-\frac{|k|}{\sqrt{2}}} \sin\frac{k}{\sqrt{2}}. \tag{268}$$

## 4.10 Odd and even extensions

If $f(x)$ is a function defined only for $0 \le x < \infty$, we cannot immediately define its Fourier transform. Often we may wish to use techniques only applicable to functions defined on the whole real line. To achieve this we may extend the function: this is usually done with either an **even extension**:

$$f^{\text{even}}(x) = \begin{cases} f(x), & 0 \le x < \infty, \\ f(-x), & -\infty < x < 0; \end{cases} \tag{269}$$

or an **odd extension**:

$$f^{\text{odd}}(x) = \begin{cases} f(x), & 0 \le x < \infty, \\ -f(-x), & -\infty < x < 0. \end{cases} \tag{270}$$

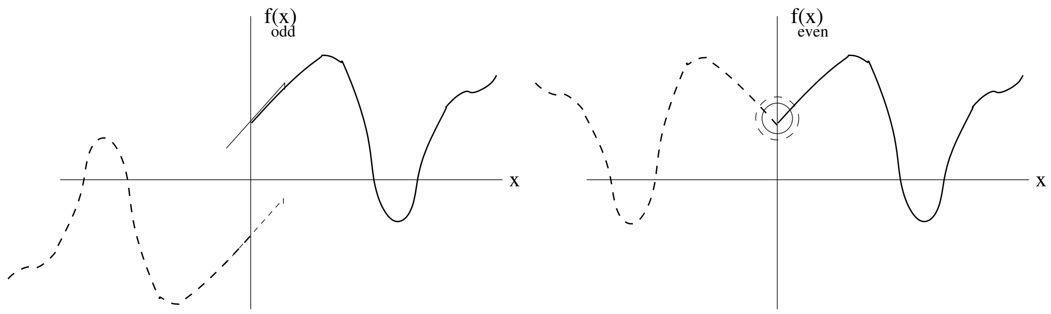



Figure 34: Odd and even extensions of a function defined on the half line.

Note that: an even extension preserves the value of $f(x)$ at $x=0$;
an odd extension preserves the gradient $f_x(x)$ at $x=0$.

Extended functions are defined on the whole real line, so we can calculate their Fourier transform. A brief calculation shows that

$$\begin{aligned}
\mathcal{F}\left[f^{\text{even}}(x)\right] &= \int_{-\infty}^{\infty} f^{\text{even}}(x)\, e^{-ikx}\, dx \\
&= \int_{0}^{\infty} f^{\text{even}}(x)\, e^{-ikx}\, dx + \int_{-\infty}^{0} f^{\text{even}}(x)\, e^{-ikx}\, dx \\
&= \int_{0}^{\infty} f(x)\, e^{-ikx}\, dx - \int_{0}^{-\infty} f(-x)\, e^{-ikx}\, dx \\
&= \int_{0}^{\infty} f(x)\, e^{-ikx}\, dx + \int_{0}^{\infty} f(X)\, e^{+ikX}\, dX \qquad (271) \\
&= 2\int_{0}^{\infty} f(x)\, \frac{\left(e^{ikx} + e^{-ikx}\right)}{2}\, dx \\
&= 2\int_{0}^{\infty} f(x)\, \cos kx dx \\
&= 2\mathcal{F}_c\left[f(x)\right].
\end{aligned}$$

Similarly, the Fourier transform of an odd extension is given by

$$\hat{f}^{\text{odd}}(k) \;=\; -2i\,\hat{f}_s(k); \qquad\qquad \text{(note the additional factor } -i). \tag{272}$$

Our choice of sine or cosine transform will depend upon the boundary conditions of the PDE problem we want to solve—more on that in the next section.

## 4.11 Application of sine and cosine transforms to PDEs

Let $f(x,t)$ be defined for $0 \le x < \infty$, $t > 0$. If $f$ and $f_x \to 0$ as $x \to \infty$, we can use integration by parts with respect to $x$ twice to find the sine transform of $f_{xx}(x,t)$ in terms of the sine transform of $f(x,t)$:

$$\begin{aligned}\mathcal{F}_s\,[f_{xx}] &= \int_0^\infty f_{xx}\sin kx\,dx \\ &= [f_x \sin kx]_0^\infty - k\int_0^\infty f_x \cos kx\,dx \\ & \qquad\qquad\qquad\qquad (\text{using } f_x \to 0,\ x\to\infty) \\ &= -\,[kf\cos kx]_0^\infty - k^2\int_0^\infty f\sin kx\,dx \\ & \qquad\qquad\qquad\qquad (\text{using } f \to 0,\ x\to\infty) \\ &= kf(0,t) - k^2\hat{f}_s(k,t).\end{aligned} \tag{273}$$

Similarly we can show for the cosine transform of $f_{xx}$ that

$$\mathcal{F}_c\,[f_{xx}] = -f_x(0,t) - k^2\hat{f}_c(k,t). \tag{274}$$

The multiplication with $-k^2$ is just as in the complex Fourier transform, but now there are also boundary terms reminiscent of those in the Laplace transform.

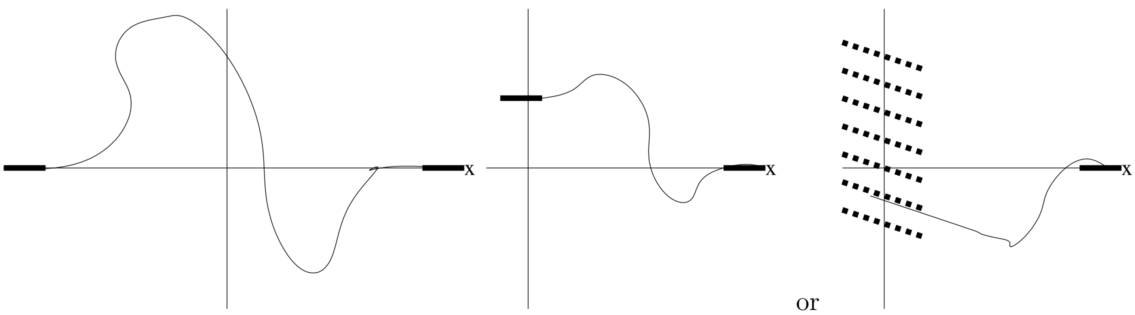



Figure 35: Boundary conditions at $-\infty$ and $+\infty$, or at 0 and $\infty$.

If we know $f(0,t)$ then we use the sine transform to get an ODE in $t$,
If we know $f_x(0,t)$ then we use the cosine transform to get an ODE in $t$.

It is important to note that the application of sine or cosine transforms is only useful in PDEs without first derivatives: If $f \to 0$ as $x \to \infty$ then

$$\begin{aligned}\mathcal{F}_s\,[f_x] &= \int_0^\infty f_x\sin kx\,dx \\ &= [f\sin kx]_0^\infty - k\int_0^\infty f\cos kx\,dx \\ &= -k\hat{f}_c(k,t),\end{aligned} \tag{275}$$

so that sine transform of $f_x$ brings in the cosine transform of $f$ and vice versa; this is of little use as $\hat{f}_c(k,t)$ and $\hat{f}_s(k,t)$ are fundamentally different functions of $k$.

**Example:** Use the Fourier cosine transform to solve $u_{xx} = c\,u_t$ $(c > 0)$ for $x > 0$, $t > 0$, with $u(x,0) = e^{\frac{-x^2}{2}}$ and where $u_x(0,t) = 0$; $u(x,t) \to 0$ as $x \to \infty$; and $u_x(x,t) \to 0$ as $x \to \infty$.
You may use the result

$$\mathcal{F}\left[e^{-\frac{x^2}{2}}\right] = \sqrt{2\pi}e^{-\frac{k^2}{2}}. \tag{276}$$

By using property (203), namely that $\mathcal{F}\left[f(a\,x)\right] = \frac{1}{|a|}\hat{f}(k/a)$ we can easily derive the more general result

$$\mathcal{F}\left[e^{-\alpha\frac{x^2}{2}}\right] = \mathcal{F}\left[e^{-\frac{(\sqrt{\alpha}\,x)^2}{2}}\right] = \sqrt{\frac{2\pi}{\alpha}}e^{-\frac{1}{\alpha}\frac{k^2}{2}}, \quad \alpha > 0. \tag{277}$$

Our result is restricted to $\alpha > 0$ because here $\sqrt{\alpha} = a \in \mathbb{R}$.

We write $\hat{u} = \mathcal{F}_c\,[u]$, so that

$$\begin{aligned} -k^2\,\hat{u} - u_x(0,t) &= c\,\hat{u}_t \qquad \text{(using BCs at } \infty) \\ -k^2\,\hat{u} &= c\,\hat{u}_t \qquad \text{(using BC at 0)} \end{aligned} \tag{278}$$

$$\hat{u}_t = -\frac{k^2}{c}\hat{u} \Rightarrow \hat{u}(k,t) = \hat{u}(k,0)\,e^{-\frac{k^2}{c}t}.$$

$$\begin{aligned} \hat{u}(k,0) &= \int_0^\infty u(x,0)\,\cos kx\,dx \\ &= \frac{1}{2}\text{Re}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,e^{-ikx}\,dx \\ &= \frac{1}{2}\text{Re}\,\mathcal{F}\left[e^{-\frac{x^2}{2}}\right] \\ &= \frac{1}{2}\text{Re}\left(\sqrt{2\pi}e^{-\frac{k^2}{2}}\right) \qquad \text{[Using Eq. (277) with } \alpha = 1] \\ &= \sqrt{\frac{\pi}{2}}e^{-\frac{k^2}{2}} \end{aligned} \tag{279}$$

Hence $\hat{u}(k,t) = \sqrt{\frac{\pi}{2}}\,e^{-\frac{k^2}{2}}\,e^{-\frac{k^2}{c}t} = \sqrt{\frac{\pi}{2}}\,e^{-(1+2\frac{t}{c})\,\frac{k^2}{2}}.$

$$\begin{aligned} u(x,t) &= \frac{2}{\pi}\int_0^\infty \sqrt{\frac{\pi}{2}}\,e^{-(1+2\frac{t}{c})\,\frac{k^2}{2}}\,\cos kx\,dk \\ &= \frac{1}{2}Re\,\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-(1+2\frac{t}{c})\,\frac{k^2}{2}}\,e^{-ikx}\,dk \\ &= \frac{1}{\sqrt{2\pi}}\,\text{Re}\,\mathcal{F}\left[e^{-(1+2\frac{t}{c})\,\frac{k^2}{2}}\right] \qquad \text{(but from } k \text{ to } x) \\ &= \frac{1}{\sqrt{2\pi}}\,\text{Re}\left(\sqrt{\frac{2\pi}{1+2\frac{t}{c}}}\,e^{\left(-\frac{1}{1+2\frac{t}{c}}\,\frac{x^2}{2}\right)}\right) \quad \left(\text{using Eq. (277) with } \alpha = 1 + 2\frac{t}{c}\right) \\ &= \sqrt{\frac{c}{c+2t}}\,e^{-\frac{c\,x^2}{2(c+2t)}}. \end{aligned} \tag{280}$$

We can confirm that all ICs and BCs are satisfied:

$$\begin{aligned} \lim_{x\to\infty} u(x,t) &= \lim_{x\to\infty}\sqrt{\frac{c}{c+2t}}\,e^{-\frac{c\,x^2}{2(c+2t)}} &= 0, \\ \lim_{x\to\infty} u_x(x,t) &= \lim_{x\to\infty} -x\left(\frac{c}{c+2t}\right)^{\frac{3}{2}}\,e^{-\frac{c\,x^2}{2(c+2t)}} &= 0, \\ u_x(0,t) &= \lim_{x\to 0} -x\left(\frac{c}{c+2t}\right)^{\frac{3}{2}}\,e^{-\frac{c\,x^2}{2(c+2t)}} &= 0, \\ u(x,0) &= \lim_{t\to 0}\sqrt{\frac{c}{c+2t}}\,e^{-\frac{c\,x^2}{2(c+2t)}} &= e^{-\frac{x^2}{2}}. \end{aligned} \tag{281}$$

The Fourier sine and cosine transforms are applicable to problems on semi-infinite domains. For example, recall the **heat conduction** equation

$$T_t = \kappa T_{xx}. \tag{282}$$

Suppose this is solved subject to $T(0,t) = T_0$ and $T(\infty, t) \to 0$; such boundary conditions imply that the Fourier sine transform is appropriate, i.e. the transformed equation is

$$\partial_t \hat{T}_s(k,t) + \kappa k^2 \hat{T}_s(k,t) = \kappa k T_0. \tag{283}$$

Another common application of Fourier sine and cosine transforms is to **potential theory**, i.e. solutions to the Laplace equation in two dimensions

$$\phi_{xx} + \phi_{yy} = 0, \tag{284}$$

on a semi-infinite domain $x > 0$ and $y > 0$. Consider the following boundary conditions for this equation: $\phi = 1$ on $x = 0$ and $y > 0$; $\phi = 0$ on $y = 0$ and $x > 0$; $\nabla\phi \to 0$ as $x, y \to \infty$.

Applying the Fourier sine transform in the $x$ direction and using the boundary condition at $x = 0$, we obtain

$$\partial_y^2 \hat{\phi}_s(k,y) - k^2 \hat{\phi}_s(k,y) + k = 0. \tag{285}$$

This is an inhomogeneous second order equation, for which the general solution is

$$\hat{\phi}_s(k,y) = \frac{1}{k} + Ae^{ky} + Be^{-ky}. \tag{286}$$

Since $\hat{\phi}_s(k,y) \to 0$ as $y \to \infty$, the integration constant $A$ must vanish. Since $\hat{\phi}_s(k,0) = 0$

$$0 = \frac{1}{k} + B \tag{287}$$

and hence

$$\hat{\phi}_s(k,y) = \frac{1}{k}(1 - e^{-ky}) \tag{288}$$

which in turn implies that

$$\phi(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{k}(1 - e^{-ky})\sin(kx)dk = \frac{2}{\pi}\tan^{-1}\left(\frac{y}{x}\right). \tag{289}$$

(Note: you would not be expected to know integral identities such as this one, but would be given the required identity on a coursework or exam question!)

# 5 Laplace transforms

The Laplace transform transforms a *real* function $f(t)$ of a real variable $t \geq 0$ (usually $t$ represents time) to a *complex* function $\bar{f}(s)$ of a *complex* variable $s$.

The Laplace transform is defined by

$$\mathcal{L}\,[f(t)] = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt. \tag{290}$$

Again we note that the convention of placing a bar above $f$ to denote its transform is not universal: some books use $F$ for the transform of $f$, others the reverse.

If $f(t)$ is piecewise continuous for $t \geq 0$ and there exist constants $M, \alpha > 0$ such that

$$|f(t)| < \; M\,e^{\alpha t} \tag{291}$$

for all $t > T$ for some $T > 0$, then $\mathcal{L}[f(t)]$ exists and is analytic in the half-plane $\text{Re}(s) \; > \; \alpha$.

One can derive the Laplace transform from the Fourier transform, but each has its own domain of usefulness. Note, for example, that a function $f(t)$ that grows exponentially as $e^{\alpha t}$ still has a Laplace transform, but that it would not have a Fourier transform.

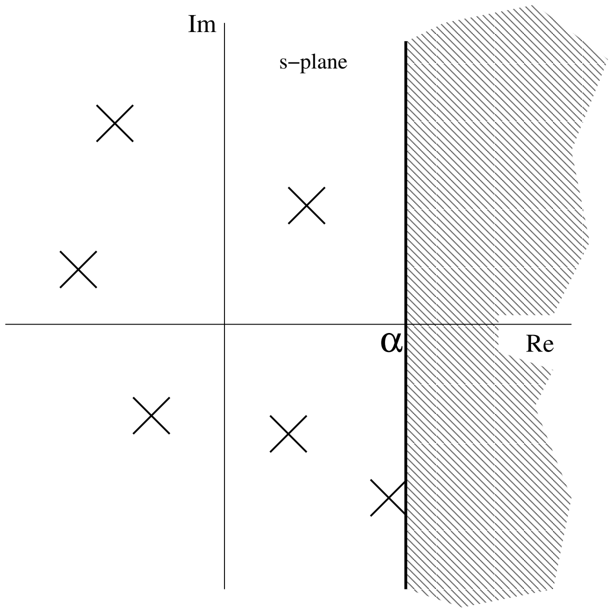



Figure 36: The half plane $\text{Re}(s) > \alpha$. All the singularities of $\bar{f}(s)$ lie to the left of $\text{Res}(s) = \alpha$.

**Example:** Find the Laplace transform of $f(t) = c$.

$$\begin{aligned}\mathcal{L}f(t) = & \int_0^\infty c\,e^{-s\,t}\,dt \;=\; c\int_0^\infty e^{-s\,t}\,dt \\ = & \; c\left[\tfrac{1}{-s}e^{-s\,t}\right]_0^\infty \;=\; \tfrac{c}{s}.\end{aligned} \tag{292}$$

**Example:** Show that $\mathcal{L}\left[\sin at\right] = \frac{a}{s^2+a^2}$.

$$\begin{aligned}\mathcal{L}\left[\sin at\right] = \bar{f}(s) = & \int_0^\infty \sin at\,e^{-s\,t}\,dt \\ = & \left[\sin at\,\frac{e^{-s\,t}}{-s}\right]_0^\infty - \int_0^\infty \frac{-a}{s}\,\cos at\,e^{-s\,t}\,dt \qquad \text{(integration by parts)} \\ = & \; 0 + \tfrac{a}{s}\int_0^\infty \cos at\,e^{-s\,t}\,dt \\ = & \; \tfrac{a}{s}\left(\left[\cos at\,\frac{e^{-s\,t}}{-s}\right]_0^\infty - \int_0^\infty -a\,\sin at\,\frac{e^{-s\,t}}{-s}\,dt\right) \quad \text{(by parts again)} \\ = & \; \tfrac{a}{s}\left(0 - \tfrac{1}{-s} - \frac{a}{s}\int_0^\infty \sin at\,e^{-s\,t}\,dt\right) \\ = & \; \tfrac{a}{s}\left(\tfrac{1}{s} - \tfrac{a}{s}\bar{f}(s)\right)\end{aligned} \tag{293}$$

Rearranging, we obtain $s^2\bar{f} = a - a^2\bar{f} \Rightarrow \bar{f} = a/\left(s^2+a^2\right)$ as required.

## 5.1 Properties

The following properties of the Laplace transform are easily shown from its definition.

1. Linearity: If $a$ and $b$ are constants, then

$$\int_0^\infty (af_1(t) + bf_2(t))\, e^{-st} dt = a \int_0^\infty f_1(t) e^{-st} dt + b \int_0^\infty f_2(t) e^{-st} dt \tag{294}$$

so that:

$$\mathcal{L}\left[af_1 + bf_2\right] = a\bar{f}_1 + b\bar{f}_2 \tag{295}$$

2. Powers of $t$:

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}. \tag{296}$$

This follows form $\mathcal{L}\left[1\right] = 1/s$ using proof by induction:

$$\begin{aligned}
\mathcal{L}\left[t^n\right] &= \int_0^\infty t^n e^{-st}\, dt \\
&= \left[t^n \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\, dt \\
&= \frac{n}{s} \int_0^\infty t^{n-1} e^{-st}\, dt \\
&= \frac{n}{s} \mathcal{L}\left[t^{n-1}\right] \\
&= \frac{n}{s} \frac{n-1}{s} \frac{n-2}{s} \dots \frac{2}{s} \frac{1}{s} \mathcal{L}[t^0] \\
&= \frac{n}{s} \frac{n-1}{s} \frac{n-2}{s} \dots \frac{2}{s} \frac{1}{s} \frac{1}{s} \;=\; \frac{n!}{s^{n+1}}.
\end{aligned} \tag{297}$$

3. Shifting in $s$:

$$\mathcal{L}\left[e^{at} f(t)\right] = \bar{f}(s-a) \tag{298}$$

**Example:** $\mathcal{L}[e^{at}]$.
If $f(t) = 1$ then $\bar{f}(s) = \frac{1}{s}$ from (292) so $\mathcal{L}[e^{at}] = \mathcal{L}[e^{at}.1] = \bar{f}(s-a) = \frac{1}{s-a}$.

4. Multiplication by powers of $t$ becomes derivatives of $\bar{f}(t)$:

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n} \bar{f}(s) \tag{299}$$

5. Shifting in $t$:

$$\mathcal{L}\left[f(t-t_0)\theta(t-t_0)\right] = e^{-t_0 s} \bar{f}(s) \tag{300}$$

where $\theta$ is the Heaviside "step function", defined by

$$\theta(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0, \end{cases} \tag{301}$$

which ensures that we only consider $f(t)$ for non-negative values of $t$, even when there is a time delay $t_0$.

6. Derivatives in $t$ become powers of $s$:

$$\mathcal{L}\left[\frac{d^n}{dt} f(t)\right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0) \tag{302}$$

Here $f^{(k)}$ denotes the $k^{\text{th}}$ derivative of $f(t)$ at $t = 0$. The Laplace transform is therefore particularly suitable for initial value problems: problems in which initial conditions on the function and its derivatives are given at $t = 0$.

7. Integration in $t$:

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{1}{s}\bar{f}(s) \tag{303}$$

8. Convolution:

$$\mathcal{L}\left[(f*g)(t)\right] = \mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right] = \bar{f}(s)\bar{g}(s) \tag{304}$$

9. If $f(t)$ is periodic with period $T$, so that $f(t+T) = f(t)$ for all $t \geq 0$, then

$$\bar{f}(s) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}. \tag{305}$$

## 5.2 Proofs

To prove the convolution theorem (8), we first form the product of the defining integrals for $\bar{f}(s)$ and $\bar{g}(s)$:

$$\bar{f}(s)\bar{g}(s) = \int_0^\infty \int_0^\infty e^{-s(\tau'+\tau)} f(\tau')g(\tau)d\tau' d\tau. \tag{306}$$

Next we introduce a change of variables, letting $\tau' = t - \tau$ and use the fact that $f$ vanishes for negative values of its arguments. Thus

$$\bar{f}(s)\bar{g}(s) = \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)d\tau\right]dt. \tag{307}$$

To prove Heaviside's shifting theorem (6) first note that

$$\int_0^\infty e^{-st} f(t-t_0)\theta(t-t_0)dt = \int_{t_0}^\infty e^{-st} f(t-t_0)dt. \tag{308}$$

Changing variables to $\tau = t - t_0$ gives

$$\int_{t_0}^\infty e^{-st} f(t-t_0)dt = \int_0^\infty e^{-s(\tau+t_0)} f(\tau)d\tau = e^{-st_0}\bar{f}(s). \tag{309}$$

To prove the periodicity property (9), note that for a function $f(t)$ such that $f(t+T) = f(t)$

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt = \sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{-st} f(t)dt \tag{310}$$

and hence

$$\bar{f}(s) = \sum_{n=0}^\infty e^{-nsT} \int_0^T e^{-s\tau} f(\tau)d\tau = (1-e^{-sT})^{-1} \int_0^T e^{-st} f(t)dt, \tag{311}$$

where the first equality follows from the change of variable $t = nT + \tau$ and the second equality follows from the known result for the sum of a geometric series and the change of variable $t = \tau$.

The **inverse formula** using the Bromwich contour can be proved using the inverse Fourier transform. The main ingredients of the proof are letting $s = a + iy$, and replacing $\int_0^\infty dt$ by $\int_{-\infty}^\infty H(t)\,dt$.

## 5.3 Uses

Laplace transforms allow us to convert complicated integro-differential equations into simpler problems.

**Example:** Consider the following equation, with boundary conditions $f(0) = 0$ and $f(\pi) = \sqrt{2}$:

$$\int_0^t f(\tau)d\tau + 4tf(t) = \frac{df(t)}{dt}. \tag{312}$$

Taking the Laplace transform we get

$$\begin{array}{ccccc} \frac{1}{s}\bar{f}(s) & + & -4\frac{d}{ds}\bar{f}(s) & = & s\bar{f}(s) - f(0) \\ \text{(by 7)} & & \text{(by 1 and 4)} & & \text{(by 6)} \end{array} \tag{313}$$

This is a 1st-order ODE for $\bar{f}(s)$. Using the first initial condition it becomes

$$\begin{aligned} 4\tfrac{d}{ds}\bar{f}(s) &= \bar{f}(s)\left(\tfrac{1}{s} - s\right) \\ \tfrac{4}{\bar{f}}d\bar{f} &= \left(\tfrac{1}{s} - s\right) ds \\ 4\ln\left(\bar{f}\right) + C &= \ln(s) - \tfrac{s^2}{2} \\ \bar{f}^4 &= c\, s\, e^{-s^2/2} \\ \bar{f} &= \tilde{c}\, s^{\frac{1}{4}}\, e^{-s^2/8} \end{aligned} \tag{314}$$

we have one constant of integration, $c$, to be determined by using the remaining boundary condition.

This method is only useful if we can get back to $f(t)$ once we have found our Laplace transform $\bar{f}(s)$. There are two ways of doing this, by:

inspection or inversion (theorem)

**Example:** Find $f$ by inspection, given $\bar{f} = 4\left(2(s-7)^3 + 9s^2 + 36\right) / \left((s^2+4)(s-7)^3\right)$.

$$\begin{aligned} \bar{f}(s) &= 4\frac{2(s-7)^3 + 9s^2 + 36}{(s^2+4)(s-7)^3} \\ &= 4\frac{2(s-7)^3 + 9(s^2+4)}{(s^2+4)(s-7)^3} \\ &= \frac{8}{s^2+4} + \frac{36}{(s-7)^3} \\ &= 4\frac{2}{s^2+2^2} + 18\,\frac{2}{(s-7)^3} \\ f(t) &= 4\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right] + 18\mathcal{L}^{-1}\left[\frac{2!}{s^3}\right] e^{7t} \\ &= 4\sin 2t \;+\; 18\, t^2\, e^{7t}. \end{aligned} \tag{315}$$

For more complicated expressions it may be necessary to invoke:

## 5.4 The complex inversion theorem

If $\mathcal{L}f(t) = \bar{f}(s)$ exists for $\operatorname{Re} s > \alpha$ and $f(t)$ is piecewise smooth, then the inverse Laplace transform of $\bar{f}(s)$ is given, for $t > 0$, by

$$\frac{1}{2}\left[f(t^+) + f(t^-)\right] = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{a-iR}^{a+iR} \bar{f}(s)\, e^{st}\, ds, \quad a > \alpha. \tag{316}$$

The inversion theorem guarantees that any continuous and piecewise smooth function is uniquely determined by its transform.

We can choose any real number $a$ such that the line $\gamma$ from $a - i\infty$ to $a + i\infty$ lies to the right of all the singularities of $\bar{f}(s)$. Since $\bar{f}(s)$ is analytic in Re s $> \alpha$, $a > \alpha$ will do.

This is called the **Bromwich integral** and, if $f(t)$ is continuous, it takes the form

$$f(t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{a-iR}^{a+iR} \bar{f}(s) e^{st}\, ds, \quad t > 0. \tag{317}$$

If $\bar{f}(s)$ has no branch points, we can evaluate the Bromwich integral by closing the contour with a large semicircle to the left. This is called the **Bromwich contour**. But for this we need to be able to show that the contribution of the large semicircle vanishes as its radius $R \to \infty$.

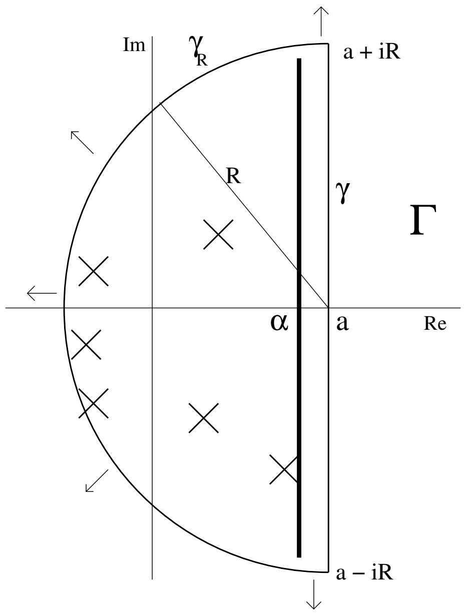



Figure 37: The Bromwich contour.

If there is a branch cut going to $-\infty$ the contour can be wrapped around the branch cut, and the contribution from the integral on both sides of the branch cut needs to be taken into account explicitly.

We can now apply the Residue Theorem to the Bromwich contour:

$$\int_\Gamma \bar{f}(s)\, e^{st}\, ds \;=\; 2\pi i \sum_{\substack{\text{singularities } a_k \\ \text{of } \bar{f}(s)e^{st} \\ \text{inside } \Gamma}} \mathrm{Res}(\bar{f}(s)\, e^{st} : a_k). \tag{318}$$

Note that the $a_k$'s that appear in the formula are the residues of $e^{st}\bar{f}(s)$, not $\bar{f}(s)$. In the limit $R \to \infty$ all the singularities will be inside $\Gamma$. We can also write the integral as

$$\int_\Gamma \bar{f}(s)\, e^{st}\, ds \;=\; \lim_{R\to\infty} \int_{a-iR}^{a+iR} \bar{f}(s)\, e^{st}\, ds \;+\; \lim_{R\to\infty} \int_{\gamma_R} \bar{f}(s)\, e^{st}\, ds. \tag{319}$$

Hence:

$$\begin{aligned} 2\pi i\, f(t) &= \lim_{R\to\infty} \int_{a-iR}^{a+iR} \bar{f}(s)\, e^{st}\, ds \\ &= \int_\Gamma \bar{f}(s)\, e^{st}\, ds \;-\; \lim_{R\to\infty} \int_{\gamma_R} \bar{f}(s)\, e^{st}\, ds \\ &= 2\pi i \sum_{a_k} \mathrm{Res}(\bar{f}(s)\, e^{st} : a_k) \;-\; \lim_{R\to\infty} \int_{\gamma_R} \bar{f}(s)\, e^{st}\, ds. \end{aligned} \tag{320}$$

If we can show that the contribution from the large semicircle $\gamma_R$ vanishes as its radius $R$ goes to $\infty$, then we have $f(t)$ in terms of residues which we can calculate.

The following Lemma tells us when we can neglect the contribution from the large semicircle:

**Lemma (Bromwich contour lemma)** If there exist constants $K > 0$, $k > 0$ and $R_0 > 0$ such that

$$|\bar{f}(s)| < \frac{K}{R^k} \tag{321}$$

for all $s$ on the large semicircle to the left $|s| = R$, Re$s < a$, for all $R > R_0$, then

$$\int_{\gamma_R} \bar{f}(s)\, e^{st}\, ds \;\to\; 0, \quad \text{as} \quad R \;\to\; \infty. \tag{322}$$

i.e. the contribution from the large semicircle vanishes.

Less rigorously put, the condition of the Lemma is that $f(s) \sim R^k$ for some $k > 0$, for $R$ sufficiently large. We can now summarise our residue calculation in the following Theorem:

**Theorem (Complex inversion theorem)** For Laplace transforms $\bar{f}$ satisfying condition (322) we have:

$$f(t) \;=\; \sum_{\substack{\text{singularities } a_k \\ \text{of } \bar{f}(s)e^{st}}} \text{Res}(\bar{f}(s)\, e^{st} : a_k). \tag{323}$$

Remember that we are summing the residues of $\bar{f}(s)e^{st}$, not just $\bar{f}(s)$. Although they will have the same singularities (because $e^{st}$ is an analytic function which does not contribute any singularities), the residues of the two functions will be different! A simple way to remember this is that: without $e^{st}$ the residues of $\bar{f}(s)$ are independent of $t$, so cannot give any time dependence in $f(t)$!

**Example:** Find $f(t)$ using the Inverse Transform Theorem, given $\bar{f}(s) = (2-s)/(s^2+1)$.

$$\bar{f}(s) = \frac{2-s}{(s+i)(s-i)} \quad \text{so } \bar{f}(s) \text{ has no branch points and two simple poles at } s = \pm i. \tag{324}$$

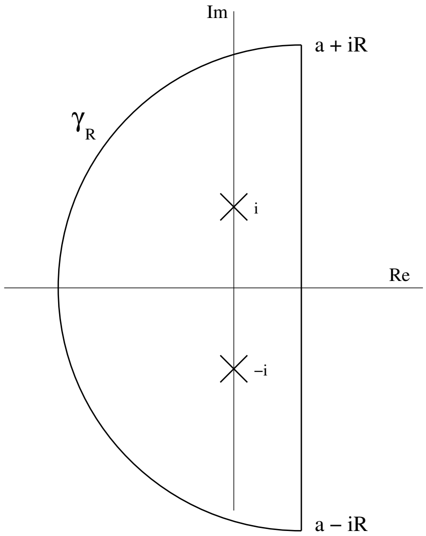



Figure 38: The Bromwich contour for $\bar{f}(s) = (2-s)/(s^2+1)$ requires $a > 0$.

On $\gamma_R$, $s = a + Re^{i\theta}$, where $\frac{\pi}{2} \le \theta < \frac{3\pi}{2}$, so

$$|\bar{f}(s)| = \left| \frac{2 - a - Re^{i\theta}}{1 + a^2 + 2aRe^{i\theta} + R^2e^{2i\theta}} \right| \le \frac{R+2+a}{R^2 - 2aR - a^2 - 1} < \frac{K}{R} \tag{325}$$

for some $K > 1$ and $R > R_0$ for some $R_0 > 0$. Note that we need to fix $R_0$ as least big enough so that the numerator and denominator in the second fraction here are both positive. $K$ then depends on how much bigger than that we make $R_0$. The sloppy reasoning would be that

$$|\bar{f}(s)| \to \frac{1}{R} \text{ as } R \to \infty, \tag{326}$$

But for the Lemma we need a rigorous inequality, and for this we need to fix a suitable $R_0$, and it is clear that we will need $K > 1$ for any $R_0$.

Hence

$$
\begin{aligned}
f(t) &= \sum_{\substack{\text{singularities } a_k \\ \text{of } \bar{f}(s)e^{st}}} \operatorname{Res}\left(\frac{2-s}{s^2+1}\, e^{st} : a_k\right) \\
&= \operatorname{Res}\left(\tfrac{2-s}{(s+i)(s-i)}\, e^{st} : i\right) \quad + \quad \operatorname{Res}\left(\tfrac{2-s}{(s+i)(s-i)}\, e^{st} : -i\right) \\
&= \tfrac{2-s}{s+i} e^{st}\Big|_{s=+\mathrm{i}} \quad + \quad \tfrac{2-s}{s-i} e^{st}\Big|_{s=-\mathrm{i}} \\
&= \tfrac{2-i}{2i} e^{it} \quad + \quad \tfrac{2+i}{-2i} e^{-it} \\
&= -\tfrac{e^{+it}+e^{-it}}{2} \quad + \quad \tfrac{2}{i}\tfrac{e^{+it}-e^{-it}}{2} \\
&= -\cos t \quad + \quad 2\sin t.
\end{aligned}
\tag{327}
$$

**Example:** Find $f(t)$ by inspection, given $\bar{f}(s) = (2-s)/(s^2+1)$.

$$f(t) = \mathcal{L}^{-1}\left(\frac{2-s}{s^2+1}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = 2\sin t - \cos t. \tag{328}$$

**Example:** Find $f(t)$ given $\bar{f}(s) = s^{-\frac{1}{2}}$.

$$\bar{f}(s) = \frac{1}{\sqrt{s}} \text{ has a branch point at } s = 0 \text{ but no other singularities.} \tag{329}$$

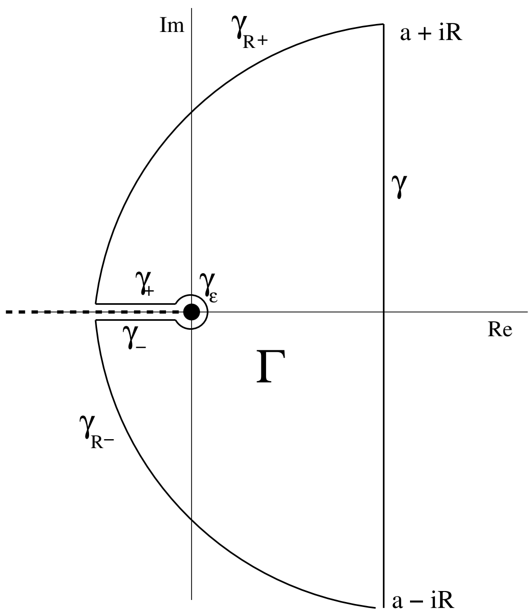



Figure 39: The Bromwich contour for $\bar{f}(s) = \frac{1}{\sqrt{s}}$.

The Complex Inversion Theorem tells us that

$$f(t) = \frac{1}{2\pi i}\int_\gamma \bar{f}(s)\, e^{st}\, ds. \tag{330}$$

The Residue Theorem tells us that

$$\int_\Gamma \bar{f}(s)\, e^{st}\, ds = 0. \tag{331}$$

Since

$$\begin{aligned} \int_\Gamma &= \int_\gamma + \int_{\gamma_R^+} + \int_{\gamma_+} + \int_{\gamma_\epsilon} + \int_{\gamma_-} + \int_{\gamma_R^-} \\ 0 &= 2\pi i f(t) + \int_{\gamma_R^+} + \int_{\gamma_+} + \int_{\gamma_\epsilon} + \int_{\gamma_-} + \int_{\gamma_R^-} \end{aligned} \tag{332}$$

it follows that

$$f(t) = \frac{-1}{2\pi i}\left[\int_{\gamma_R^+} + \int_{\gamma_+} + \int_{\gamma_\epsilon} + \int_{\gamma_-} + \int_{\gamma_R^-}\right]\bar{f}(s)\, e^{st}\, ds. \tag{333}$$

Considering each contour in turn:

On $\gamma_R$, $s = a + Re^{i\theta}$ so $\bar{f}(s) = \bar{f}\left(a + Re^{i\theta}\right) = \dfrac{1}{\sqrt{a + Re^{i\theta}}} \sim R^{-\frac{1}{2}} \leq M\, R^{-K}$ for $K = \dfrac{1}{2}$.

Hence the contribution of $\gamma_R$ disappears: we have shown that $\displaystyle\lim_{R\to\infty}\int_{\gamma_{R+}} = \lim_{R\to\infty}\int_{\gamma_{R-}} = 0.$ (334)

On $\gamma_\epsilon$, $s = \epsilon e^{i\theta}$ so

$$\begin{aligned} \left|\int_{\gamma_\epsilon} \bar{f}(s) e^{st}\, ds\right| &= \left|\int_{+\pi}^{-\pi} \left(\epsilon e^{i\theta}\right)^{-\frac{1}{2}} e^{\epsilon\, e^{i\theta}\, t}\, i\epsilon e^{i\theta} d\theta\right| \leq \underset{\mathrm{L}}{\underset{\uparrow}{(2\pi)}} \underset{\mathrm{M}}{\underset{\uparrow}{\left|\epsilon^{\frac{1}{2}} e^{\epsilon t\cos\theta} e^{i\epsilon t\sin\theta}\right|}} \\ &\sim \epsilon^{+\frac{1}{2}} e^{\epsilon\, t} \to 0,\ \epsilon \to 0. \end{aligned} \tag{335}$$

(here using $ML$ estimates).

On the negative axis we have $s = x$ for $-\infty < x < 0$. However, since this is a branch cut for the multifunction $\bar{f}(s)$, it will have different values on either side.

Just above the cut, $s = -x$ may be written in polar form as $s = xe^{i\pi}$ for $\infty > x > 0$,
Just below the cut, $s = -x$ may be written in polar form as $s = xe^{-i\pi}$ for $0 < x < \infty$.

Therefore, on $\gamma_+$ we have $\sqrt{s} = \sqrt{x}e^{i\frac{\pi}{2}} = i\sqrt{x}$, and on $\gamma_-$ we have $\sqrt{s} = \sqrt{x}e^{-i\frac{\pi}{2}} = -i\sqrt{x}$.

Calculating both integrals on either side of the branch cut together (this may prove useful, as there often exists symmetry) we get

$$\begin{aligned} &\int_{\gamma_+} \bar{f}(s)\, ds + \int_{\gamma_-} \bar{f}(s)\, ds \\ &= \int_\infty^0 \frac{e^{-xt}}{i\sqrt{x}}(-dx) + \int_0^\infty \frac{e^{-xt}}{-i\sqrt{x}}(-dx) \\ &= -i\int_0^\infty \frac{e^{-xt}}{\sqrt{x}}\, dx + -i\int_0^\infty \frac{e^{-xt}}{\sqrt{x}}\, dx \\ &= -2\mathrm{i}\int_0^\infty \frac{e^{-xt}}{\sqrt{x}}\, dx \end{aligned} \tag{336}$$

If we let $X^2 = x$ then $X = \sqrt{x}$ and $dx = 2X\, dX$. To substitute we need to change limits: $x = 0 \Rightarrow X = 0$, $x = \infty \Rightarrow X = \infty$.

$$\begin{aligned} -2\mathrm{i}\int_0^\infty \frac{e^{-xt}}{\sqrt{x}}\, dx &= -2\mathrm{i}\int_0^\infty \frac{e^{-X^2 t}}{X}\, 2X\, dX = -4\mathrm{i}\int_0^\infty e^{-X^2 t}\, dX \\ &= -2\mathrm{i}\int_\infty^\infty e^{-X^2 t}\, dX \qquad \text{(using symmetry)} \\ &= -2i\sqrt{\tfrac{\pi}{t}} \qquad \text{(standard result)} \end{aligned} \tag{337}$$

Finally we have, remembering (333):

$$f(t) \;=\; \frac{-1}{2\pi i}\left[-2i\,\sqrt{\frac{\pi}{t}}\;+\;0\right] \;=\; \frac{1}{\sqrt{\pi t}}. \tag{338}$$

**Example:** $\bar{f}(s) = \cos s/(s^2+3)$.
On $\gamma_R$, $s = a + Re^{i\theta}$ so

$$\begin{aligned} \bar{f}\left(a+Re^{i\theta}\right) &= \frac{\cos\left(a+Re^{i\theta}\right)}{\left(a+Re^{i\theta}\right)^2+3} = \frac{\cos\left(a+R\cos\theta+iR\sin\theta\right)}{a^2+3+2aRe^{i\theta}+R^2e^{i2\theta}} \\ &\sim \frac{\cos iR}{R^2} \sim \frac{\cosh R}{R^2} \to \infty, \text{ as } R\to\infty. \end{aligned} \tag{339}$$

Hence we cannot justify using the Complex Inversion Theorem on $\bar{f}(s)$ in the formula (323) above, because (322) does not apply.

## 5.5 Using Laplace transforms to solve partial differential equations

Since $x$ is a constant with respect to integration in time $t$, it follows that

$$\mathcal{L}\left[u(x,t)\right] \;=\; \int_0^\infty u(x,t)\;e^{-st}\,dt \;=\; \bar{u}(x,s). \tag{340}$$

Differentiating with respect to $x$ may also be taken into or out of the integral, so that

$$\begin{aligned} \mathcal{L}\left[u_x(x,t)\right] &= \bar{u}_x(x,s), \\ \mathcal{L}\left[u_{xx}(x,t)\right] &= \bar{u}_{xx}(x,s). \end{aligned} \tag{341}$$

The properties of Laplace transforms (#6) tell us that the transforms of time derivatives are given by

$$\begin{aligned} \mathcal{L}\left[u_t(x,t)\right] &= s\bar{u}(x,s) - u(x,0), \\ \mathcal{L}\left[u_{tt}(x,t)\right] &= s^2\bar{u}(x,s) - su(x,0) - u_t(x,0). \end{aligned} \tag{342}$$

These results can of course be extended to higher order, and used to find mixed derivatives such as

$$\mathcal{L}\left[u_{xt}(x,t)\right] \;=\; s\bar{u}_x(x,s) - u_x(x,0). \tag{343}$$

This way, a second-order partial differential equation can be transformed into a second-order *ordinary* differential equation (ODE). Very often it is easier to solve the ODE and then transform back from $\bar{u}(x,s)$ to $u(x,t)$ rather than solve the PDE directly. Notice that the Laplace transform of $u_t(x,t)$ brings in $u(x,0)$, the Laplace transform of $u_{tt}(x,t)$ brings in $u(x,0)$ and $u_t(x,0)$, and so on. That is why Laplace transform methods are so suitable for dealing with *initial value problems.*

To solve a PDE using Laplace Transforms we need to:
**I** Transform: both the equation *and* boundary conditions;
**II** Solve: the resulting ODE for $\bar{u}(x,s)$;
**III** Invert: To find $u(x,t)$, the solution to the original problem.

**Example:** Solve $u_x = 2u_t + u$ for $t, x > 0$, given $u(x,0) = e^{-3x}$ and $u(x,t) \to 0, x \to \infty$.
**I** Transform equation:

$$\begin{aligned} u_x &= 2u_t \;+\; u \\ \bar{u}_x &= 2\left(s\bar{u} \;-\; u(x,0)\right) \;+\; \bar{u} \\ \bar{u}_x - (1+2s)\bar{u} &= -2e^{-3x} \end{aligned} \tag{344}$$

Transform boundary conditions:

$$u(x,t) \to 0,\; x\to\infty \;\Rightarrow\; \bar{u}(x,s) \to 0, x\to\infty. \tag{345}$$

**II** Solve ODE in $x$:

$$\begin{aligned}\frac{d}{dx}\left[\bar{u}\,e^{-(1+2s)x}\right] &= -2e^{-3x}\,e^{-(1+2s)x} && \text{(apply integration factor)}\\ \bar{u}\,e^{-(1+2s)x} + C(s) &= \int -2e^{-(2s+4)x}\,dx && \text{(To treat this as an ODE in } x \text{ requires constants of integration to be functions of } s)\\ \bar{u}\,e^{-(1+2s)x} &= -\frac{2e^{-(2s+4)x}}{-(2s+4)} - C(s)\\ \bar{u}(x,s) &= \frac{e^{-3x}}{s+2} - C(s)\,e^{+(1+2s)x}\end{aligned} \tag{346}$$

We *cannot* invert the solution yet, because $C(s)$ is an unknown function of $s$ which will affect the dependence of $u$ on $t$. To obtain $C(s)$ we next use the BC:

**II** Solution of ODE with BC:

$$0 = \lim_{x\to\infty}\bar{u}(x,s) = \lim_{x\to\infty}\left[\frac{e^{-3x}}{s+2} - C(s)\,e^{+(1+2s)x}\right] = 0 + C(s)\lim_{x\to\infty}e^{(2s+1)x}. \tag{347}$$

$e^{(2s+1)x} \to 0$ as $x\to\infty$ if and only if $Re(2s) > -1$, which does not hold for all s. Hence, for boundary condition to be satisfied, we must have $C(s) = 0$.

**III** Invert:

$$\begin{aligned}\bar{u}(x,s) &= \tfrac{e^{-3x}}{s+2}\\ u(x,t) &= e^{-3x}\,\mathcal{L}^{-1}\left[\tfrac{1}{s+2}\right]\\ u(x,t) &= e^{-3x}\,e^{-2t}\\ u(x,t) &= e^{-3x-2t}\end{aligned} \tag{348}$$

**Example:** Solve $u_{tt} = u_{xx}$, for $0 < x < l$, $t > 0$ with initial conditions : $u(x,0) = 0$, $u_t(x,0) = \sin\frac{\pi x}{l}$ and boundary conditions: $u(0,t) = 0$, $u(l,t) = 0$.

Note:
2 I.C.s as 2nd order in $t$
2 B.C.s as 2nd order in $x$

**I** Transform: Equation (using initial conditions):

$$\begin{aligned}s^2\bar{u} - su(x,0) - u_t(x,0) &= \bar{u}_{xx}\\ s^2\bar{u} - 0 - \sin\tfrac{\pi x}{l} &= \bar{u}_{xx}\\ \bar{u}_{xx} - s^2\bar{u} &= -\sin\tfrac{\pi x}{l}\end{aligned} \tag{349}$$

Boundary conditions: $u(0,t) = 0 \Rightarrow \bar{u}(0,s) = 0$, $u(l,t) = 0 \Rightarrow \bar{u}(l,s) = 0$.

**II** Solve:

Homogeneous equation: $\frac{d^2\bar{u}}{dx^2} - s^2\bar{u} = 0 \Rightarrow \bar{u} = Ae^{sx} + Be^{-sx}$

Particular solution: RHS$= -\sin\frac{\pi x}{l}$ so try $u = \alpha\sin\frac{\pi x}{l} + \beta\cos\frac{\pi x}{l}$.

$$\begin{aligned}\bar{u} &= \alpha\sin\tfrac{\pi x}{l} + \beta\cos\tfrac{\pi x}{l}\\ \bar{u}_x &= \tfrac{\pi}{l}\alpha\cos\tfrac{\pi x}{l} - \tfrac{\pi}{l}\beta\sin\tfrac{\pi x}{l}\\ \bar{u}_{xx} &= -\tfrac{\pi^2}{l^2}\alpha\sin\tfrac{\pi x}{l} - \tfrac{\pi^2}{l^2}\beta\cos\tfrac{\pi x}{l}\end{aligned} \Rightarrow \begin{aligned}&\tfrac{d^2\bar{u}}{dx^2} - s^2\bar{u} =\\ &-\tfrac{\pi^2}{l^2}\alpha\sin\tfrac{\pi x}{l} - \tfrac{\pi^2}{l^2}\beta\cos\tfrac{\pi x}{l} - s^2\alpha\sin\tfrac{\pi x}{l} - s^2\beta\cos\tfrac{\pi x}{l}\end{aligned} \tag{350}$$

Hence,

$$\left(-s^2 - \left(\frac{\pi}{l}\right)^2\right)\left(\alpha\sin\frac{\pi x}{l} + \beta\cos\frac{\pi x}{l}\right) = -\sin\frac{\pi x}{l} \tag{351}$$

$$\begin{aligned}\left(s^2 + \left(\tfrac{\pi}{l}\right)^2\right)\alpha &= 1\\ \left(s^2 + \left(\tfrac{\pi}{l}\right)^2\right)\beta &= 0\end{aligned} \Rightarrow \begin{aligned}\alpha &= \tfrac{1}{s^2 + \left(\frac{\pi}{l}\right)^2}\\ \beta &= 0\end{aligned} \tag{352}$$

$$\bar{u}(x,s) = A(s)e^{sx} + B(s)e^{-sx} + \frac{\sin\frac{\pi x}{l}}{s^2 + \frac{\pi^2}{l^2}}. \tag{353}$$

Now impose boundary conditions:
$\bar{u}(0,s) = 0 = A(s) + B(s)$, $\bar{u}(l,s) = 0 = A(s)e^{ls} + B(s)e^{-ls}$,

$\Rightarrow A(s) = -B(s) \Rightarrow A(s)\left(e^{ls} - e^{-ls}\right) = 0 \Rightarrow A(s) = 0 \Rightarrow B(s) = 0.$

The specific solution satisfying both ICs and BCs is therefore

$$\bar{u}(x,s) = \frac{\sin\frac{\pi x}{l}}{s^2 + \frac{\pi^2}{l^2}}. \tag{354}$$

**III** Invert:
By inspection:

$$\begin{aligned} u(x,t) &= \mathcal{L}^{-1}\left[\frac{\sin\frac{\pi x}{l}}{s^2+\frac{\pi^2}{l^2}}\right] \\ u(x,t) &= \sin\frac{\pi x}{l}\,\mathcal{L}^{-1}\left[\frac{1}{s^2+\frac{\pi^2}{l^2}}\right] \\ u(x,t) &= \frac{l}{\pi}\sin\frac{\pi x}{l}\,\mathcal{L}^{-1}\left[\frac{\frac{\pi}{l}}{s^2+\frac{\pi^2}{l^2}}\right] \\ u(x,t) &= \frac{l}{\pi}\sin\frac{\pi x}{l}\sin\frac{\pi t}{l}. \end{aligned} \tag{355}$$

Alternatively, we can apply the Complex Inversion theorem, noticing $\left|\frac{1}{s^2+\frac{\pi^2}{l^2}}\right| < M\,R^{-2}$:
$\bar{f}$ has no branch points and two simple poles at $s = \pm\frac{\pi}{l}i$, hence

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^2+\frac{\pi^2}{l^2}}\right] &= \text{Res}\left(\frac{e^{st}}{(s-i\pi/l)(s+i\pi/l)} : +\frac{\pi}{l}i\right) + \text{Res}\left(\frac{e^{st}}{(s-i\pi/l)(s+i\pi/l)} : -\frac{\pi}{l}i\right) \\ &= \frac{e^{i(\pi/l)t}}{2i\pi/l} + \frac{e^{-i(\pi/l)t}}{-2i\pi/l} = \frac{l}{\pi}\left(\frac{e^{i\frac{\pi}{l}t} - e^{-i\frac{\pi}{l}t}}{2i}\right) = \frac{l}{\pi}\sin\frac{\pi t}{l}. \end{aligned} \tag{356}$$

**Example:** Solve $u_t = u_{xx}$, for $0 < x < 1$, $t \geq 0$ with
boundary conditions: $u_x(0,t) = 0$, $u(1,t) = u_1$ (a constant), — 2 B.C.s as 2nd order in $x$
and initial condition: $u(x,0) = u_0$ (a constant). — 1 I.C. as 1st order in $t$

**I** Transform Equation: $s\bar{u} - u(x,0) = \bar{u}_{xx} \Rightarrow \bar{u}_{xx} - s\bar{u} = -u_0$.
Transform Boundary conditions:
$u_x(0,t) = 0 \Rightarrow \bar{u}_x(0,s) = 0$,
$u(1,t) = u_1 \Rightarrow \bar{u}(1,s) = \frac{u_1}{s}$.
**II** Solve homogeneous equation: $\frac{d^2\bar{u}}{dx^2} - s\bar{u} = 0 \Rightarrow \bar{u} = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}$
Particular solution: Try $\bar{u} = \frac{\alpha}{s} \Rightarrow -u_0 = \bar{u}_{xx} - s\bar{u} = 0 - s\frac{\alpha}{s} \Rightarrow \alpha = u_0$.

$$\bar{u} = A(s)e^{x\sqrt{s}} + B(s)e^{-x\sqrt{s}} + \frac{u_0}{s}. \tag{357}$$

Applying BCs:
$0 = \bar{u}_x(0,s) = \sqrt{s}A(s) - \sqrt{s}B(s) + 0.$
$u_1/s = \bar{u}(1,s) = A(s)e^{\sqrt{s}} + B(s)e^{-\sqrt{s}} + \frac{u_0}{s}.$

$$\Rightarrow A(s) = B(s) = \frac{u_1/s - u_0/s}{\left(e^{\sqrt{s}} + e^{-\sqrt{s}}\right)} = \frac{u_1 - u_0}{s\left(e^{\sqrt{s}} + e^{-\sqrt{s}}\right)}. \tag{358}$$

Hence,

$$\bar{u} = \frac{u_1 - u_0}{s}\left[\frac{e^{x\sqrt{s}} + e^{-x\sqrt{s}}}{e^{\sqrt{s}} + e^{-\sqrt{s}}}\right] + \frac{u_0}{s} \tag{359}$$

$$= \frac{u_1 - u_0}{s}\frac{\cosh\sqrt{s}x}{\cosh\sqrt{s}} + \frac{u_0}{s}. \tag{360}$$

**III** Invert: $u(x,t) = u_0\mathcal{L}^{-1}\left[\frac{1}{s}\right] + (u_1 - u_0)\,\mathcal{L}^{-1}\left[\frac{\cosh\sqrt{s}x}{s\cosh\sqrt{s}}\right]$.

By inspection: $\mathcal{L}^{-1}\left[\frac{1}{s}\right]=1$.

Consider the inversion of $\bar{f}(x,s)=\frac{\cosh\sqrt{s}x}{s\cosh\sqrt{s}}$ using the Complex Inversion Theorem. First check that the theorem is applicable:

$$\left|\bar{f}(x,s)\right| \;=\; \left|\frac{\cosh\sqrt{s}x}{s\cosh\sqrt{s}}\right| \;\sim\; \underset{\text{(On arc, for } R\gg 1)}{\frac{1}{R}\frac{e^{x\sqrt{R}}}{e^{\sqrt{R}}}} \;\le\; \underset{\text{(for } K=1 \text{ and since } x<1)}{MR^{-K}} \tag{361}$$

Hence

$$f(x,t) \;=\; \sum_{\substack{\text{singularities } a_k\\ \text{of } \bar{f}(x,s)e^{st}}} \operatorname{Res}(\bar{f}(x,s)\,e^{st}:a_k). \tag{362}$$

The singularities of $\bar{f}(s)$ are a simple pole at $s=0$ and the roots of $\cosh\sqrt{s}$.

$\cosh\sqrt{s}=0 \Rightarrow \sqrt{s}=\left(\frac{2n+1}{2}\right)\pi i,\ n\in\mathbb{Z}$, (Remember $\cosh z=\cos\frac{z}{i}$)
$\Rightarrow s=a_n=-\left(\frac{2n+1}{2}\right)^2\pi^2,\ n\in\mathbf{Z}^+$.
We take only $n\ge 0$ as the values of $a_n$ for $n<0$ would be repetitive. Note $\cosh\sqrt{z}$, unlike $\sqrt{z}$, is single valued:

$$\cosh\sqrt{z}=1+\frac{\left(\sqrt{s}\right)^2}{2!}+\frac{\left(\sqrt{s}\right)^4}{4!}+\ldots=1+\frac{s}{2!}+\frac{s^2}{4!}+\ldots \tag{363}$$

Note that within our boundaries, $0<x<1$, the above singularities do not cancel any root of the numerator, $\cosh x\sqrt{s}$ (remember that we are interested in singularities with respect to $s$; not $x$ or $t$).

The singularities $a_n$ are all simple poles, with residues given by

$$\operatorname{Res}(\bar{f}e^{st}:a_n) \;=\; \lim_{s\to a_n}\frac{e^{st}\cosh x\sqrt{s}}{\frac{\sqrt{s}}{2}\sinh\sqrt{s}} \quad \left(\text{using } \frac{g}{h'} \text{ formula}\right) \tag{364}$$

Hence:

$$\begin{aligned}
\operatorname{Res}\left(\bar{f}e^{st}:-\left(\tfrac{2n+1}{2}\right)^2\pi^2\right) &= 2\frac{e^{\left(-\left(\frac{2n+1}{2}\right)^2\pi^2\right)t}\cosh x\sqrt{-\left(\frac{2n+1}{2}\right)^2\pi^2}}{\sqrt{-\left(\frac{2n+1}{2}\right)^2\pi^2}\sinh\sqrt{-\left(\frac{2n+1}{2}\right)^2\pi^2}}\\
&= 2\,e^{-\left(\frac{2n+1}{2}\right)^2\pi^2 t}\frac{\cosh\left[\left(\frac{2n+1}{2}\right)\pi x i\right]}{\left[\frac{2n+1}{2}\pi i\right]\sinh\left[\left(\frac{2n+1}{2}\right)\pi i\right]}\\
&= 4e^{-\left(\frac{2n+1}{2}\right)^2\pi^2 t}\frac{\cos\left[\left(\frac{2n+1}{2}\right)\pi x\right]}{-(2n+1)\pi\sin\left[\left(\frac{2n+1}{2}\right)\pi\right]}\\
&= -\frac{4}{(2n+1)\pi}e^{-\left(\frac{2n+1}{2}\right)^2\pi^2 t}\frac{\cos\left[\left(\frac{2n+1}{2}\right)\pi x\right]}{(-1)^n}
\end{aligned} \tag{365}$$

For the simple pole $s=0$ we have: $\operatorname{Res}(\bar{f}e^{st}:0) \;=\; \lim_{s\to 0}\frac{\cosh x\sqrt{s}}{\cosh\sqrt{s}}e^{st} \;=\; 1.$

Substituting these values into the Complex Inversion Theorem formula (362) gives:

$$f(x,t) \;=\; 1+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}4}{(2n+1)\pi}\cos\left(\frac{2n+1}{2}\pi x\right)e^{-\left(\frac{2n+1}{2}\right)^2\pi^2 t}. \tag{366}$$

The solution to the original PDE is therefore given by

$$\begin{aligned}
u(x,t) &= u_0 \;+\; (u_1-u_0)\left[1+4\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)\pi}\cos\left(\frac{2n+1}{2}\pi x\right)e^{-\left(\frac{2n+1}{2}\right)^2\pi^2 t}\right] &(367)\\
&= u_1 \;+\; 4(u_1-u_0)\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)\pi}\cos\left(\frac{2n+1}{2}\pi x\right)e^{-\left(\frac{2n+1}{2}\right)^2\pi^2 t}. &(368)
\end{aligned}$$

It is useful to check that our solution satisfies all the necessary conditions:

$$\begin{aligned} u_x(0,t) &= 0 - \left(\tfrac{2n+1}{2}\pi\right) 4(u_1-u_0)\textstyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)\pi} \sin(0)\, e^{-\left(\frac{2n+1}{2}\right)^2 \pi^2 t} &= 0; \\ u(1,t) &= u_1 + 4(u_1-u_0)\textstyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)\pi} \left[\cos\left(\tfrac{2n+1}{2}\pi\right)\right] e^{-\left(\frac{2n+1}{2}\right)^2 \pi^2 t} &= u_1. \end{aligned} \tag{369}$$

Note that as $t \to \infty$, $e^{-t} \to 0$ so that

$$\lim_{t\to\infty} u(x,t) = u_1, \tag{370}$$

which continues to satisfy both boundary conditions.

It is more difficult to check the initial condition: At $t = 0$ we have

$$u(x,0) = u_1 + 4(u_1-u_0)\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)\pi} \cos\left(\frac{2n+1}{2}\pi x\right); \tag{371}$$

so $u(x,0) = u_0$ would imply

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos\left(\frac{2n+1}{2}\pi x\right) = \frac{\pi}{4} \tag{372}$$

for all values of $x \in (0,1)$. That this indeed holds can be confirmed numerically—see Figure 40.

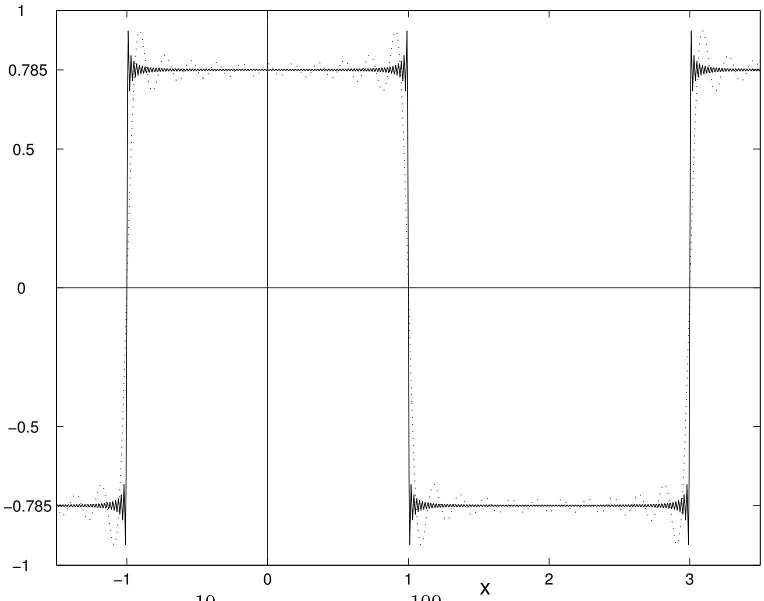



Figure 40: The sum (372) for $\sum_0^{10}$ (dotted) and $\sum_0^{100}$ (solid). In the relevant domain $(0 < x < 1)$ the sum seems to converge to $\sim 0.785 \sim \pi/4$.

The expression (372) is in fact a Fourier series solution of a square wave with amplitude $\frac{\pi}{4}$, extended as an even function. This result suggests that our solution (368) is a Fourier Series of a much simpler (closed form) expression for $u(x,t)$. The inversion of Fourier series solutions is beyond this course, but it is worth noting that Transform methods and Series solutions are closely related (see section 4.2).

## 5.6 Applications

In this section we briefly discuss physical problems that can be described by partial differential equations of the types analysed in the previous section.

**Heat conduction:** Consider a solid located at $x \geq 0$, i.e. with a boundary at $x = 0$. Let the solid be initially (time $t = 0$) at a temperature $T_i$ and suppose that the boundary $x = 0$ is maintained at a temperature $T_0$ for time $t > 0$. The rate at which heat is transferred across a plane section, of constant $x$, in the direction of increasing $x$ is $-KT_x$, where $K$ is the thermal conductivity and $T_x$ is temperature gradient.

The rate at which heat is transferred into a slab bounded by $x$ and $x+dx$ is $KT_{xx}$ per unit area (in the $yz$ plane). This rate must be equal to the rate at which the slab is gaining heat, namely $\rho C T_t dx$, where $\rho$ is the density, $C$ is the specific heat and $dx$ is the volume per unit of transverse area. Equating these two rates we obtain the *diffusion equation*

$$T_t = \kappa T_{xx} \tag{373}$$

where

$$\kappa = \frac{K}{\rho C} \tag{374}$$

is called the diffusivity.

This equation is indeed of the form analysed in the previous section. We are given initial conditions for all $x > 0$, i.e. $T = T_i$ at $t = 0$, and we are given a boundary condition $T = T_0$ at $x = 0$. Since the equation has two spatial derivatives, we will need a second boundary condition in $x$. We can consider either a finite size slab, in which case we will often be told the temperature at the second boundary $x = L$, or an infinite slab, in which case we will usually be told that $T \to 0$ as $x \to \infty$.

**Wave propagation along bars:** Consider a uniform bar of cross section $A$, extended along the $x$ axis. The bar has one end fixed at $x = 0$ and has length $l$. Suppose a stress is applied to the bar in the direction of the positive $x$ axis, causing the bar to make small longitudinal displacements $h(x, t)$ along the $x$ axis. These displacements satisfy a wave equation

$$h_{tt} = c^2 h_{xx} \tag{375}$$

where

$$c^2 = \frac{E}{\rho} \tag{376}$$

with $E$ the Young modulus and $\rho$ the density. Again this equation is of a type analysed in the previous section. To solve the equation we will need two initial conditions ($h(x, 0)$ and $h_t(x, 0)$) and two boundary conditions. One boundary condition follows from the fact the end is fixed at $x = 0$, i.e. $h(0, t) = 0$; the second boundary condition might be $h_t(0, t)$ or $h_t(l, t)$.

**Wave propagation along strings:** Consider a string extended along the $x$ axis. Suppose that a stress is applied, causing the string to make small displacements in a transverse direction, i.e. $y(x, t)$ . Again these displacements satisfy a wave equation

$$y_{tt} = c^2 y_{xx} \tag{377}$$

where the wave speed is now given by

$$c^2 = \frac{T}{\mu} \tag{378}$$

with $T$ the tension of the string and $\mu$ the density of the string. The equation will require two initial conditions and two boundary conditions.

# 6 Hankel transforms

The complex Fourier transform and Laplace transform are by far the two most used integral transforms, but there are many other integral transforms that are useful for transforming PDEs into ODEs. Here we give only one more example: For problems in cylindrical polar coordinates we can often use Hankel transforms. Their integral kernel is not simply $e^{-st}$ or $e^{-ikx}$, but the more complicated *Bessel functions*.

## 6.1 The Bessel equation

Consider the wave equation in two space dimensions:

$$f_{tt} = f_{xx} + f_{yy} \tag{379}$$

When we go from the Cartesian coordinates $x$ and $y$ to cylindrical polar coordinates $r$ and $\varphi$, defined by

$$x = r\cos\varphi, \quad y = r\sin\varphi \tag{380}$$

then the wave equation transforms into

$$f_{tt} = f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\varphi\varphi}. \tag{381}$$

Now we can use the technique of separation of variables to write $f(r,\varphi,t)$ as a product of three functions of one variable each:

$$f(r,\varphi,t) = e^{i\omega t}e^{in\varphi}u(r) \tag{382}$$

where $\omega$ is a real constant and $n$ is an integer. We find that $u(r)$ obeys the ODE

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(\omega^2 - \frac{n^2}{r^2}\right)u = 0 \tag{383}$$

This seems to depend on two parameters $\omega$ and $n$ but if we make the substitution $\omega r = x$ we obtain the simpler equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left(1 - \frac{n^2}{x^2}\right)u = 0 \tag{384}$$

This equation is called the **Bessel equation of order** $n$.

## 6.2 Bessel functions

The Bessel equation is a second order equation and therefore has two independent solutions. The one that is well-behaved at the origin $r = 0$ is called the **Bessel function of order** $n$ and can be given as

$$J_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\tau - ir\sin\tau}\,d\tau = \frac{1}{\pi}\int_0^{\pi}\cos(n\tau - r\sin\tau)\,d\tau. \tag{385}$$

These are called **Bessel functions of the first kind**. There is another, independent, set of solutions to the Bessel equation, called *Bessel functions of the second kind* and denoted $Y_n(r)$. These are singular at $r = 0$ and so occur less often in solutions of real life problems. One can verify that the Bessel functions $J_n(r)$ obey the Bessel equation for any integer $n$ by substituting Eq. (385) in Eq. (384) and integrating by parts.

We can obtain the **general regular solution** of our original wave equation by summing over the solutions for each $n$ and $\omega$, with each multiplied by a coefficient $C_n(\omega)$. Since $t$ takes all real values, the summation over $\omega$ is actually a Fourier integral and since $f(r,\varphi,t)$ is periodic in $\varphi$, the sum over $n$ is a Fourier series.

$$f(r,\varphi,t) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} C_n(\omega)e^{in\varphi}e^{i\omega t}J_n(\omega r)\,d\omega \tag{386}$$

The constants $C_n(\omega)$ are arbitary. Note that if the problem is cylindrically symmetric, so that $f$ depends only on $r$ and $t$, then we only need the term with $n = 0$ in the series, so that

$$f(r,t) = \int_{-\infty}^{\infty} C_0(\omega)e^{i\omega t}J_0(\omega r)\,d\omega. \tag{387}$$

## 6.3 The Hankel transform

The **Hankel transform of order** $n$ of a function $f(r)$ is defined by

$$\mathcal{H}_n f(r) = \tilde{f}_n(\alpha) = \int_0^\infty J_n(\alpha r) f(r) r dr. \tag{388}$$

The kernel of this transform is therefore $rJ_n(\alpha r)$.

One can show that the Hankel transform is its own inverse:

$$f(r) = \int_0^\infty J_n(\alpha r) \tilde{f}_n(\alpha) \alpha \, d\alpha, \tag{389}$$

for any integer $n$.

## 6.4 Relation between FT and HT

We have seen that the Laplace transform can be derived from the Fourier transform. The Hankel transform can also be derived from the Fourier transform, and we show this here as it is neat.

Consider the FT in two dimensions, in Cartesian coordinates:

$$\hat{f}(k_1, k_2) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(k_1 x_1 + k_2 x_2} f(x_1, x_2) \, dx_1 \, dx_2. \tag{390}$$

If we now transform to polar coordinates, then we expect this to break up into a Fourier series and a Hankel transform. So define

$$x_1 \;:=\; r\cos\theta, \quad x_2 := r\sin\theta, \tag{391}$$

$$k_1 \;:=\; \alpha\cos\phi, \quad k_2 := \alpha\sin\phi. \tag{392}$$

In a slight abuse of notation, define $\hat{f}(k_1, k_2) = \hat{f}(\alpha, \varphi)$, and $f(x_1, x_2) = f(r, \theta)$. Define the Fourier series coefficients $f_n(r)$ by

$$f(r, \theta) =: \sum_{-\infty}^{\infty} f_n(r) e^{in\theta}. \tag{393}$$

Then one can show that

$$\hat{f}(\alpha, \varphi) = \sum_{-\infty}^{\infty} \hat{f}_n(\alpha) e^{in\phi}, \tag{394}$$

where

$$\hat{f}_n(\alpha) = (-i)^n \mathcal{H}_n \left[ f_n(r) \right] (\alpha). \tag{395}$$

## 6.5 Application of the Hankel transform to PDEs

As we have seen, the second order differential operator

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2} \tag{396}$$

comes up when one writes the Laplace equation, diffusion equation or wave equation in cylindrical polar coordinates; see for example equation (381). The Hankel transform is useful because it takes a simple form under this differential operator. Let $\mathcal{H}_n f(r) = \tilde{f}_n(\alpha)$ be the Hankel transform of order $n$ of the function $f(r)$. One can show (by two integrations by parts) that

$$\mathcal{H}_n \left( f_{rr} + \frac{1}{r} f_r - \frac{n^2}{r^2} f \right) = -\alpha^2 \tilde{f}_n(\alpha) \tag{397}$$

provided that $rf(r)$ and $rf_r(r)$ tend to zero as $r \to \infty$. (Note that this identity is given on the formula sheet and you are not expected to either remember it or to prove it.)

This property is similar to the Fourier transform of $f_{xx}$ being $-k^2\hat{f}$, but in cylindrical polar coordinates. Here, too, it allows converting a differential expression for $f$ into an algebraic expression for its transform $\tilde{f}_n$.

The Hankel transform is not as easy to calculate as the Fourier or Laplace transform because its integral kernel is more complicated. There are, however, tables of Hankel transforms and, in simple cases, we can find what we need in the tables of Laplace and Fourier transforms.

**Example** Solve

$$f_{tt}(r,t) = f_{rr}(r,t) + \frac{1}{r} f_r(r,t) \tag{398}$$

for $f(r,t)$ for $r \geq 0$ and $t \geq 0$ subject to the following initial conditions:

$$f(r,0) = 0 \qquad f_t(r,0) = e^{-kr} \tag{399}$$

with $k$ a constant. The boundary conditions are

$$rf(r,t) \to 0 \qquad rf_r(r,t) \to 0 \tag{400}$$

as $r \to \infty$, so the above identity is applicable.

Looking at the right hand of the differential equation, we can see that we should use the Hankel transform of order zero, i.e.

$$\mathcal{H}_0 \left( f_{rr}(r,t) + \frac{1}{r} f_r(r,t) \right) = -\alpha^2 \tilde{f}_0(\alpha, t) \tag{401}$$

Implicitly we use the boundary conditions when applying this expression. Transforming the left hand side of the equation is straightforward, as $t$ derivatives commute with the transform:

$$\mathcal{H}_0 \left( f_{tt}(r,t) \right) = \frac{\partial^2}{\partial t^2} \left( \tilde{f}_0(\alpha, t) \right). \tag{402}$$

Thus the transformed equation is an ordinary differential equation in time:

$$\frac{\partial^2}{\partial t^2} \left( \tilde{f}_0(\alpha, t) \right) = -\alpha^2 \tilde{f}_0(\alpha, t) \tag{403}$$

We also need to transform the initial conditions.

$$\mathcal{H}_0(f(r,0)) = \tilde{f}_0(\alpha, 0) = 0 \tag{404}$$

while

$$\frac{\partial}{\partial t} \left( \tilde{f}_0(\alpha, 0) \right) = \mathcal{H}_0 \left( e^{-kr} \right). \tag{405}$$

The latter can be shown to be

$$\frac{k}{(k^2 + \alpha^2)^{\frac{3}{2}}}. \tag{406}$$

We can now solve the transformed equation subject to these initial conditions. The general solution of the transformed equation is

$$\tilde{f}_0(\alpha, t) = A(\alpha) \sin(\alpha t) + B(\alpha) \cos(\alpha t) \tag{407}$$

and using the initial conditions we obtain

$$\tilde{f}_0(\alpha, t) = \frac{k}{\alpha (k^2 + \alpha^2)^{\frac{3}{2}}} \sin(\alpha t) \tag{408}$$

which implies that

$$f(r,t) = \int_0^\infty \frac{k}{(k^2 + \alpha^2)^{\frac{3}{2}}} \sin(\alpha t) J_0(\alpha r) d\alpha \tag{409}$$

(The integral over $\alpha$ can be computed using tables, but it cannot be found using elementary methods.)

# 7 Nyquist stability theory

## 7.1 Stability of circuits

In this section we will apply the theory of Laplace transforms to circuits.

Suppose an input voltage $V_{\rm in}(t)$ and an output voltage $V_{\rm out}(t)$ of a circuit are related by the ordinary differential equation

$$a_n V_{\rm out}^{(n)}(t) + a_{n-1} V_{\rm out}^{(n-1)}(t) + ... + a_1 V_{\rm out}^{(1)} + a_0 V_{\rm out}(t) = b_m V_{\rm in}^{(m)}(t) + ... + b_0 V_{\rm in}(t), \tag{410}$$

where $n > m$ and $a_n, b_m$ are constants. Suppose that $V_{\rm in}$ and $V_{\rm out}$, and all their time derivatives up to (not including) order $n$ vanish at $t = 0$. Taking the Laplace transform of equation (410) and using the derivative property of the Laplace transform, we find that

$$a_n\, s^n\, \bar{V}_{\rm out}(s) + a_{n-1}\, s^{n-1}\, \bar{V}_{\rm out}(s) + ... + a_1\, s\, \bar{V}_{\rm out}(s) + \; a_0\, \bar{V}_{\rm out}(s) \;\; = \;\; b_m\, s^m \bar{V}_{\rm in}(s) + ... + b_0\, \bar{V}_{\rm in}(s). \tag{411}$$

We can rearrange this to get

$$\frac{\mathcal{L}\left[V_{\rm out}(t)\right]}{\mathcal{L}\left[V_{\rm in}(t)\right]} \;\; = \;\; \frac{\bar{V}_{\rm out}(s)}{\bar{V}_{\rm in}(s)} \;\; = \frac{b_m s^m + ... + b_0}{a_n s^n + ... + a_0} \;\; \equiv \;\; \bar{A}(s). \tag{412}$$

$\bar{A}(s)$ is called the **system transfer function**. From the convolution property of the Laplace transform

$$\bar{V}_{\rm out} = \bar{V}_{\rm in} \bar{A}(s) \;\; \Rightarrow \;\; V_{\rm out}(t) = \mathcal{L}^{-1}[\bar{V}_{in} \bar{A}] = \int_0^t A(t-\tau) V_{\rm in}(\tau)\, d\tau. \tag{413}$$

We see that $A(t)$ is the reaction of the circuit to a $\delta$-function blip in the input at $t = 0$. Because of linearity, the reaction of the circuit to an arbitrary input $V_{\rm in}(t)$ is just an integral over such blips. The function $A(t-\tau)$ is known as the *Green function* of the circuit.

This method is useful for determining the reaction to an arbitrary input, but here we shall only ask if the circuit is stable or not. "Stable" here means that a bounded input gives a bounded output, and this, in turn, means that $A(t)$ does not blow up as $t \to \infty$.

We shall now determine the stability of the circuit based on $\bar{A}(s)$.

Because $\bar{A}(s)$ is a rational function, the residue of $e^{st}\bar{A}(s)$ at $s = a$ is $e^{at}$ times a polynomial in $t$. (The order of the polynomial is the order of the pole minus 1, as can be shown by the derivative rule formula for the calculation of residues.) Therefore, poles in $\bar{A}(s)$ with $\mathrm{Re}\, s < 0$ give contributions to $A(t)$ that decay exponentially at late times, while poles with $\mathrm{Re}\, s > 0$ give contributions that blow up exponentially at late times and make the circuit unstable.

Poles on the imaginary axis are critically stable and require further investigation. Simple poles on the imaginary $s$-axis give contributions to $A(t)$ that are constant or oscillate at late times - these are marginally stable. Higher poles on the imaginary axis give blowup as a power of $t$ and hence are unstable.

Henceforth we assume that all critically stable poles are marginally stable, since the case when there exists an unstable pole on the imaginary axis is trivial (the circuit is unstable!):

*The circuit is stable if $\bar{A}(s)$ has no poles in the right hand complex plane.*

## 7.2 Closed loop feedback circuits

Now we consider two circuits, with transfer functions $\bar{A}(s)$ and $\bar{B}(s)$, put together in a **closed loop system** with negative feedback (see figure 42).

The original system function $\bar{A}(s)$ is now called the *forward transfer function.* The output $V_{\rm out}$ is monitored and fed into a feedback path with transfer function $\bar{B}(s)$. The feedback signal $V_{\rm fb}$ is fed into a *comparator* (C), which modifies the input to $A$ according to

$$V_{\rm in} = V_{\rm in,total} - V_{\rm fb}, \tag{414}$$

where $V_{\rm in,total}$ is the total input to the system.

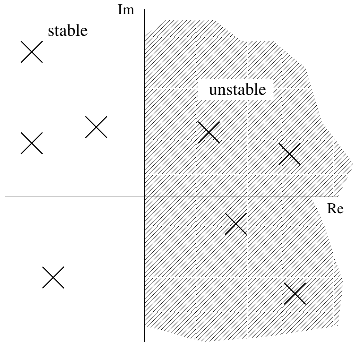



Figure 41: Stabilizing and destabilizing poles.

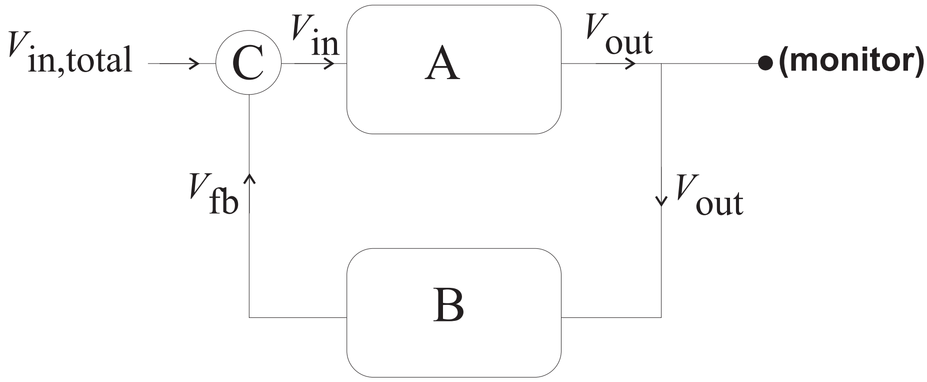



Figure 42: The closed loop circuit.

The total transfer function of the system is, by definition

$$\bar{f}(s) = \frac{\bar{V}_{\text{out}}(s)}{\bar{V}_{\text{in,total}}(s)} = \frac{\bar{A}(s)\bar{V}_{\text{in}}}{\bar{V}_{\text{in}}(s) + \bar{V}_{\text{fb}}(s)}, \tag{415}$$

where we used $\bar{V}_{\text{out}}(s)/\bar{V}_{\text{in}}(s) = \bar{A}(s)$. Noticing that $\bar{V}_{\text{fb}}(s)/\bar{V}_{\text{in}}(s) = (\bar{V}_{\text{fb}}/\bar{V}_{\text{out}})(\bar{V}_{\text{out}}/\bar{V}_{\text{in}}) = \bar{A}(s)\bar{B}(s)$, we obtain the total transfer function of the system as

$$\bar{f}(s) \;=\; \frac{\bar{A}(s)}{1+\bar{A}(s)\bar{B}(s)}. \tag{416}$$

The loop is stable *if $\bar{f}(s)$ has no poles in the right hand plane.*

It is usually assumed that both systems $A$ and $B$ are stable on their own—only then does it make sense to ask about stability of the integrated feedback loop. We henceforth thus assume that neither $\bar{A}(s)$ nor $\bar{B}(s)$ have poles in the right hand plane. In this case, the only poles of $\bar{f}(s)$ are possible zeros of

$$\bar{g}(s) \;\equiv\; 1+\bar{A}(s)\bar{B}(s). \tag{417}$$

[Note that zeros of $\bar{g}(s)$ cannot cancel out with zeros of the numerator, $\bar{A}(s)$, if $\bar{B}(s)$ has no poles.] The feedback circuit will be stable *if and only if $\bar{g}(s)$ has no zeros in the right hand plane.*

If $\bar{g}(s)$ is a simple enough polynomial, we can find its zeros directly. In more complicated cases we can use the *Principle of the Argument*, with a suitable contour, to determine the number of zeros $\bar{g}(s)$ has in the right half plane. (For our purpose we don't actually need to find those zeros, only to tell if there exist any!)

## 7.3 Nyquist's stability criterion

To implement the Principle of the Argument in the above stability problem we apply the **Nyquist contour**. This is a contour section $\Gamma$ which runs down the imaginary axis from $+i\infty$ to $-i\infty$. It is then closed at infinity with a semicircular arc ($\gamma_R$), such that the contour covers the entire right hand plane as the radius goes to infinity (see figure 43). The closed contour is usually denoted $C$.

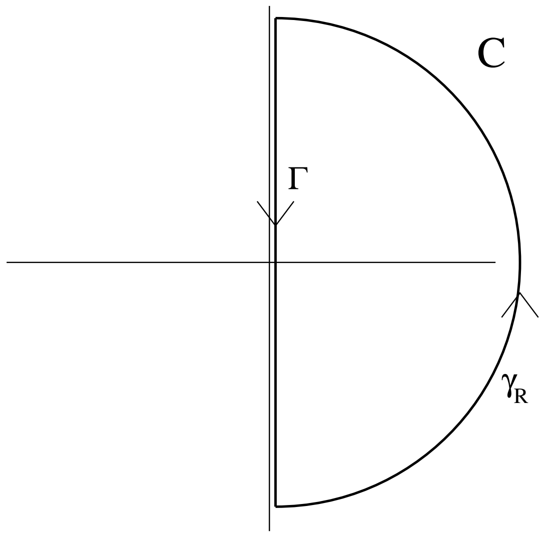



Figure 43: The Nyquist contour, closed with a semicircular arc.

All the zeros and poles in the right hand plane will be inside the contour $C$. If there is a zero or pole on the imaginary axis we exclude it with an $\epsilon$-semicircle (see Figure 44): we wish to only count unstable poles as critically stable poles are considered separately.

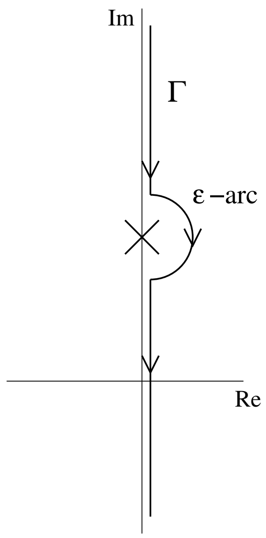



Figure 44: Deformation of the Nyquist contour to exclude critical values.

The principle of the argument states that, for a contour $C$ and function $\bar{g}(s)$

$$W = N - P, \tag{418}$$

where $N$ = number of zeros of $\bar{g}$ inside $C$,
$P$ = number of poles of $\bar{g}$ inside $C$, and
$W$ = number of times the $\bar{g}(s)$ winds around the origin in the $\bar{g}$ ($\bar{g}s = 0$) as $s$ moves around the contour $C$.

Recall that for the feedback circuit considered above we have stability if and only if $\bar{g}(s) = 1 + \bar{A}(s)\bar{B}(s)$ has no *zeros* in the right hand plane. Namely, the circuit is stable if and only if

$$N = P + W = 0. \tag{419}$$

If both systems $A$ and $B$ are stable, $\bar{A}(s)$ and $\bar{B}(s)$ have no poles in the right hand plane. In this case, $\bar{g}(s)$, too, has no poles in the right hand plane, and we have $P = 0$. (In solving problems it is necessary to first check that $A$ and $B$ are stable before assuming $P = 0$!)

If $P = 0$ then the stability criterion becomes simply

$$W = 0 \tag{420}$$

In this case, one therefore only needs to calculate the winding number of the contour $C$.

Most textbooks define

$$\bar{h}(s) \;=\; \bar{g}(s) - 1 \;=\; \bar{A}(s)\,\bar{B}(s), \tag{421}$$

and then ask how many times $\bar{h}(s)$ winds around $\bar{h}(s) = -1$ as $s$ goes around $C$, instead of asking how many times $\bar{g}(s)$ winds around $\bar{g}(s) = 0$ as $s$ passes around the contour $C$.

To answer this question one should sketch the path $\bar{h}(C)$, as in the following example.

**Example:** Determine the stability of a feedback system formed from a circuit with transfer function $\bar{A}(s) = \frac{1-s}{1+s}$ and a feedback circuit with transfer function $\bar{B}(s) = \frac{k}{s}$ where $k > 0$ is a (real) constant.

$\bar{A}(s)$ has pole at $s = -1$ $\Rightarrow$ no poles in the RH plane $\Rightarrow$ $\bar{A}(s)$ is stable,
$\bar{B}(s)$ has pole at $s = 0$ $\Rightarrow$ no poles *inside* the RH plane $\Rightarrow$ $\bar{B}(s)$ is stable.

Hence $\bar{h}(s) = \bar{A}(s)\bar{B}(s)$ has no poles in the RH plane, and to determine the zeros of $\bar{h}$ we need only consider the winding number of $\bar{h}(C)$.

The Nyquist contour for

$$\bar{h}(s) \;=\; \bar{A}(s)\,\bar{B}(s) \;=\; \frac{(1-s)}{(1+s)}\frac{k}{s}, \tag{422}$$

goes down the imaginary axis avoiding the simple pole at $s = 0$ with an $\epsilon$-semicircle (see Figure 45).

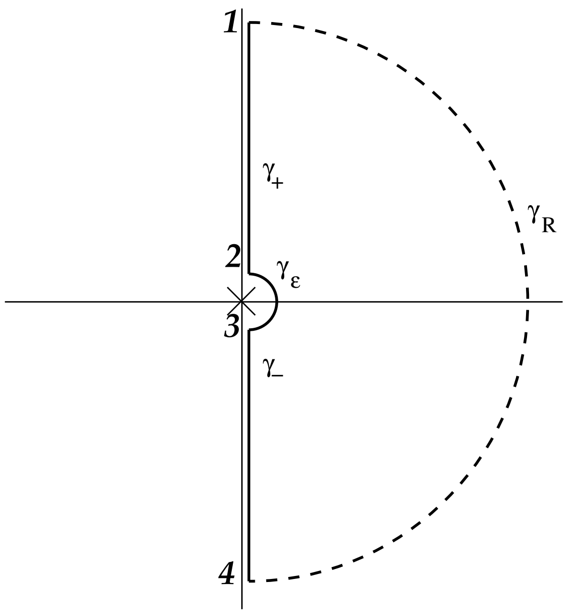



Figure 45: The Nyquist contour used for $\bar{h}(s) = \frac{k(1-s)}{s(1+s)}$.

To sketch $\bar{h}(C)$ we will consider the contour $C$ in parts.

On $\gamma_{\rm R}$ we have $s = Re^{i\theta}$ with $\theta \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$ and $R \to \infty$. Then,

$$\bar{h}(\gamma_{\rm R}) = \lim_{R\to\infty} \frac{(1 - Re^{i\theta})k}{(1 + Re^{i\theta})Re^{i\theta}} = \lim_{R\to\infty}\left(-\frac{ke^{-i\theta}}{R}\right) = 0. \tag{423}$$

(So the image of $\bar{h}(s)$ along the entire semi-circle $\gamma_{\rm R}$ shrinks to a point.)

On $\gamma_+$ we have $s = i\omega$ with $\omega > 0$. Then,

$$\bar{h}(\gamma_+) = k\frac{(1-i\omega)}{i\omega(1+i\omega)} = -\frac{2k}{1+\omega^2} + i\frac{k(\omega^2-1)}{\omega(\omega^2+1)}. \tag{424}$$

At point 1 ($\omega \to +\infty$) we have already found $\bar{h} = 0$.

At point 2 ($\omega \to 0$) we have $\bar{h} \to -2k - \frac{k}{\omega}i.$, which is an asymptote $x = -2k$ with $y \to -\infty i$.

On $\gamma_\epsilon$, $s = \epsilon\, e^{-i\theta}$ for $\theta \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$, since we move clockwise around the semicircle, so:

$$\bar{h}(s) = \frac{k}{\epsilon\, e^{-i\theta}}\frac{1-\epsilon e^{-i\theta}}{1+\epsilon e^{-i\theta}} \sim \frac{k}{\epsilon}e^{+i\theta} \text{ for } \epsilon \ll 1. \tag{425}$$

Hence $\bar{h}(\gamma_\epsilon)$ is still a semicircle, but orientated anticlockwise from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ and with radius $\frac{k}{\epsilon} \to \infty$ as $\epsilon \to 0$.

Finally, on $\gamma_-$ we have $s = -i\omega$ with $\omega > 0$. We notice then from Eq. (424) that $\bar{h}(\gamma_-)$ is complex conjugate to $\bar{h}(\gamma_+)$. This means that the image of $\bar{h}(\gamma_-)$ can be obtained from the image of $\bar{h}(\gamma_+)$ by merely reflecting through the x axis.

To be able to plot the image of $\bar{h}(s)$ it is also useful to look at points where $\bar{h}(s)$ crosses the axes:

$\bar{h}(\gamma_\pm)$ crosses the *real* axis when

$$\operatorname{Im}\left[\bar{h}(s)\right] = \pm\frac{k(\omega^2-1)}{\omega(\omega^2+1)} = 0, \tag{426}$$

i.e., when $\omega = 1$, for which $x = -\frac{2k}{1+1^2} = -k$.

Hence $\bar{h}$ cuts the real axis at $\bar{h}(s) = -k$ when $s = \pm i$.

$\bar{h}(\gamma_\pm)$ crosses the *imaginary* axis when

$$\operatorname{Re}\left[\bar{h}(s)\right] = -\frac{2k}{1+\omega^2} = 0, \tag{427}$$

i.e., when $\omega = \pm\infty$, for which $y = 0$.

Hence $\bar{h}$ cuts the imaginary axis at $\bar{h}(s) = 0$ when $s = \pm i\infty$.

We are now ready to sketch $\bar{h}\,(C)$—see figure 46.

Our sketch indicates that $W = \begin{cases} 2, & \text{if } -k < -1, \\ 0, & \text{if } -k > -1. \end{cases}$ Hence

$$N = \begin{cases} 2, & \text{if } k > 1, \\ 0, & \text{if } k < 1. \end{cases} \tag{428}$$

The circuit is therefore stable if and only if $k < 1$. Combining these results we see that the circuit is only stable for $0 < k < 1$.

**Example:** Determine the stability of the above closed loop circuit by considering the poles of $\bar{f}$ directly.

$$\bar{f}(s) = \frac{\bar{A}(s)}{1+\bar{A}(s)\bar{B}(s)} = \frac{\frac{1-s}{1+s}}{1+\frac{k}{s}\frac{1-s}{1+s}} = \frac{s(1-s)}{s^2+(1-k)s+k}. \tag{429}$$

The poles of $\bar{f}$ are the zeros of

$$s^2 + (1-k)s + k. \tag{430}$$

The roots are given by

$$s_\pm = \frac{1}{2}\left(k - 1 \pm \sqrt{1-6k+k^2}\right), \tag{431}$$

and the circuit is stable provided the real parts of $s_+$ and $s_-$ are both not positive.

Fig. 47 shows a plot of $\operatorname{Re}(s_\pm)$ as a function of $k$. The plot was obtain by evaluating Eq. (431) numerically. The plot confirms our finding that the system is only stable when $0 < k < 1$.

In examples where the transfer function $\bar{f}$ is more complicated, it may be very difficult to obtain the poles of $\bar{f}$ directly, as in the last example. In such cases, applying the Nyquist stability analysis may require much less work and will usually be more effective.

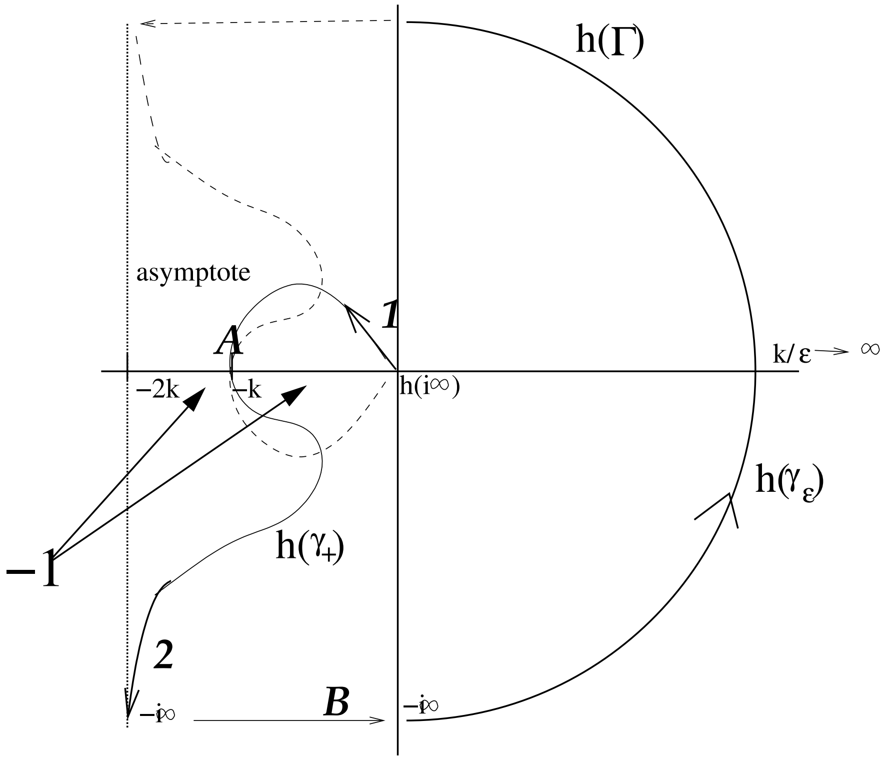



Figure 46: The path of $\bar{h}(s)$ in the complex plane as $s$ travels along $C$.

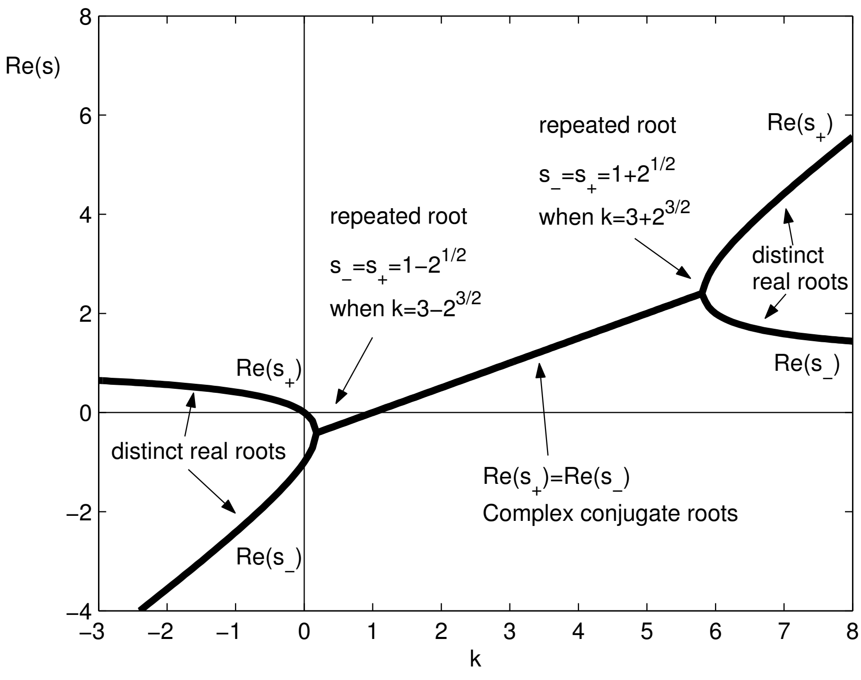



Figure 47: The real part of the roots $s_+$ and $s_-$, as functions of $k$.