MATH3083/MATH6163
Advanced Partial Differential Equations

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Contents

1 Overview

1.1 Notation and terminology

In these lecture notes I use **boldface** to highlight technical terms when they are being defined, either informally by using them, or in a formal definition. These terms also appear in the index at the end of these notes. I use *italics* to highlight something that is important. In equations, either $a := b$ or $b =: a$ means that a is being defined in terms of b. Three dots ... in a mathematical expression signify something that I do not write out in full because it repeats something earlier – what, should be clear from the context. Vectors are written in boldface in the printed notes, for example $\boldsymbol{x}$, but on the board I will underline them.

1.1.1 Partial derivatives and index notation

In these notes, I often abbreviate partial derivatives by commas, as in

$$u_{,x} := \frac{\partial u}{\partial x}, \quad u_{,xy} := \frac{\partial^2 u}{\partial x \partial y} \tag{1}$$

Obviously, $u_{,xy} = u_{,yx}$, as partial derivatives commute. It is helpful to stick to some preferred order (say x first). To take an example of this notation, the most general linear second-order PDE in one dependent variable u and two independent variables (or coordinates) x and y can be written as

$$au_{,xx} + 2bu_{,xy} + cu_{,yy} + pu_{,x} + qu_{,y} = f. \tag{2}$$

(This is linear if the coefficients a, b, c, p, q and f do not depend on u.)

Remark 1.1. In the PDE literature (for example in Renardy and Rogers), it is more common to write u_x, u_{xy} etc., without the comma, but I find it helpful to distinguish partial derivatives from other indices.

To write a PDE even more concisely when discussing general theory, we may number the independent variables, for example

$$x_1 := x, \quad x_2 := y, \tag{3}$$

and do the same for the partial derivatives, obviously in the same numbering:

$$u_{,1} := u_{,x}, \quad u_{,2} := u_{,y}. \tag{4}$$

We do the same for the coefficients in (2),

$$a^{11} := a, \quad a^{12} = a^{21} := b, \quad a^{22} := c, \quad b^1 := p, \quad b^2 := q \tag{5}$$

and then we can write our example PDE (2) as

$$\sum_{i=1}^{2}\sum_{j=1}^{2} a^{ij}u_{,ij} + \sum_{i=1}^{2} b^i u_{,i} = f, \tag{6}$$

or in more relaxed notation

$$\sum_{i,j} a^{ij}u_{,ij} + \sum_{i} b^i u_{,i} = f. \tag{7}$$

We can, and always will, assume that a_{ij} is symmetric. Make sure you understand where the factor of 2 in the coefficient $2b$ of (2) has gone when it is written in the form (6). The form (6) has the advantage that it can be easily generalised to an arbitrary number of independent variables. We will use this in definitions and theorems later.

1.1.2 Systems of PDEs

More generally, we may want to solve a **PDE system** of $N \geq 2$ PDEs for N dependent variables $(u_1, u_2, \ldots, u_N)$, in n independent variables $(x_1, x_2, \ldots x_n)$. Note that typically there will be as many PDEs as dependent variables (unknowns) (N of each), but the number of independent variables n has nothing to do with that. It is then useful to write the dependent variables and the equations as components of column vectors.

A simple example with $N = 2$ and $n = 2$ are the **Cauchy-Riemann equations**

$$u_{,y} + v_{,x} = 0, \tag{8}$$
$$u_{,x} - v_{,y} = 0. \tag{9}$$

If we write the dependent variables u and v as the vector

$$\boldsymbol{u} := \begin{pmatrix} u \\ v \end{pmatrix}, \tag{10}$$

we can write the two PDEs as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{u}_{,x} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{u}_{,y} = 0, \tag{11}$$

or more abstractly as

$$\sum_{i=1}^{2} A^i \boldsymbol{u}_{,i} = 0, \tag{12}$$

where

$$A^x := A^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A^y := A^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

As another example, the most general second-order linear system in N dependent and n independent variables can, in an abstract way, be written as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A^{ij} \boldsymbol{u}_{,ij} + \sum_{i=1}^{n} B^i \boldsymbol{u}_{,i} = \boldsymbol{f}, \tag{14}$$

where each A^{ij} and each B^i (say, $A^{12} = A^{21}$, or B^3) is an $N \times N$ matrix. One sometimes refers to $\boldsymbol{u} = (u_1, \ldots u_N)$ as a vector in **state space** to distinguish it from a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ in **physical space**.

1.1.3 PDE problems

A PDE (or PDE system) is not usually solved in isolation. We must specify a **domain** in space, or in space and time, on which we want the PDE to hold. If the domain is **bounded**, then we will need to impose one or several **boundary conditions** on each **boundary**. For a time-dependent problem, boundary conditions at the initial time $t = 0$ are also called **initial conditions**. (For other purposes, it may make sense to refer to boundary conditions in space and initial conditions together as "boundary conditions".) It is important to understand that the initial and boundary conditions are independent of the PDE (system) in the interior.

If the domain is **unbounded**, for example all of $\mathbb{R}^n$, then we still need boundary conditions, but they may be less obvious. For example, we may want the solution to fall off (go to zero) sufficiently fast as we approach infinity.

Definition 1.2. A **PDE problem** is a PDE (system), together with its domain and all necessary boundary and/or initial conditions.

1.1.4 Revision: Linear PDEs

Probably all the PDEs you have seen until now were **linear PDEs**. This means that they are linear expressions (containing the first power or the zeroth power) in the unknown and its derivatives. Our example PDE (2) is linear if the coefficients a, b, c, p, q, f are known functions of x and y (they may be constants), but not of u. All terms in it are of the first power in the unknown u, except for s, which is zeroth power. If there is a zeroth-power term, also called the **inhomogeneous term** or **source term**, the linear PDE is called **inhomogeneous**. If this is absent ($f = 0$ in our example), the linear PDE is **homogeneous**. A PDE system is linear if each PDE is linear in the vector $\boldsymbol{u}$ of unknowns.

Sometimes it is useful to write an inhomogeneous linear PDE abstractly as

$$Lu = f, \tag{15}$$

where L is a homogeneous linear differential operator (say the Laplace operator) and f is the inhomogeneous term. For a PDE system, we could write $L\boldsymbol{u} = \mathbf{f}$.

The general solution (without imposing any boundary conditions) of any inhomogeneous linear PDE (or PDE system) is of the form $u = u_{\mathrm{CF}} + u_{\mathrm{PI}}$. Here the **complementary function** u_{CF} is the general solution of the corresponding homogeneous linear PDE $Lu = 0$, and the **particular integral** u_{PI} is any one solution of the original inhomogeneous linear PDE (15). This is just the same as for linear ODEs. It works because

$$Lu = L(u_{\mathrm{CF}} + u_{\mathrm{PI}}) = Lu_{\mathrm{CF}} + Lu_{\mathrm{PI}} = 0 + f = f. \tag{16}$$

We have used the linearity of L in the second equality. Recall that any operator L is linear if $L(f + g) = L(f) + L(g)$ and $L(cf) = cL(f)$, where c is a constant. For example $L(f) = df/dx$ or $L(f) = h(x)f$ are linear, but $L(f) = f^2$ would not be. Moreover, it is customary in mathematics to write homogeneous linear functions as prefix operators, that is, we write $L(u)$ as Lu if L is homogeneous linear.

Similarly, in linear PDE problems we distinguish **homogeneous boundary conditions** and **inhomogenous boundary conditions**. $u = 0$ at the boundary is an example of a homogenous boundary condition, while $u = g$, with $g(\boldsymbol{x})$ a given function, would be inhomogeneous. This $g(\boldsymbol{x})$ is an example of what is called **boundary data**. With inhomogeneous boundary conditions and a source term, we can then again write the general solution as the sum of a particular integral that solves the PDE problem with the source term and with inhomogeneous boundary conditions, and a complementary function that is the general solution of the corresponding PDE with zero source term and homogeneous boundary conditions.

1.2 Revision: Vector calculus

1.2.1 Notation

In (14), we have written the dependent variables as a vector $\boldsymbol{u}$, but we have used index notation for the independent variables. In other contexts, it may be useful to write the *independent variables* as a vector $\boldsymbol{x}$, especially if they represent a position vector in three-dimensional Euclidean space. A dependent variable may itself be a direction vector in Euclidean space, for example the velocity vector $\boldsymbol{v}(\boldsymbol{x})$ in fluid dynamics. We then write

$$\boldsymbol{x} := (x_1, x_2, x_3) := (x, y, z), \tag{17}$$

$$\boldsymbol{v}(\boldsymbol{x}) := (v^1, v^2, v^3) := (v^x, v^y, v^z), \tag{18}$$

$$\nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{19}$$

We can use ∇ (pronounced **nabla**) to define the **gradient** of the scalar function u as

$$\nabla u := (u_{,x}, u_{,y}, u_{,z}) \tag{20}$$

and the **divergence** of the vector-valued function $\boldsymbol{v}$ as

$$\nabla \cdot \boldsymbol{v} := v^x_{,x} + v^y_{,y} + v^z_{,z} = \sum_i^3 v^i{}_{,i}. \tag{21}$$

Note that

$$\nabla \cdot \boldsymbol{x} = \sum_{i=1}^{n} x^i{}_{,i} = n \tag{22}$$

gives the number of space dimensions.

Remark 1.3. In this course, I use the notation of denoting a vector by a boldface letter both for vectors in three (or two) space dimensions such as $\boldsymbol{x}$, short for x_i with $i = 1, \ldots, n$, and for vectors of variables such as $\boldsymbol{u}$, short for u^α with $\alpha = 1, \ldots, N$. I use ∇ only to denote derivatives with respect to space (the independent variables). (By contrast, Renardy and Rogers use it also for derivatives with respect to the dependent variables.)

1.2.2 The Laplace operator

The second-order derivative operator $\nabla \cdot \nabla$ is called the Laplace operator and is usually denoted by either ∇^2 or Δ. In three-dimensional **Cartesian coordinates** $\boldsymbol{x} = (x_1, x_2, x_3)$ the Laplace operator is

$$\Delta_{(3)} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \tag{23}$$

In two or one dimensions we have

$$\Delta_{(2)} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \qquad \Delta_{(1)} = \frac{\partial^2}{\partial x^2}. \tag{24}$$

Without proof, we state a few other formulas. In **cylindrical polar coordinates** (r, θ, z), where

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z, \tag{25}$$

we have

$$\Delta_{(3)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \tag{26}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}. \tag{27}$$

By just removing the coordinate z we obtain polar coordinates in the plane

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \tag{28}$$

where

$$\Delta_{(2)} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{29}$$

In **spherical polar coordinates** (r, θ, φ) where

$$x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \quad x_3 = r\cos\theta, \tag{30}$$

we have

$$\Delta_{(3)} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] \tag{31}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]. \tag{32}$$

In the following, Δ will be used to denote any of $\Delta_{(3)}$, $\Delta_{(2)}$ or $\Delta_{(1)}$. Note that in curvilinear coordinates, such as the polar coordinates we have used here, the Laplace operator generally contains first, as well as second, derivatives.

All of these formulas can be derived from the **chain rule of partial derivatives**

$$\frac{\partial}{\partial x'^i} = \sum_{j=1}^{n} \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}, \tag{33}$$

where x^j is one system of coordinates and x'^i the other. As a simple example, where $n = 2$, take the transformation from coordinates (x, y) to (ξ, η), given by

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y}, \tag{34}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y}. \tag{35}$$

1.2.3 The divergence theorem

There is more to vector calculus than concise notation. One result we will need is

Theorem 1.4 (**Divergence theorem**)**.** Let V be a volume and $S := \partial V$ its surface. Let dV be the **volume element** and dS the **surface element**, and let $\boldsymbol{n}(\boldsymbol{x})$ be the **outward-pointing unit normal vector** at each point of S. Then

$$\int\int\int_V \nabla \cdot \boldsymbol{f}\, dV = \int\int_S \boldsymbol{n} \cdot \boldsymbol{f}\, dS, \tag{36}$$

It is is easy to prove the theorem in the simple case in three dimensions where V is the rectangular box $x_0 \le x \le x_1$, $y_0 \le y \le y_1$, $z_0 \le z \le z_1$, and hence S is the union of six rectangles. Using the **fundamental theorem of calculus**,

$$\int_{x_0}^{x_1} \frac{df}{dx}\, dx = f(x_1) - f(x_0), \tag{37}$$

separately in the x, y and z directions, one can show that

$$\begin{aligned}
&\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy \int_{z_0}^{z_1} dz\, (f^x_{,x} + f^y_{,y} + f^z_{,z}) \\
= &\int_{y_0}^{y_1} dy \int_{z_0}^{z_1} dz\, [f^x(x_1, y, z) - f^x(x_0, y, z)] \\
+ &\int_{x_0}^{x_1} dx \int_{z_0}^{z_1} dz\, [f^y(x, y_1, z) - f^y(x, y_0, z)] \\
+ &\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\, [f^z(x, y, z_1) - f^z(x, y, z_0)]
\end{aligned} \tag{38}$$

In vector calculus notation this can be written as (36), where $\boldsymbol{n} = (0, -1, 0)$ for all points in the part of S given by $y = y_0$, $x_0 \le x \le x_1$, $z_0 \le z \le z_1$, and so on for the other five faces of the box.

The actual divergence theorem above generalises our simple result in two ways. First, one can prove that (36) holds for any volume V with boundary S, not only a rectangular box. Secondly, ∇, $\boldsymbol{f}$, dV, dS and $\boldsymbol{n}$ are *geometric objects*, in the sense that the two integrals on either side of (36) can be defined and evaluated not only in Cartesian coordinates, but in arbitrary coordinates.

You have previously learned to evaluate (36) in some simple situations, but for this course we mostly need the abstract version.

1.3 Revision: Separation of variables

In MATH2038, MATH2047, MATH2048 or MATH2015 you have learned to solve PDEs using separation of variables. Keep in mind that this works only under the following conditions (we initially assume there are only $n = 2$ independent variables):

1. *The PDE is homogeneously linear and separable.* This means that an **ansatz** like

$$u(x, y) = X(x)\, Y(y) \tag{39}$$

turns the linear PDE

$$Lu(x, y) = 0, \tag{40}$$

where L is a homogeneous linear differential operator into

$$L_x X(x) = L_y Y(y), \tag{41}$$

where L_x and L_y are homogeneous linear differential operators. Hence both sides must be equal to some constant K, and so we have the ODEs

$$L_x X_K(x) = K, \qquad L_y Y_K(y) = K. \tag{42}$$

These ODEs are then solved for arbitrary **separation constant** K. Typically, homogeneous boundary conditions allow only discrete values of K. The solution $u(x, y)$ is obtained as the sum

$$u(x, y) = \sum_K c_K X_K(x) Y_K(y). \tag{43}$$

The constants c_K are typically determined by initial conditions or inhomogeneous boundary conditions.

2. *The domain of the problem is a coordinate rectangle.* In other words, the domain has the form $a < x < b$ and $c < y < d$. The important consequence of this is that each part of the boundary affects only one of the ODEs above. So a boundary condition at $x = 0$ becomes a boundary condition for $X(0)$ but does not affect $Y(y)$. This in turn means that the ODEs for X and Y can be solved separately.

A coordinate rectangle does not have to be a rectangle in physical space. For example, a sphere in spherical coordinates is given by

$$0 \le r \le 1, \qquad 0 \le \theta \le \pi, \qquad 0 \le \varphi < 2\pi, \tag{44}$$

which is again a coordinate rectangle, but now in spherical polar coordinates.

We do not review basic separation of variables any further here, but it is worth reminding you of a few more advanced aspects below.

More than two independent variables: Separate in one variable first and introduce a first separation constant. You now have an ODE, and a PDE in one fewer independent variable. Repeat.

Inhomogenous boundary conditions: Say we want to solve the Laplace equation $u_{,xx} + u_{,yy} = 0$ on a coordinate rectangle, with inhomogenous (non-zero) boundary conditions on each side. We do this by writing the solution u as the sum of four solutions, each of which obeys homogeneous boundary conditions on three sides and the given inhomogeneous one on one side.

Another method works more generally. Here we turn inhomogeneous boundary conditions into a source term. Let u_1 be any function that obeys the inhomogenous boundary condition, but is not actually a solution of the PDE, i.e. $Lu_1 \neq f$. Then write $u = u_1 + u_2$ where u_2 obeys the corresponding homogeneous boundary condition but has a non-vanishing source term, given by

$$Lu_2 = -Lu_1 + f. \tag{45}$$

You should convince yourself that this works.

1.4 Complex Fourier series and Fourier transforms

In this course, we will need the complex Fourier transform for two purposes: formally, in the classification of PDEs, and in solving PDEs by separation of variables where the domain in, say, x is not an interval $a \le x \le b$ but the real line $-\infty < x < \infty$. You have already seen the FT briefly in MATH2047, MATH2048 or MATH2015, but not in MATH2038.

1.4.1 Motivation from separation of variables: PDE problems with fall-off boundary conditions

In separation of variables on an interval, we typically have a boundary condition on each end of the interval. As a simple example, assume we want to solve some PDE for $u(x, y, z)$ with homogeneous boundary conditions $u(0, y, z) = u(2\pi, y, z) = 0$ on u. After separation of variables with the usual ansatz $u(x, y, z) = X(x)Y(y)Z(z)$ this gives the boundary conditions $X(0) = X(2\pi) = 0$ on

X. Then we would make each $X_n(x)$ obey these boundary condition, by trying $X_n(x) = \sin nx$, $n = 1, 2, \ldots$. Note that this choice of $X_n(x)$ also implies the simple relation $X_n'' = -n^2 X_n$.

If instead of an interval we have the real line, we typically have a fall-off condition, $u(x, y, z) \to 0$ as $|x| \to \infty$. In principle it would be possible to make each $X_k(x)$ vanish separately at infinity. But then we would lose the useful property that X_k'' is related to X_k in a simple way, namely $X_k'' = -k^2 X$. Instead we use $X_k(x) = e^{ikx}$. This is periodic (with period $2\pi/k$), and so does *not* fall off at infinity. It also obeys $X_k'' = -k^2 X$ again. However, by superimposing a continuum of such functions we *can* make a function that falls off at infinity. This leads us to the concept of a Fourier transform.

1.4.2 Revision: Complex Fourier series

You have already covered the complex **Fourier series** in the prequisite course for this one. If $f(x)$ is a periodic function with period L, that is $f(x + L) = f(x)$, then we can write it as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x}, \qquad k_n := \frac{2\pi n}{L}, \tag{46}$$

where the complex **Fourier coefficients** c_n are given by

$$c_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-ik_n x}\, dx. \tag{47}$$

Note the integral could be taken over any other full period, for example from 0 to L. If $f(x)$ is real, then $c_n = c_{-n}^*$, where the star denotes the complex conjugate.

[You can verify (47) directly by changing n to m and then substituting the expression (46) for $f(x)$. After integrating, you get $c_m = c_n$, as required.]

1.4.3 Derivation of the Fourier transform from the complex Fourier series

From the complex Fourier series we can obtain the Fourier transform as a limit. Define

$$\hat{f}(k_n) := \frac{c_n L}{\sqrt{2\pi}}, \qquad \Delta k := k_{n+1} - k_n = \frac{2\pi}{L}. \tag{48}$$

With this notation, (46) and (47) become

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}(k_n) e^{ik_n x}\, \Delta k, \tag{49}$$

$$\hat{f}(k_n) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-ik_n x}\, dx. \tag{50}$$

So far, the function $\hat{f}(k)$ is defined only at the discrete points k_n. Now as we take the limit $L \to \infty$, we have $\Delta k \to 0$ and we can think of it as dk under an integral, that is $\sum \Delta k \to \int dk$. At the same time, $\hat{f}(k)$ is then defined on the continuous real line. We have motivated

Definition 1.5 (**Fourier transform**).

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk, \tag{51}$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx. \tag{52}$$

Note the pleasing symmetry. When $f(x)$ is real, then $\hat{f}(k) = \hat{f}(-k)^*$ (the complex conjugate).

1.4.4 Use of the Fourier transform: derivatives and fall-off boundary conditions

We can now go back to the motivation from separation of variables. First, taking derivatives. It is easy to see that

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ik\, \hat{f}(k) e^{ikx}\, dk, \tag{53}$$

so we know how to differentiate.

Second, fall-off boundary conditions at infinity. It is a theorem that $f(x)$ admits a Fourier transform if and only if

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty. \tag{54}$$

We say that f is square-integrable, or that it is in the function space $L^2(\mathbb{R})$. But this means that $f(x)$ must vanish at infinity sufficiently fast for this integral to exist. Hence if we we *assume* that $f(x)$ can be written as an inverse Fourier transform (51), $f(x)$ is square-integrable and we automatically have the **fall-off condition** $f(x) \to 0$ as $|x| \to \infty$. In other words, if we solve a PDE using a Fourier transform in the variable x with range $-\infty < x < \infty$, then we automatically impose a fall-off condition $f(x) \to 0$ as $|x| \to \infty$.

1.5 Common linear PDEs

1.5.1 Laplace, Poisson and Helmholtz equations

We start with PDEs that do not have time derivatives. The **Laplace equation** is

$$\Delta u = 0. \tag{55}$$

The **Poisson equation** is just the Laplace equation with a (given) source term, or

$$\Delta u = f(\boldsymbol{x}). \tag{56}$$

The (inhomogeneous) **Helmholtz equation**, which we will derive below, is

$$(\Delta + k^2)u = f(\boldsymbol{x}), \tag{57}$$

Let us assume that we want to solve one of these linear PDEs on a bounded domain V with boundary S, with unit normal vector $\boldsymbol{n}$. At each point on the boundary we can impose precisely one boundary condition. The standard classes of conditions that can be imposed are

- **Dirichlet boundary conditions**: Specify the value of u on S.
- **Neumann boundary conditions**: Specify the value of the **normal derivative**

$$u_{,n} := \boldsymbol{n} \cdot \nabla u = \sum_i n^i u_{,i} \tag{58}$$

  on S, where $\boldsymbol{n}$ is the outward-pointing unit normal vector.
- **Robin boundary conditions**: Specify the value of a linear combination of u and $u_{,n}$ on S.

For example, the Poisson equation with inhomogeneous Robin boundary conditions specifies the PDE problem

$$\begin{aligned} \Delta u &= f(\boldsymbol{x}) \text{ in } V, & (59)\\ \beta u_{,n} + \alpha u &= g(\boldsymbol{x}) \text{ on } S. & (60) \end{aligned}$$

Here the special value $\beta = 0$ gives Dirichlet boundary conditions and $\alpha = 0$ gives Neumann boundary conditions. Here, and in similar examples, $g(x) \neq 0$ specifies an inhomogeneous boundary condition and $g(x) = 0$ the corresponding homogeneous boundary condition.

Remark 1.6. For the Poisson equation the solution with Neumann boundary conditions is only unique up to an arbitrary constant, for if u_1 satisfies the problem so too does $u_2 = u_1 + C$ where C is constant. This is so because the derivative of a constant is zero, and so both $\Delta C = 0$ and $C_{,n} = 0$.

Remark 1.7. For the Poisson or Laplace equation with Neumann boundary conditions,

$$\Delta u = f(\boldsymbol{x}) \text{ in } V, \tag{61}$$
$$u_{,n} = g(\boldsymbol{x}) \text{ on } S, \tag{62}$$

we must have

$$\int_V f\,dV = \int_V \Delta u\,dV = \int_V \nabla\cdot(\nabla u)\,dV = \int_S (\nabla u)\cdot\boldsymbol{n}\,dS = \int_S u_{,n}\,dS = \int_S g\,dS \tag{63}$$

for the source term f and boundary data g to be compatible. Otherwise, this problem has no solution. (We will come back to this in Sec. 8.4.3.)

Remark 1.8. Dirichlet, Neumann or Robin boundary conditions, homogeneous or inhomogeneous, all apply only at a boundary at finite distance. If the domain of the PDE is infinite, then most likely we will want to impose a **fall-off condition** $u(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$ in some direction, or in any direction. Note that there is only one type of fall-off condition, and in particular we should think of it as a (homogeneous) **boundary condition at infinity**. It turns out that if, for a given PDE, we can impose either Dirichlet, Neumann or Robin boundary condition on a *finite* volume, as a fourth alternative we can also impose a free space boundary condition on any open side of an *infinite* (unbounded) volume. If the domain is all of space, and we have fall-off at infinity in all directions, this type of boundary condition is also called a **free space boundary condition**. This is often the simplest boundary condition to work with. In particular, we will use it a lot when we find Green's functions for PDEs.

The Laplace, Poisson and Helmholtz equations, in any number of space dimensions, are examples of a class of PDEs that are called **elliptic** equations. They are, in fact, the most important examples of linear second-order elliptic equations. A formal definition of ellipticity will be given in Sec. 3 below.

Remark 1.9. A key feature of elliptic PDEs is that they smoothe out their data: the interior solution is more often differentiable than the boundary data.

1.5.2 Heat equation

For a time-dependent problem on a bounded domain, with a PDE that contains one or more time derivatives, we must specify both initial data throughout the domain (volume) V at some time $t = 0$, and boundary data on the boundary of the domain (surface) S for $t \geq 0$; see Fig. 1. The problem is then solved for $t > 0$ inside the volume V. This is called the **Cauchy problem**.

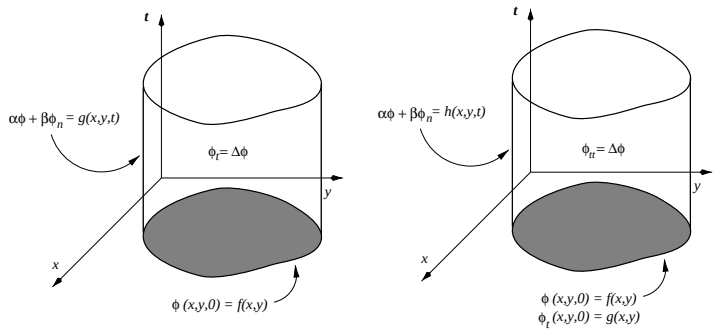

Figure 1: The Cauchy problem for the heat equation (left) and the wave equation (right)

We begin by looking at the **heat equation**, also called the **diffusion equation**,

$$u_{,t} = \kappa \Delta u. \tag{64}$$

Here $\kappa > 0$ is a constant with dimension length^2/time, the **diffusion constant**. To see this, note that both terms in this equation must have the same dimension. The left-hand side has whatever the dimension of u is, divided by time. The right-hand side has the dimension of u, times the dimension of κ, divided by length^2. The dimension of u cancels.

Because the heat equation gives us $u_{,t}(\boldsymbol{x}, 0)$ if we know $u(\boldsymbol{x}, 0)$, it is intuitively clear that we need to specify precisely $u(\boldsymbol{x}, 0)$ as initial data. A particular class of solutions of the heat equation are those that are independent of time: they then obey the Laplace equation. This suggests that the heat equation requires the same boundary conditions as the Laplace equation. Thus, the Cauchy problem for the heat equation is

$$\begin{aligned} u_{,t} &= \kappa \Delta u, \quad t > 0 && (65)\\ u(\boldsymbol{x}, 0) &= f(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in V && (66)\\ \beta u_{,n} + \alpha u &= g(\boldsymbol{x}, t) \text{ for } \boldsymbol{x} \in S. && (67) \end{aligned}$$

(Either α or β can be zero, to give Neumann or Dirichlet boundary conditions.)

The heat equation, in any number of space dimensions, is an example of a class of PDEs that are called **parabolic**. We give a formal definition of parabolic second-order scalar PDEs in Sec. 3, but if $Lu(\boldsymbol{x}) = 0$ is a linear elliptic equation for $u(\boldsymbol{x})$, then $u_{,t} = Lu(\boldsymbol{x}, t)$ is a parabolic equation for $u(\boldsymbol{x}, t)$. Going from the Laplace equation (elliptic) to the heat equation (parabolic) is one example of this. One can prove that a parabolic equation with Dirichlet, Neumann or Robin boundary conditions and initial data for u has a unique solution.

Remark 1.10. Parabolic PDEs have these key features:

a) They smoothe out their initial data: the solution is more often differentiable than the initial data and boundary data.

b) They have infinite propagation speed. Even if the initial data at $t = 0$ is non-zero only in a finite region of space, at any $t > 0$ the solution is typically non-zero in the entire domain: in this sense, the solution has spread from the initial data at infinite speed.

c) They cannot be run backwards. If the time evolution problem with initial data at $t = 0$ is well-posed for $t > 0$, it is ill-posed for $t < 0$. (Well-posedness will be defined below).

1.5.3 Wave equation

The wave equation, in any number of space dimensions, is

$$u_{,tt} = c^2 \Delta u, \tag{68}$$

where $c > 0$ is the wave speed. For the wave equation to have consistent dimensions, c must have dimension length/time.

The wave equation contains two time derivatives, and so we need initial data for both $u(\boldsymbol{x}, 0)$ and $u_{,t}(\boldsymbol{x}, 0)$ at $t = 0$. To understand this intuitively, consider the wave equation that describes the motion of a string, that is the wave dimension in one space dimension

$$u_{,tt} = c^2 u_{,xx}. \tag{69}$$

(Here x is the position along the string, and u the transversal displacement of the string from its rest position, assumed small.) We need to specify the initial position $u(x, 0,)$ and velocity $u_{,t}(x, 0)$ of each part of the string. Newton's force law and Hooke's law of elasticity then give us the acceleration and we can evolve in time. (By contrast, heat has no inertia, so it is sufficient to specify the initial temperature.) If one considers a time-periodic solution to the wave equation, it reduces to the Helmholtz equation. This suggests that the wave equation needs Dirichlet, Neumann or Robin boundary conditions, like the Helmholtz equation.

All these conditions together constitute the Cauchy problem for the wave equation (on a volume V with boundary S, and assuming Robin boundary conditions):

$$\begin{aligned} u_{,tt} &= c^2\Delta u \quad \text{for } \boldsymbol{x} \in V, \quad t \geq 0, && (70)\\ u(\boldsymbol{x},0) &= f(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in V, && (71)\\ u_{,t}(\boldsymbol{x},0) &= g(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in V, && (72)\\ \beta u_{,n} + \alpha u &= h(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x} \in S. && (73) \end{aligned}$$

The wave equation is an example of a class of PDEs called linear second-order **hyperbolic** equations. We give a formal definition below. However, if $Lu = 0$ is an elliptic equation for some differential operator L, then $c^{-2}u_{,tt} + bu_{,t} = Lu$ is a hyperbolic second-order PDE, for any $c > 0$ and real b. The wave equation is obviously in this class, but we shall see that the class of hyperbolic PDEs is much larger.

Remark 1.11. Hyperbolic PDEs have these key features:

a) Any features in the initial data move about in space at one or several specific velocities.

b) There is no smoothing: the solution is typically as often differentiable as the initial data and boundary data.

c) They can be run backwards: the same initial data at $t = 0$ can in general be evolved to $t > 0$ or $t < 0$.

Remark 1.12. The one dimensional wave equation (69) is unusual among wave equations in that, on the infinite domain $-\infty < x < \infty$, it can be solved in closed form. Its general solution, known as the **d'Alembert solution**, is

$$u(x,t) = F(x - ct) + E(x + ct). \tag{74}$$

This can be easily verified by using the chain rule.

D'Alembert's solution does not help us for more general hyperbolic problems, but it provides a nice explicit illustration of all points of Remark 1.11: $F(x - ct)$ represents a wave of shape $F(x)$ (at time $t = 0$) moving in the direction of increasing x at constant speed c. That is, $F(x - ct)$ represents the same shape as $F(x)$, but with the origin moved to $x = ct$, i.e. the same shape as $F(x)$ but translated a distance ct in the positive x direction. (If you are confused about the signs, imagine the example where F has its maximum when its argument is zero, say.) Conversely, $E(x + ct)$ represents a wave of shape $E(x)$ going in the negative x direction at constant speed c. It is also obvious that the solution $u(x,t)$ is just as often differentiable as the initial data $u(x,0)$ and once more than $u_{,t}(x,0)$.

Remark 1.13. The Helmholtz equation is derived from the wave equation by separating the time variable (assuming a periodic time dependence) as follows. Consider the wave equation with a time-periodic source term. For simplicity, we assume the source has only a single (angular) frequency ω, and we use complex notation:

$$-\frac{1}{c^2}\psi_{,tt} + \Delta\psi = e^{-i\omega t}f(\boldsymbol{x}) \tag{75}$$

If we look for a time-periodic solution of this, called a **standing wave**,

$$\psi(\boldsymbol{x},t) = e^{-i\omega t}u(\boldsymbol{x}), \tag{76}$$

we find that $u(\boldsymbol{x})$ satisfies (57), where

$$k = \frac{\omega}{c} = \frac{2\pi\nu}{c} \tag{77}$$

is the **wave number**. Here c is the wave speed, ν the **frequency** and $\omega := 2\pi\nu$ the **angular frequency**. The **wavelength** of a plane wave of frequency ν is

$$\lambda = \frac{c}{\nu} = \frac{2\pi}{k}. \tag{78}$$

1.6 Cauchy data and the Cauchy-Kowalewski solution

The initial data for the Cauchy problem consist of specifying u at $t = 0$ for a PDE that contains only a first time derivative, u and $u_{,t}$ for a PDE that contains up to the second time derivative, and so on: one fewer time derivative in the initial data than in the PDE. These initial data are called **Cauchy data**. The following two remarks are meant to motivate why Cauchy data can be expected to give rise to well-posed initial-value problems.

Remark 1.14. If we solve the heat equation numerically, we start with data $u(\boldsymbol{x}, 0)$at $t = 0$. The PDE then gives us $u_{,t}(\boldsymbol{x}, 0)$. We use this to obtain an approximation for $u(\boldsymbol{x}, h)$, where h is some short time interval. In the simplest case, called **forward Euler**, this is just the zeroth and first term of a Taylor series,

$$u(\boldsymbol{x}, h) \simeq u(\boldsymbol{x}, 0) + h u_{,t}(\boldsymbol{x}, 0) = u(\boldsymbol{x}, 0) + h\kappa\Delta u(\boldsymbol{x}, 0). \tag{79}$$

Then consider this as new initial data and repeat the process to get $u(\boldsymbol{x}, 2h)$, then $u(\boldsymbol{x}, 3h)$ and so on. This is the basic idea behind all numerical methods for solving time-dependent problems.

Remark 1.15. A related theoretical way of looking at Cauchy data is to try and write the solution for small $t > 0$ by expanding $u(\boldsymbol{x}, t)$ into its Taylor series about $t = 0$, that is

$$u(\boldsymbol{x}, t) = u(\boldsymbol{x}, 0) + u_{,t}(\boldsymbol{x}, 0)\, t + \frac{1}{2!} u_{,tt}(\boldsymbol{x}, 0)\, t^2 + O(t^3). \tag{80}$$

To do this, we need to find all t-derivatives of u at $t = 0$. In principle, this can be done by replacing successive time derivatives by spatial derivatives and using the fact that partial derivatives, and in particular space and time derivatives, commute. In the example of the heat equation, we have

$$u_{,t} = \kappa\Delta u \quad \Rightarrow \quad u_{,tt} = (\kappa\Delta u)_{,t} = \kappa\Delta u_{,t} = \kappa\Delta(\kappa\Delta u) = \kappa^2\Delta^2 u, \tag{81}$$

and so on for higher time derivatives. If we evaluate this at $t = 0$ and substitute it into (80), we obtain the **Cauchy-Kowalewski solution**: an infinite series in powers of t, with coefficients that contain arbitrarily many spatial derivatives of the initial data. For the heat equation this is

$$u(\boldsymbol{x}, t) = u(\boldsymbol{x}, 0) + \kappa\Delta u(\boldsymbol{x}, 0)\, t + \frac{1}{2!}\kappa^2\Delta^2 u(\boldsymbol{x}, 0)\, t^2 + O(t^3). \tag{82}$$

However, the Cauchy-Kowalewski solution is just a *formal* solution. In the first place, it only exists if u is infinitely often differentiable in t and x. (The technical term for infinitely often differentiable is **smooth**.) More importantly, in almost any interesting situation, the infinite series does not converge for any finite $t > 0$, and so this solution does not make sense. We have introduced it here to show why a k-th order evolution equation requires the first $k - 1$ derivatives as Cauchy data.

1.7 Weak solutions

The d'Alembert solution of the wave equation gives us a glimpse of a general fact: hyperbolic PDEs have discontinuous solutions. To derive (74) we have assumed that $u(x, t)$ is twice differentiable, but intuitively the d'Alembert solution makes sense for non-differentiable and even discontinuous functions F and E. For example, one could consider initial data that correspond to a wave moving right and then make these initial data more and more square until they become a step function, which is discontinuous.

Remark 1.16. It is clear that a discontinuity in the solution arises when either F or E is a discontinuous function. Say $F(z)$ jumps at z_0, then $F(x-ct)$ jumps at $x-ct = z_0$. Similarly, a jump in $E(z)$ at z_0 becomes a jump at $x + ct = z_0$. These lines are the **characteristic hypersurfaces** of (69). This is a simple example of a general fact: the solution of a linear hyperbolic equation can be discontinuous across its characteristic hypersurfaces. (A hypersurface has one dimension fewer than the space it is in, so a hypersurface in (t, x, y, z) is three-dimensional, but a hypersurface in (t, x) is a curve.)

In what sense does such a discontinuous solution still obey the wave equation? Consider initially a solution $u(x,t)$ of the wave equation (69) that is at least twice differentiable (and so really can be said to obey the wave equation), and an arbitrary function $\phi(x,t)$ that is smooth and vanishes at infinity, together with all its derivatives. (More precisely, ϕ and all its derivatives must have finite L^2 norm.) Such a functions is called a **test function**. Then $u_{,tt} = c^2 u_{,xx}$ for all x and t implies that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-u_{,tt} + c^2 u_{,xx})\phi\, dx\, dt = 0 \tag{83}$$

for *any* such function ϕ, simply because the round bracket is zero. We can imagine in particular that ϕ, while smooth, is sharply peaked about the point (x,t). It then probes if u obeys $u_{,tt} = c^2 u_{,xx}$ at that point.

Integrating the first term by parts in t twice, and the second part twice in x gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-\phi_{,tt} + c^2 \phi_{,xx})u\, dx\, dt = 0. \tag{84}$$

The boundary terms in the integration by parts vanish by the assumption that ϕ and its derivatives vanish at infinity. But ϕ was an arbitrary function, and this integral no longer requires u to be differentiable in order to be defined. This observation motivates

Definition 1.17. $u(x,t)$ is a **weak solution** of the wave equation (69) if it obeys (84) for *all* test functions $\phi(x,t)$.

Definition 1.18. A **strong solution** of (69) is a twice differentiable function $u(x,t)$ for which (69) holds for every x and t.

Remark 1.19. Any strong solution is also a weak solution, as we have just shown by integrating by parts and considering all possible ϕ. The reverse is not true [because we may not be able to differentiate u, and hence may not be able to go back from (84) to (83).]

Weak solutions are important for hyperbolic (wave equation-like) problems for two reasons. First, the solution of a linear hyperbolic equation with discontinuous initial data often makes physical sense. Secondly, the solution of a *nonlinear* hyperbolic equation typically becomes discontinuous in finite time even if the initial data are everywhere smooth. We will see examples of this when we discuss conservation laws.

1.8 Exercises

1. **Revision problem 1:** a) Solve the following PDE problem (Laplace equation in two dimensions on a square, with homogeneous Dirichlet boundary conditions, on three sides of the square, and inhomogeneous Dirichlet boundary conditions on one side):

$$\begin{aligned} u_{,xx} + u_{,yy} &= 0, \quad 0 \le x \le \pi, \quad 0 \le y \le \pi, && (85)\\ u(0,y) &= 0, \quad 0 \le y \le \pi, && (86)\\ u(\pi,y) &= 0, \quad 0 \le y \le \pi, && (87)\\ u(x,0) &= 0, \quad 0 \le x \le \pi, && (88)\\ u(x,\pi) &= f(x), \quad 0 \le x \le \pi. && (89) \end{aligned}$$

   b) Hence show that the solution for $f(x) = 1$ is

$$u(x,y) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi \sinh[(2m+1)\pi]} \sin[(2m+1)x] \sinh[(2m+1)y] \tag{90}$$

   c) Use Maple, Matlab, Mathematica, or some other software, to plot the solution series with the first 1, 2, 10, 100 terms. Observe that the solution is differentiable in the interior even though the boundary data jumps (between 0 and 1) at the two corners $(x,y) = (0,\pi)$ and (π,π).

2. **Revision problem 2:** a) Solve the following PDE problem (one-dimensional heat equation on an interval, with homogeneous Dirichlet boundary conditions at both ends of the interval):

$$\begin{aligned} u_{,xx} - u_{,y} &= 0, \quad 0 \le x \le \pi, \quad y \ge 0 && (91)\\ u(0,y) &= 0, \quad y \ge 0, && (92)\\ u(\pi,y) &= 0, \quad y \ge 0, && (93)\\ u(x,0) &= f(x), \quad 0 \le x \le \pi. && (94) \end{aligned}$$

b) Hence show that the solution for $f(x) = 1$ is

$$u(x,y) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin[(2m+1)x] e^{-(2m+1)^2 y} \tag{95}$$

c) Use Maple, Matlab, Mathematica, or some other software, to plot the solution series with the first 1, 2, 10, 100 terms for $0 \le y \le \pi$. Observe that the solution is differentiable in the interior even though the there is a jump from 0 to 1 between the initial data and the boundary data at the corners $(x,y) = (0,0$ and $(\pi,0)$. (Or put differently, the initial data do not obey the boundary condition.)

3. **Revision problem 3:** a) Solve the following PDE problem (one-dimensional wave equation on an interval, with homogeneous Dirichlet boundary conditions at both ends of the interval):

$$\begin{aligned} u_{,xx} - u_{,yy} &= 0, \quad 0 \le x \le \pi, \quad y \ge 0 && (96)\\ u(0,y) &= 0, \quad y \ge 0, && (97)\\ u(\pi,y) &= 0, \quad y \ge 0, && (98)\\ u(x,0) &= f(x), \quad 0 \le x \le \pi, && (99)\\ u_{,y}(x,0) &= g(x), \quad 0 \le x \le \pi. && (100) \end{aligned}$$

b) Hence show that the solution for $f(x) = 1$, $g(x) = 0$ is

$$u(x,y) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin[(2m+1)x] \cos[(2m+1)y] \tag{101}$$

c) Use Maple, Matlab, Mathematica, or some other software, to plot the solution series with the first 1, 2, 10, 100 terms for $0 \le y \le \pi$. Observe that the solution is discontinuous, and that these discontinuities propagate with speeds ± 1.

4. **Homework 1:** By applying the chain rule of partial derivatives, transform the 2-dimensional Laplacian in Cartesian coordinates to polar coordinates. In other words, show that

$$u_{,xx} + u_{,yy} = u_{,rr} + \frac{1}{r} u_{,r} + \frac{1}{r^2} u_{,\theta\theta}. \tag{102}$$

[Hint: solve for $r(x,y)$ and $\theta(x,y)$ first.]

5. **Homework 2:** Find the solution of the wave equation $u_{,tt} = c^2 u_{,xx}$ on the line $-\infty < x < \infty$ in terms of the initial data $u(x,0) = f(x)$, $u_{,t}(x,0) = g(x)$. [Hint: use the d'Alembert solution. Your answer will contain $f(x)$ and an integral over $g(x)$.]

6. **Homework 3:** a) Write down the Cauchy-Kowalewski solution for the one-dimensional wave equation $u_{,tt} = c^2 u_{,xx}$. Use this to find the solution with the Cauchy data $u(x,0) = \sin x$, $u_{,t}(x,0) = 0$ as an infinite series. Show that the series sums to $u(x,t) = \cos(ct)\sin(x)$. b) Then write this also as a d'Alembert solution.

7. Write out

$$\nabla \cdot (u \nabla u) = \nabla u \cdot \nabla u + u \Delta u \tag{103}$$

in Cartesian coordinates (x,y,z).

8. Consider the answer (101) to a previous wave equation problem - it does not matter now how we found that. Write this as

$$u(x,y) = E(x+y) + F(x-y) \tag{104}$$

for two functions $E(z)$ and $F(z)$. Give these functions both as Fourier series, and explicitly. [Hint: For the first part, use a trig identity for $\sin a \cos b$. For the second part, note first from the Fourier series that E and F are periodic functions. Then use part of the answer to Revision Problem 3 to determine what the value of these functions is over the interval $0 \le z \le \pi$. Hence work out what they are for all z.]

9. Using separation of variables, find the solution of

$$\begin{aligned} -u_{,tt} + u_{,xx} &= 0, \quad a \le x \le b, \quad t \ge 0, && (105)\\ \alpha_L u(a,t) - \beta_L u_{,x}(a,t) &= 0, \quad t \ge 0, && (106)\\ \alpha_R u(b,t) + \beta_R u_{,x}(b,t) &= 0, \quad t \ge 0, && (107)\\ u(x,0) &= f(x), \quad a \le x \le b, && (108)\\ u_{,t}(x,0) &= g(x), \quad a \le x \le b && (109) \end{aligned}$$

(wave equation on an interval with Robin BCs). (I have written $-\beta_L$ because $u_{,n} = -u_{,x}$ at the left boundary, but that is just a convention.)

Hint: Your main challenge in this problem is to find the basis functions $X_n(x)$. Sturm-Liouville theory then tells you that the $X_n(x)$ must obey the orthogonality conditions

$$\int_a^b X_n(x)\bar{X}_m(x)\,dx = \begin{cases} 0, & n \ne m, \\ N_n, & n = m, \end{cases} \tag{110}$$

where $\bar{X}$ denotes the complex conjugate (if you have chosen to use complex notation for the X_n) for some coefficients $N_n > 0$. In your answer you can use N_n without finding them explicitly – we leave that for the next problem.

10. Show explicitly by integration that (110) holds for the basis functions $X_n(x)$ from your solution of the previous Problem and find the coefficients N_n.

Hint: Use integration by parts twice, and the boundary condition.

11. Using separation of variables, find the solution of

$$\begin{aligned} -u_{,tt} + u_{,xx} &= 0, \quad -\infty < x < \infty, \quad t \ge 0 && (111)\\ u(x,t) &\to 0, \quad |x| \to \infty, \quad t \ge 0 && (112)\\ u(x,0) &= f(x), \quad -\infty \le x \le \infty, && (113)\\ u_{,t}(x,0) &= g(x), \quad -\infty \le x \le \infty, && (114) \end{aligned}$$

(1-dimensional wave equation with free space boundary conditions).

Hint: Here of course you have a Fourier transform instead of a Fourier series, and this problem is much simpler than the same problem on $a \le x \le b$. It is included here for comparison, and to reassure you that Fourier transforms are not intrinsically difficult.

2 Well-posedness

2.1 Main definition

Recall the definition of a PDE problem in Sec. 1.1.3. Some boundary conditions are unsuitable for certain types of PDE in that they can lead to unphysical behaviour. For example, Cauchy type conditions are unsuitable for the Laplace equation and Dirichlet conditions are unsuitable for the wave equation. This leads to the notion of a *well-posed problem.* Problems which arise in practical applications are usually well-posed boundary value problems (for PDEs in space only) or well-posed Cauchy problems (for PDEs in time and space). It is always a PDE *problem* that is well-posed (or not), not a PDE on its own.

Definition 2.1. A PDE problem is **well-posed** if and only if

1. a solution exists (**existence**);
2. for given data there is only one solution (**uniqueness**);
3. a small change in the data (boundary data, initial data, source terms) produces only a small change in the solution (**continuous dependence on the data**).

The need for existence and uniqueness seems only common sense, but in practice they may be hard to prove, and typically need to be proved separately. Only for very simple problems can one prove existence by explicitly deriving the solution. The third condition, continuity, will be less familiar to you. To give it a precise meaning requires some technical preparation. For motivation, we look at a a few simple but illustrative examples of ill-posed problems first.

2.2 Examples of ill-posed problems

2.2.1 Wave equation with Dirichlet boundary conditions: many solutions

Consider the wave equation

$$u_{,tt} = u_{,xx}, \qquad 0 < x < \pi, \qquad 0 < t < \pi, \tag{115}$$

with the homogeneous Dirichlet boundary conditions in space

$$u(0,t) = 0, \qquad u(\pi,t) = 0, \tag{116}$$

and the homogeneous initial *and final* Dirichlet boundary conditions in time,

$$u(x,0) = 0, \qquad u(x,\pi) = 0. \tag{117}$$

Separation of variables leads us to look for solutions of the form $u(x,t) = X(x)T(t)$, and it is easy to see that there are infinitely many such solutions which obey all boundary conditions, namely

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \, \sin nt \tag{118}$$

all the A_n are arbitrary. Thus there are infinitely many solutions to this problem. It is ill-posed.

We have not considered inhomogeneous boundary conditions, or a source term, but we can immediately see that those problems are also ill-posed. Suppose they have a solution (if not, they are ill-posed). As this is a linear PDE, we can add any solution of the problem with homogeneous boundary conditions and zero source term, and get another solution. But, as we have just shown, there are infinitely many of these. So the problem with a source term and/or inhomogeneous boundary conditions is also ill-posed.

2.2.2 Wave equation with Dirichlet boundary conditions: no continuous dependence on the boundary data

Perhaps what was wrong with the previous example was that we imposed final conditions at the special time $t = \pi$. So let us impose them at a general value $t = a$ instead, with a not an integer multiple of π. We now also allow for inhomogeneous final conditions. Hence consider

$$u_{,tt} = u_{,xx}, \qquad 0 < x < \pi, \qquad 0 < t < a, \tag{119}$$

with homogeneous Dirichlet boundary conditions in space

$$u(0,t) = 0, \qquad u(\pi,t) = 0, \tag{120}$$

as before, and the homogeneous initial and inhomogeneous final Dirichlet boundary conditions in time,

$$u(x,0) = 0, \qquad u(x,a) = f(x). \tag{121}$$

It is easy to see that (118) again obeys the wave equation and all three homogeneous boundary conditions. Substituting this into the inhomogeneous boundary condition gives

$$\sum_{n=1}^{\infty} A_n \sin nx \, \sin na = f(x), \tag{122}$$

and we can solve this using by determining the constants $\sin naA_n$ as the coefficients of a sine series:

$$A_n = \frac{1}{\sin na} \frac{2}{\pi} \int_0^{\pi} \sin nx \, f(x) \, dx. \tag{123}$$

Hence we have a unique solution for any given function $f(x)$. For $f(x) = 0$ this unique solution is just $u(x,t) = 0$. So we avoid the problem of the previous example, as we suspected.

But now there is another problem instead. Focus on the factor $1/\sin na$. As we have assumed that a is not an integer multiple of π, $\sin na$ is not zero for any n. But how small can it be? One can prove that it can become arbitrarily small, and hence A_n can become arbitrarily large, in the following technical sense. Fix any $\epsilon > 0$, as small as you like. Then there exists some positive n (which depends on ϵ), such that $|\sin na| < \epsilon$. Now consider $f(x) = \sin nx$. The absolute value of these data is bounded by 1. But the solution is bounded only by $1/\epsilon$, which we can make arbitrarily large. In this sense, the solution does not depend continuously on the data. The problem is ill-posed.

Note that it was sufficient to show ill-posedness to look at one (suitably chosen) particular family of data, here $f(x) = \sin nx$.

2.2.3 Cauchy problem for the Laplace equation: no continuous dependence on the boundary data

We now look at another problem that is ill-posed because its solution does not depend continuously on the data. Consider the solution of the Laplace equation

$$u_{,xx} + u_{,yy} = 0, \qquad 0 < x < \pi, \qquad y > 0, \tag{124}$$

with the homogeneous Dirichlet boundary conditions

$$u(0,y) = 0, \qquad u(\pi,y) = 0, \tag{125}$$

and the *Cauchy data*

$$u(x,0) = 0, \qquad u_{,y}(x,0) = \sin nx. \tag{126}$$

where $n > 0$ is an integer. Separation of variables suggests that we look for a solution of the form $u(x,y) = X(x)Y(y)$. Doing this, and taking into account all boundary conditions, we find the unique solution

$$u(x,y) = \frac{1}{n} \sin nx \sinh ny. \tag{127}$$

Now, as $n \to \infty$, the Cauchy data remain finite and bounded between 1 and -1, but the solution for $y > 0$ diverges at almost every x, because for any $y > 0$,

$$\lim_{n\to\infty} \frac{\sinh ny}{n} = \infty. \tag{128}$$

[We say at *almost* every x, because obviously $u(m\pi/n, y) = 0$, for any integer m.] Hence once again, we have given an example of a family of data (with parameter n) for which the solution does not depend continuously on the data, and so the problem is ill-posed.

2.3 Continuous dependence on the data

We now introduce some concepts that allows us to define the intuitive notion in which the solution did not depend continuously on the data in the previous example.

Definition 2.2. A **real vector space** V is a set whose elements (called **vectors**) can be added and multiplied by real numbers (called **scalars** in contrast to vectors). Hence if $X, Y \in V$ and $c \in \mathbb{R}$ then $cX \in V$ and $X + Y \in V$.

Example 2.3. The prototypical vector space that you know already is $\mathbb{R}^n$. However, when using this example keep in mind that a norm or an inner product is not part of the definition of a vector space.

Example 2.4. Functions from $\mathbb{R}$ to $\mathbb{R}$ also form a vector space, with $f+g$ defined by $(f+g)(x) := f(x)+g(x)$ and cf defined by $(cf)(x) = cf(x)$. More generally, we could consider the vector space of functions from $\mathbb{R}^n$ to $\mathbb{R}^N$, or in other words, vector-valued functions on $\mathbb{R}^n$. Clearly a vector space of functions is infinite-dimensional in the sense that it is not spanned by a finite number of basis vectors, but we can still do many of the same things that we can do with finite-dimensional vector spaces such as $\mathbb{R}^n$.

Definition 2.5. A **norm** on a real vector space V is a function from V to the non-negative real numbers with the properties

1. $\|X\| = 0$ if and only if $X = 0$;
2. $\|cX\| = |c|\ \|X\|$ for all $X \in V$ and $c \in \mathbb{R}$;
3. $\|X + Y\| \le \|X\| + \|Y\|$ for all $X, Y \in V$ (the **triangle inequality**).

Example 2.6. The **Euclidean norm**, also called the l^2 **norm**, on $\mathbb{R}^n$ is given by

$$|\boldsymbol{x}| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{129}$$

I use $|\boldsymbol{x}|$ instead of $\|\boldsymbol{x}\|$ for this norm because that is the established notation for this particular norm. Also, for $n = 1$, this norm is just the absolute value of real number x. Convince yourself that the square root is necessary for property 2 above to hold.

This is the most important norm on finite-dimensional vector spaces, and we mention two more only to illustrate the concept of norm.

Example 2.7. The **maximum norm** on $\mathbb{R}^n$ is given by

$$\|\boldsymbol{x}\|_{\max} := \max(|x_1|, |x_2|, \ldots, |x_n|), \tag{130}$$

where of course $|x_1|$ stands for the absolute value.

Example 2.8. The l^1 **norm** on $\mathbb{R}^n$ is given by

$$\|\boldsymbol{x}\|_{l^1} := |x_1| + |x_2| + \cdots + |x_n|. \tag{131}$$

Convince yourself that properties 1 and 2 above hold for these norms. (Property 3 is a little harder to check.)

We will only introduce one norm on a function space:

Example 2.9. The L^2 **norm** on functions from $\mathbb{R}^n$ to $\mathbb{R}^N$ defined by

$$\|\boldsymbol{f}(\cdot)\|_{L^2} := \sqrt{\int_{\mathbb{R}^n} |\boldsymbol{f}(\boldsymbol{x})|^2 \, d^n x}, \tag{132}$$

where $|\boldsymbol{f}(\boldsymbol{x})|$ denotes the Euclidean norm on $\mathbb{R}^N$ (over the components of the vector $\boldsymbol{f}$, at constant $\boldsymbol{x}$). The notation $\boldsymbol{f}(\cdot)$ indicates that the function norm depends on the value of $\boldsymbol{f}(\boldsymbol{x})$ for all $\boldsymbol{x}$. Functions from $\mathbb{R}^n$ to $\mathbb{R}$ with finite L^2 norm are said to be in the vector space $L^2(\mathbb{R}^n)$, but there is no space in this course to give a formal definition.

Definition 2.10. The function f from the vector space V with norm $\|\cdot\|_V$ to the vector space W with norm $\|\cdot\|_W$ is **Lipshitz continuous** with Lipshitz constant $K > 0$ if

$$\|f(X_1) - f(X_2)\|_W \leq K \|X_1 - X_2\|_V \tag{133}$$

for all pairs of vectors $X_1, X_2 \in V$.

Remark 2.11. You will get the right intuitive idea if you simply take both V and W to be $\mathbb{R}$ with the usual absolute value of real numbers $|x|$ as the norm: the change in f cannot be bigger than K times the change in x.

We are now ready to formally define **continuous dependence on the initial data**. For simplicity, we consider an evolution problem that is first order in time and has homogeneous BCs, so that $\boldsymbol{u}(\boldsymbol{x}, 0)$ are the only data. Consider two solutions $\boldsymbol{u}_1(\boldsymbol{x}, t)$ and $\boldsymbol{u}_2(\boldsymbol{x}, t)$ with initial data $\boldsymbol{u}_1(\boldsymbol{x}, 0)$ and $\boldsymbol{u}_2(\boldsymbol{x}, 0)$, respectively.

Definition 2.12. The solution $\boldsymbol{u}(\boldsymbol{x}, t)$ of a first-order in time evolution problem depends continuously on the initial data $\boldsymbol{u}(\boldsymbol{x}, 0)$ in the function norm $\|\cdot\|$ if and only if there exists a function $K(t) > 0$ such that

$$\|\boldsymbol{u}_1(\cdot, t) - \boldsymbol{u}_2(\cdot, t)\| \leq K(t) \|\boldsymbol{u}_1(\cdot, 0) - \boldsymbol{u}_2(\cdot, 0)\| \tag{134}$$

for all pairs of initial data $\boldsymbol{u}_1(\boldsymbol{x}, 0), \boldsymbol{u}_2(\boldsymbol{x}, 0)$, and with $K(t)$ independent of these data.

Note that continuity, and hence well-posedness, is defined only in some particular function norm. An inequality such as (134) in PDE theory is called an **estimate**. The notion of continuity we have used here is Lipshitz continuity, where the Lipshitz constant $K(t)$ is allowed to depend on t. The choice of function norm is an art, and depends on the problem in hand. (You will not be asked to find estimates in this course, just to check and use them.)

Remark 2.13. I have kept the definition as simple as possible by restricting it to a first-order in time evolution problem, and by using the same norm for the solution as for the initial data. In practice, it may be necessary to use different function norms on the initial data and the solution in order to prove an estimate at all, or in order to get the most useful one. For second-order in time evolution problems, we need a norm on $\boldsymbol{u}$ and $\boldsymbol{u}_{,t}$ for the initial data, and if there are boundary data, we need a norm for those as well, and the solution must be bounded by the sum of all these norms. For a system of PDEs, the norm combines a norm over state space $\boldsymbol{u}$ with a function norm over position $\boldsymbol{x}$, as in the example (132).

Proposition 2.14. Continuous dependence on all the data implies uniqueness.

Proof. We again restrict to a problem with only initial data for simplicity. If there are two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ that have the same data, then the right-hand side of (134) is zero. As the left-hand side cannot be negative, it must also be zero, so $\boldsymbol{u}_1 - \boldsymbol{u}_2$ has zero norm for all $t \geq 0$. But the only function that has zero norm is the zero function, and so $\boldsymbol{u}_1(\boldsymbol{x}, t) - u_2(\boldsymbol{x}, t)$ vanishes for all t. Hence $\boldsymbol{u}_1 = \boldsymbol{u}_2$ for all t. □

Remark 2.15. If the PDE or PDE system is linear, then the difference $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ of any two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is itself a solution of the corresponding homogeneous problem (zero inhomogeneous term) with homogeneous BCs (zero boundary data). Hence we can reformulate the continuity condition for *linear* problems as

$$\|\boldsymbol{u}(\cdot, t)\| \leq K(t) \|\boldsymbol{u}(\cdot, 0)\| \tag{135}$$

for all initial data $\boldsymbol{u}(\cdot, 0)$, with $K(t)$ independent of these data.

Remark 2.16. The condition (135) can also be viewed as a form of **stability** with respect to the initial data, where by stability we mean that the solution cannot grow arbitrarily rapidly. It is clear that often solutions will grow in time, so we cannot demand that $K(t)$ is a constant. But it is essential that it is independent of the initial data $\boldsymbol{u}(\boldsymbol{x}, 0)$, or else the condition would be trivial. In fact, a typical way in which an evolution problem fails to be well-posed is that initial data that vary more rapidly in space lead to solutions that grow more quickly in time. We have seen an example of this in Sec. 2.2.3.

Remark 2.17. In fact, (134) is often too strong for nonlinear problems. Instead one may want the problem to be well-posed only for solutions $\boldsymbol{u}$ close to a given reference solution $\boldsymbol{u}_0$. The relevant criterion is then

$$\|\delta\boldsymbol{u}(\cdot, t)\| \leq K(t)\|\delta\boldsymbol{u}(\cdot, 0)\|, \tag{136}$$

where $\delta\boldsymbol{u} := \boldsymbol{u} - \boldsymbol{u}_0$ denotes any sufficiently small perturbation about a reference solution $\boldsymbol{u}_0$. One then says that the problem is well-posed in a neighbourhood of $\boldsymbol{u}_0$.

2.4 Examples of well-posedness results

2.4.1 An energy estimate for the heat equation

Consider now the 1-dimensional heat equation

$$u_{,t} = u_{,xx} \tag{137}$$

on all of space. (For simplicity we have set $\kappa = 1$.) Define the "energy"

$$\tilde{E}(t) := \frac{1}{2}\int_{-\infty}^{\infty} u^2\, dx. \tag{138}$$

and again consider initial data where $\tilde{E}(0)$ is finite. We then have

$$\begin{aligned} \frac{d\tilde{E}}{dt} &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t} u^2\, dx \\ &= \int_{-\infty}^{\infty} u u_{,t}\, dx \\ &= \int_{-\infty}^{\infty} u u_{,xx}\, dx \\ &= -\int_{-\infty}^{\infty} u_{,x}^2\, dx \leq 0. \end{aligned} \tag{139}$$

In the first equality we have differentiated under the integral, and in the second we have used the product rule. We have used (137) in the third equality, and in the fourth equality we have integrated by parts and used that finite E implies that u vanishes $x = \pm\infty$. Hence we have obtained the estimate

$$\|u(\cdot, t)\|_{\tilde{E}} \leq \|u(\cdot, 0)\|_{\tilde{E}} \tag{140}$$

in the norm defined by

$$\|u(\cdot, t)\|_{\tilde{E}} := \sqrt{\tilde{E}(t)}. \tag{141}$$

The square root is required here to obtain the propert $\|cu\| = |c|\ \|u\|$ of a norm. In this example we have $K(t) = 1$, and we use the same norm for the initial data and the solution, but neither is typical.

With this estimate, we have shown continuous dependence of the solution on the initial data in the energy norm, for (differentiable) solutions. From Prop. 2.14 we also have uniqueness for (differentiable) solutions.

2.4.2 An energy estimate for the wave equation

Consider the 1-dimensional wave equation

$$u_{,tt} = u_{,xx} \tag{142}$$

on all of space. (For simplicity we have set $c = 1$.) Define the energy

$$E(t) := \frac{1}{2}\int_{-\infty}^{\infty} (u_{,t}^2 + u_{,x}^2)\, dx, \tag{143}$$

and consider initial data where $E(0)$ is defined and is finite. We then have

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}(u_{,t}^2 + u_{,x}^2)\, dx \\ &= \int_{-\infty}^{\infty} (u_{,t}u_{,tt} + u_{,x}u_{,xt})\, dx \\ &= \int_{-\infty}^{\infty} (u_{,t}u_{,xx} + u_{,x}u_{,xt})\, dx \\ &= \int_{-\infty}^{\infty} (u_{,t}u_{,x})_{,x}\, dx = 0. \end{aligned} \tag{144}$$

In the first equality we have differentiated under the integral, and in the second we have used the product rule. We have used (142) in the third equality, and in the fourth equality we have used the fact that finite E implies that $u_{,t}$ and $u_{,x}$ must vanish at $x = \pm\infty$. This gives us the estimate

$$\|u(\cdot, t)\|_E = \|u(\cdot, 0)\|_E \tag{145}$$

in the **energy norm** defined by

$$\|u(\cdot, t)\|_E := \sqrt{E(t)}. \tag{146}$$

As for the heat equation, we have $K(t) = 1$, and we use the same norm for the initial data and the solution. The inequality (135) is actually a strict equality here. Finally, we note that E is really the physical energy (potential energy plus kinetic energy) carried by the wave. This has lent the name "energy norm" to this type of function norm.

As for the heat equation, we have shown continuous dependence on the initial data, and hence uniqueness, for strong (differentiable) solutions.

2.4.3 Uniqueness of solutions of the boundary value problem for the Poisson equation

Theorem 2.18. The solution of the Poisson equation (59) with either Dirichlet boundary conditions, or Robin boundary conditions (60) with $\alpha\beta > 0$, is unique. With Neumann boundary conditions, the solution is unique up to addition of a constant.

Proof. Suppose we have two solutions u_1 and u_2 that have the same boundary conditions and source term but differ in V. We define their difference

$$u := u_1 - u_2. \tag{147}$$

Convince yourself that if u_1 and u_2 are solutions of the Poisson equation with some inhomogeneous boundary condition, u satisfies Laplace's equation with the homogeneous boundary condition of the same type (say Dirichlet) as in the original problem. Hence we have $\Delta u = 0$, and therefore

$$\int_V u\Delta u\, dV = 0. \tag{148}$$

Now $\nabla \cdot (u\nabla u) = u\Delta u + \nabla u \cdot \nabla u$, so that

$$\int_V \left[\nabla \cdot (u\nabla u) - |\nabla u|^2\right] dV = 0 \tag{149}$$

for any volume V. The divergence theorem gives us

$$\int_V \nabla \cdot (u \nabla u)\, dV = \int_S u u_{,n}\, dS, \tag{150}$$

and hence

$$\int_V |\nabla u|^2\, dV = \int_S u u_{,n} dS \tag{151}$$

for any volume V with surface S. Now if our original boundary conditions were inhomogeneous Dirichlet boundary conditions then $u = 0$ on the boundary, and if they were Neumann, then $u_{,n} = 0$ on the boundary. Either way, the right-hand side of (151) vanishes. Finally, if $\alpha\beta > 0$, then $uu_{,n} = -(\alpha/\beta)u^2 \leq 0$. But then the left-hand side of (151) is non-negative and the right-hand side is non-positive, so both must be zero. Hence u must be constant. But for either Dirichlet or Robin boundary conditions, this constant must be zero. Hence $u_1 = u_2$, and the solution is unique. For Neumann boundary conditions, $u = u_1 - u_2$ can be constant, and so the solution of the Neumann problem is unique only up to addition of a constant. □

Remark 2.19. Here we have assumed that the solutions are twice differentiable, so we have proved uniqueness only in the space of strong solutions.

2.5 The importance of well-posedness

Why is well-posedness important?

If the solution exists and is unique but if arbitrarily small changes in data result in large changes in the solution, then it is more likely that the solution is useless for engineering and science because, in practice all data come from measurements which have small errors in them. Similarly, any *numerical* solution of the PDE is only an approximation and so will have small errors in it. Consider some initial data at t_0 and evolve them to t_1. The final data at t_1 then have some numerical error in them, and so we have a small random change of initial data when we evolve further from t_1 to t_2. If this gives rise to a very large change in the solution later on, the numerical solution becomes nonsense.

There is a large mathematical literature on the well-posedness or ill-posedness of PDEs. You need to understand the basic concept of well-posedness so that you know when to worry about possible ill-posedness and consult this literature. There are several reasons why an ill-posed PDE problem may not be recognised as such:

- The mathematical problems that you encounter in this course are both standard problems, and well-posed, but in modelling an engineering system, a system of ODEs and PDEs may well arise that has never been investigated mathematically before and that may or may not be well-posed.

- Many PDEs or systems of PDEs that one can write down are neither elliptic, hyperbolic or parabolic, and then the question of well-posedness becomes tricky.

- A physical or engineering problem may be "physically well posed" or may "clearly have a unique solution" but the mathematical problem one solves may be ill-posed because it does not completely reflect the physical situation one had in mind. As a (further) example, the solutions of the Navier-Stokes equations in the limit of small viscosity are completely different, mathematically and in fact (they have boundary layers), from solutions of the Euler equations (where the viscosity is set to zero), but the only difference is a tiny amount of dissipation that one might naively consider irrelevant.

- Trying to solve an ill-posed problem numerically, there is a risk that you get an answer that looks reasonable, but is actually nonsense. However, there is one important clue: a numerical solution to a well-posed problem will converge with increasing resolution (finer numerical mesh). A numerical simulation to an ill-posed problem will actually get worse, by the mechanism of Sec. 2.2.3: a finer mesh allows for $\sin nx$ with larger n to be resolved.

2.6 Exercises

12. **Homework 4:** (A reworking of a similar example in Lecture 4, and practice with Fourier transforms) Find the solution of the Cauchy problem (sic!) for the Laplace equation on the half-plane

$$u_{,xx} + u_{,yy} = 0, \qquad -\infty < x < \infty, \qquad y \geq 0, \tag{152}$$

$$u(x,0) = f(x), \qquad u_{,y}(x,0) = g(x), \qquad -\infty < x < \infty, \tag{153}$$

$$u(x,y) \to 0 \quad \text{as} \quad x \to \pm\infty. \tag{154}$$

Show that it is ill-posed (because the solution can grow arbitrarily rapidly compared to the initial data).

13. **Homework 5:** (Tests your understanding of the properties of a norm, but not much calculation required) Show that the Cauchy problem for the *backwards in time* heat equation $u_{,t} = -\kappa u_{,xx}$ with $\kappa > 0$ is ill-posed, because the solution cannot depend continuously on the initial data in any possible function norm over x. [For definiteness, you may assume the Dirichlet boundary conditions $u(0,t) = u(2\pi,t) = 0$, but the answer will not depend on the boundary conditions.]

14. **Homework 6:** (A variation on the example in Lecture 6) Derive the values of the constants α_L, β_L, α_R and β_R for which the 1-dimensional heat equation on the interval $[a,b]$ with Robin boundary conditions

$$u_{,t} = u_{,xx}, \quad a \leq x \leq b, \quad t \geq 0, \tag{155}$$

$$\alpha_L u(a,t) + \beta_L u_{,x}(a,t) = 0, \quad t \geq 0, \tag{156}$$

$$\alpha_R u(b,t) + \beta_R u_{,x}(b,t) = 0, \quad t \geq 0, \tag{157}$$

is well-posed in the energy norm

$$||u(\cdot,t)||_{\tilde{E}} := \sqrt{\tilde{E}(t)}, \qquad \text{where} \quad \tilde{E}(t) := \int_a^b \frac{1}{2}u^2\,dx. \tag{158}$$

15. In a previous problem we showed that the Cauchy problem for the *backwards in time* heat equation $u_{,t} = -\kappa u_{,xx}$ with $\kappa > 0$ is ill-posed, because the solution does not depend continuously on the initial data. In the model answer we neglected boundary conditions. Extend that answer assuming that the domain is $0 \leq x \leq \pi$, $t \geq 0$, with initial data and inhomogeneous Dirichlet boundary data

$$u(x,0) = f(x), \quad u(0,t) = g(t), \quad u(\pi,t) = h(t). \tag{159}$$

[Hint: write $u = u_1 + u_2$, where

$$u_1(x,0) = f(x), \quad u_1(0,t) = 0, \quad u_1(\pi,t) = 0 \tag{160}$$

and

$$u_2(x,0) = 0, \quad u_2(0,t) = g(t), \quad u_2(\pi,t) = h(t), \tag{161}$$

so that u_1 takes care of the initial data and u_2 of any inhomogeneous boundary conditions. You can use the triangle inequality

$$||X + Y|| \geq \Big|\,||X|| - ||Y||\,\Big| \tag{162}$$

(the right-hand side denotes the absolute value of the difference of the two norms).]

16. a) How would you define the l^p norm on the vector space $\mathbb{R}^n$, where p is any positive integer, such that l^1 and l^2 give the special cases we have already seen?

b) [Much harder, but might be a coursework problem for first-year analysis for mathematicians] With the answer to the previous question, can you show that the limit as $p \to \infty$ of the l^p norm is the maximum norm? (It is in fact also called the l^∞ norm).

17. **Homework 7:** State and prove an energy estimate for the wave equation on $\mathbb{R}^2$,

$$u_{,tt} = u_{,xx} + u_{,yy}, \quad -\infty < x < \infty, \quad -\infty < y < \infty \tag{163}$$

where u and its first derivatives vanish as $x \to \pm\infty$ or $y \to \pm\infty$.

18. Show that the energy norm for the wave equation can be written as

$$E(t) = \frac{1}{\sqrt{2}} ||\boldsymbol{U}(\cdot, t)||_{L^2} \tag{164}$$

where

$$\boldsymbol{U}(x,t) := (u_{,t}(x,t), u_{,x}(x,t)) \tag{165}$$

Hence conclude that $E(t)$ obeys the three conditions for a norm, as long as we exclude constant solutions.

3 Classification of PDEs from their symbol

3.1 Introduction

In Sec. 1.5 we highlighted some typical features of what we called, provisionally, elliptic, parabolic and hyperbolic PDEs. In Sec. 2.2 we saw examples of boundary conditions that make these PDEs well-posed or ill-posed. Both are closely related to the coefficients of the highest derivatives in the PDE.

In this Section we give formal definitions of elliptic PDEs and PDE systems, and of hyperbolic PDEs and PDE systems. These definitions are quite general and abstract, and we will try to motivate them from simple examples, and then check that simple examples fall into these categories.

Initially, we look at linear PDEs with constant coefficients. It turns out that this captures the essence of what we are trying to do, and it allows us to use two essential technical tools, the Fourier transform (for functions of several variables) and, based on the Fourier transform, the concept of the symbol of a PDE.

We then look at an example of an elliptic PDE, review why it has a well-bosed boundary value problem but not a well-posed Cauchy problem, and from there make a definition of ellipticity for a linear scalar PDE with constant coefficients.

Next, we make two generalisations: to linear PDEs with coefficients that depend on $\boldsymbol{x}$, and to systems of linear PDEs.

We then move on to hyperbolic PDEs. Again we start with an example to motivate the general definition that follows.

Then we look at two important classes of scalar PDEs (i.e. not PDE systems), namely first-order and second-order linear PDEs. We will see that first-order scalar PDEs are all hyperbolic, and we will see that determining if second-order scalar PDEs are elliptic or hyperbolic can be done with an elegant shortcut. For second-order scalar PDEs (only) we also formally define parabolicity.

Finally, we generalise to nonlinear PDEs (or systems). We do not consider the most general type of nonlinear PDE, but only the most common type, called quasilinear PDEs. The conservation laws we encounter in the next section will be quasilinear, for example.

We do not state or prove any well-posedness theorems for the classes of PDEs that we define, as even stating them rigorously goes beyond this course.

3.2 The symbol of a linear PDE

3.2.1 The Fourier transform in $\mathbb{R}^n$

For any (vector-valued) function $\boldsymbol{u}(\boldsymbol{x})$ defined on all of $\mathbb{R}^n$, we define its **Fourier transform**

$$\hat{\boldsymbol{u}}(\boldsymbol{k}) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \boldsymbol{u}(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \, d^n x, \tag{166}$$

where

$$\boldsymbol{k} \cdot \boldsymbol{x} := \sum_{i=1}^{n} k_i x^i \tag{167}$$

is the standard inner product on $\mathbb{R}^n$. $\boldsymbol{k}$ is called the **wave number** (even though for $n > 1$ it is not a number but a vector). If the Fourier transform exists, one can show that it has the unique inverse

$$\boldsymbol{u}(\boldsymbol{x}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\boldsymbol{u}}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \, d^n k. \tag{168}$$

In the case $n = 1$, the formulas (166) and (168) reduce to the formulas (52) and (51) above.

3.2.2 Definition of the symbol

Definition 3.1. We can write any (scalar) linear PDE in the abstract form

$$L(\boldsymbol{x}, \nabla) u = f, \tag{169}$$

where L is a **linear derivative operator** acting to the right on $\boldsymbol{u}(\boldsymbol{x})$. More generally, we can write any system of N linear PDEs for N unknowns u^α in the form

$$L(\boldsymbol{x}, \nabla)\boldsymbol{u} = \boldsymbol{f}, \tag{170}$$

where L is now an $N \times N$ matrix of linear derivative operators, or equivalently a matrix-valued derivative operator. We denote by L^p the **principal part** of L, comprising only the highest derivatives.

Definition 3.2. The **symbol** of the linear differential operator $L(\boldsymbol{x}, \nabla)$ is the algebraic expression $L(\boldsymbol{x}, i\boldsymbol{k})$. (For a system of PDEs, the symbol is matrix-valued). The **principal symbol** $L^p(\boldsymbol{x}, i\boldsymbol{k})$ is just the symbol of the principal part.

Example 3.3. The second-order linear PDE

$$\sum_{ij} a^{ij}(\boldsymbol{x})u_{,ij} + \sum_i b^i(\boldsymbol{x})u_{,i} + c(\boldsymbol{x})u = f \tag{171}$$

can be written in operator form as $Lu = f$, where

$$L(\boldsymbol{x}, \nabla) = \sum_{ij} a^{ij}(\boldsymbol{x}) \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + \sum_i b^i(\boldsymbol{x})\frac{\partial}{\partial x^i} + c(\boldsymbol{x}). \tag{172}$$

(To avoid ambiguity about what the partial derivatives act on, we write all coefficients to the left of all partial derivatives, so that the partial derivatives act only on u.) Its symbol is therefore

$$L(\boldsymbol{x}, i\boldsymbol{k}) = -\sum_{ij} a^{ij}(\boldsymbol{x})k_i k_j + i\sum_i b^i(\boldsymbol{x})k_i + c(\boldsymbol{x}), \tag{173}$$

where k_i are the components of the vector $\boldsymbol{k}$. This is simply a quadratic expression in $\boldsymbol{k}$. (Note the inhomogeneous term f does not appear in the symbol.) The principal symbol is

$$L^p(\boldsymbol{x}, i\boldsymbol{k}) = -\sum_{ij} a^{ij}(\boldsymbol{x})k_i k_j, \tag{174}$$

that is, a homogeneous quadratic expression in $\boldsymbol{k}$.

3.2.3 The symbol and plane wave solutions

One useful property of the Fourier transform is that taking a partial derivative in real space $\boldsymbol{x}$ corresponds to multiplication in Fourier space $\boldsymbol{k}$. Namely

$$\begin{aligned}
\frac{\partial}{\partial x^i}\boldsymbol{u}(\boldsymbol{x}) &= \frac{\partial}{\partial x^i}(2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{\boldsymbol{u}}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^n k \\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{\boldsymbol{u}}(\boldsymbol{k})\frac{\partial}{\partial x^i}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^n k \\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \hat{\boldsymbol{u}}(\boldsymbol{k})(ik_i)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^n k.
\end{aligned} \tag{175}$$

So we have shown (if the Fourier transform of $\boldsymbol{u}$ exists) that

$$\widehat{(\boldsymbol{u}_{,i})}(\boldsymbol{k}) = \widehat{\left(\frac{\partial}{\partial x^i}\boldsymbol{u}\right)}(\boldsymbol{k}) = (ik_i)\,\hat{\boldsymbol{u}}(\boldsymbol{k}). \tag{176}$$

We can use this again to get the formula for a second derivative,

$$\widehat{(\boldsymbol{u}_{,ij})}(\boldsymbol{k}) = \widehat{\left(\frac{\partial^2}{\partial x^i \partial x^j}\boldsymbol{u}\right)}(\boldsymbol{k}) = (ik_i)(ik_j)\,\hat{\boldsymbol{u}}(\boldsymbol{k}) = -k_i k_j \hat{\boldsymbol{u}}(\boldsymbol{k}), \tag{177}$$

and so on.

Now assume that the coefficients of the PDE (system) are constant (independent of $\boldsymbol{x}$). Hence we can write $L(\boldsymbol{x}, \nabla) = L(\nabla)$. Then we can pull the coefficients of each derivative into the integral together with the derivative, and so we have

$$\begin{aligned} L(\nabla)\boldsymbol{u}(\boldsymbol{x}) &= L(\nabla)(2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\boldsymbol{u}}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^n k \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\boldsymbol{u}}(\boldsymbol{k}) L(\nabla) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^n k \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\boldsymbol{u}}(\boldsymbol{k}) L(i\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, d^n k. \end{aligned} \tag{178}$$

We have shown the following:

Lemma 3.4. If the Fourier transform of $L(\nabla)u$ exists, then it is given by

$$\widehat{(L(\nabla)\boldsymbol{u})}(\boldsymbol{k}) = L(i\boldsymbol{k})\, \hat{\boldsymbol{u}}(\boldsymbol{k}) \tag{179}$$

In words, one obtains the Fourier transform of $L\boldsymbol{u}$ by multiplying the Fourier transform of $\boldsymbol{u}$ by the symbol of L.

Remark 3.5. Conversely, assume we have a homogeneous linear scalar PDE with constant coefficients $L(\nabla)u = 0$. Then we can try to find a **plane wave solution** of the form

$$u(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{180}$$

Clearly

$$L(\nabla)u(\boldsymbol{x}) = L(\nabla)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = L(i\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 0 \quad \Leftrightarrow \quad L(i\boldsymbol{k}) = 0. \tag{181}$$

So for each solution of the algebraic equation $L(i\boldsymbol{k}) = 0$ we have a plane wave solution of the PDE $L(\nabla)u$.

Remark 3.6. Note that if there is no *real* wave number vector $\boldsymbol{k}$ such that $\det L(i\boldsymbol{k}) = 0$, there can be no purely *oscillating* plane wave solutions.

We can then construct the general solution by adding together (integrating over) different plane wave solutions, with arbitrary coefficients. Basically, this integral is an inverse Fourier transform. We can combine plane wave solutions and their complex conjugates to obtain real solutions, along the line of $\exp(ikx) + \exp(-ikx) = 2\cos kx$. In this context, the complex plane wave solutions or their real counterparts are also called **elementary solutions**, as they are the building blocks of the general solution.

If we apply separation of variables to a scalar PDE with constant coefficients, then once again $L(i\boldsymbol{k}) = 0$ appears: those components of $\boldsymbol{k}$ that can be chosen freely are the separation constants.

Example 3.7. Take the linear scalar PDE with constant coefficients in three variables $\boldsymbol{x} := (x, y, z)$

$$au_{,xx} + bu_{,yy} + cu_{,zz} = 0, \tag{182}$$

and look for solutions of the form

$$u(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = e^{i\xi x + i\eta y + i\zeta z}, \tag{183}$$

where $\boldsymbol{k} := (\xi, \eta, \zeta)$. This is a solution of (182) if and only if

$$a\xi^2 + b\eta^2 + c\zeta^2 = 0. \tag{184}$$

Now look at this from the point of view of separation of variables: we try

$$u(x, y, z) = X(x)Y(y)Z(z). \tag{185}$$

Separating off x first (the order does not matter in this example) gives us

$$a\frac{X''}{X} = A, \qquad b\frac{Y''}{Y} + c\frac{Z''}{Z} = -A. \tag{186}$$

Then we separate y, getting

$$a\frac{X''}{X} = A, \qquad b\frac{Y''}{Y} = B, \qquad c\frac{Z''}{Z} = -A - B. \tag{187}$$

If we now rewrite rewrite A and B as $A = -a\xi^2$, $B = -b\eta^2$, and introduce the shorthand $\zeta^2 = (A + B)/c$, we have

$$X'' = -\xi^2 X, \quad Y'' = -\eta^2 Y, \quad Z'' = -\zeta^2 Z \tag{188}$$

By solving these ODEs we again get (183), and from our definitions of ξ, η and ζ we get (184).

Remark 3.8. Consider now a linear PDE *system*, still with constant coefficients, and look for plane wave solutions of the form

$$\boldsymbol{u}(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{r}, \tag{189}$$

where $\boldsymbol{r}$ is a constant vector in state space $\mathbb{R}^N$. (But $\boldsymbol{x}$ and $\boldsymbol{k}$ are vectors in physical space $\mathbb{R}^n$!) Then

$$L(\nabla)\boldsymbol{u} = 0 \quad \Leftrightarrow \quad L(i\boldsymbol{k})\boldsymbol{r} = 0 \quad \Rightarrow \quad \det L(i\boldsymbol{k}) = 0. \tag{190}$$

The middle equality states that $\boldsymbol{r}$ is an eigenvector of the $N \times N$ matrix $L(i\boldsymbol{k})$ with eigenvalue 0, and the last equality is a necessary condition for this.

3.3 Ellipticity

3.3.1 A model elliptic PDE

Consider the Laplace equation in two dimensions,

$$u_{,xx} + u_{,yy} = 0, \tag{191}$$

without worrying about the domain or boundary conditions at this point. Using separation of variables, or looking for plane wave solutions, we find, for example, the particular elementary solution

$$u(x, y) = \sin kx \sinh ky, \tag{192}$$

for any real constant k. This solution remains finite and oscillates in the x-direction, with periodic zeros (at $x = n\pi$), and and it increases in the y-direction, with only one zero (at $y = 0$). More generally, any elementary solution, any linear combination of elementary solutions, and in fact any solution of (191), may vanish at two values of x and at one value of y, but can never vanish at two values of x and two values of y.

So there is no solution of (191) that vanishes on all four sides of a rectangle in the xy-plane. This implies uniqueness of the Dirichlet problem on a rectangle for this PDE: if there were two different solutions that obeyed the same Dirichlet boundary conditions, their difference would be a non-zero solution that vanished everywhere on the boundary. Uniqueness is actually true for the Laplace equation on a domain V of arbitrary shape, and with other types of boundary condition, as we have already shown in Sec. 2.4.3.

Now consider the Cauchy problem for the same PDE, considering y as the time variable. Because the PDE is second order in y, we specify both u and $u_{,y}$ on $y = 0$. For example, (192) is a solution with $u(x, 0) = 0$ and $u_{,y}(x, 0) = k \sin kx$. But now we run into the problem we have encountered already in Sec. 2.2.3: There is a unique solution for these Cauchy data, but it grows without bound as $k \to \infty$ because $\sinh ky \to \infty$ for any $y > 0$. Hence there is no continuous dependence on the initial data.

Remark 3.9. It is very important to understand that the problem for the well-posedness of the Cauchy problem for (191) is not that the solution (192) grows exponentially with increasing y. The problem is that it grows without bound with k at fixed y. As u can grow as $\sinh ky$, for any k, the same is true for $||u(\cdot, y)||$. This means that there can be no norms $\|u\|_{\text{data}}$ and $\|u\|_{\text{solution}}$ such that

$$\|u(\cdot, y)\|_{\text{solution}} \leq K(y)\|u(\cdot, 0)\|_{\text{data}}, \tag{193}$$

where $K(y)$ does not depend on the data *and so does not depend on* k in this example.

But how do we know that there is not *some* norm for which (193) does hold? From the property of a norm that $\|cu\| = |c|\|u\|$ for any constant c it follows that any norm over x of $\sin kx \sinh ky$ must be proportional to $\sinh ky$. So we would have the problem of unlimited growth in any norm. [This trick only works because (192) is the product of a function of x and a function of y. But that is fair enough: we only need one counterexample to prove that something does not hold.]

Remark 3.10. *The key property of elliptic PDEs is that every solution grows in at least one direction.* This gives us uniqueness of the pure boundary-value problem problem, but at the same time destroys the continuous dependence on the data for the Cauchy problem: the well-posed problem for this PDE is a boundary value-problem, not a Cauchy problem.

3.3.2 Formal definition of ellipticity

Definition 3.11. (Provisional definition) The linear PDE $L(\nabla)u = 0$ in n independent variables *with constant coefficients* is called **elliptic** if and only if

$$L^p(i\boldsymbol{k}) \neq 0 \qquad \forall \boldsymbol{k} \in \mathbb{R}^n \neq 0. \tag{194}$$

Remark 3.12. From Remark 3.6, the absence of *real* $\boldsymbol{k}$ that solve $L(i\boldsymbol{k}) = 0$ means that there are no plane wave solutions that oscillate in every direction – any plane wave solution must be growing exponentially in at least one direction (the direction in which $\boldsymbol{k}$ has an imaginary part). But the general solution is made from a superposition of plane wave solutions, via a Fourier transform or Fourier series. We saw in Sec. 3.3.1 how this growth in at least one direction gives both uniqueness of the boundary problem and ill-posedness of the Cauchy problem.

Going to systems, Remark 3.8 motivates the following definition:

Definition 3.13. (Provisional definition) The linear PDE system with constant coefficients $L(\nabla)\boldsymbol{u} = 0$ for N dependent variables in n independent variables is called **elliptic** if and only if

$$\det L^p(i\boldsymbol{k}) \neq 0 \qquad \forall \boldsymbol{k} \in \mathbb{R}^n \neq 0. \tag{195}$$

3.3.3 The high-frequency approximation

What about linear PDEs with coefficients that do depend on $\boldsymbol{x}$? If a PDE fails to be well-posed, it is often because solutions are badly behaved when they vary very rapidly in space (or space and time). For example, we saw in Sec. 3.3.1 that continuous dependence of the solution on the data typically breaks down because the solution becomes arbitrarily large relative to its data if we consider arbitrarily large wave numbers $\boldsymbol{k}$.

Denote by ℓ_{coeff} the typical distance on which the coefficients of the PDE vary. Consider solutions $u(\boldsymbol{x})$ which vary on a typical length scale ℓ_{soln}, with

$$\ell_{\text{soln}} \ll \ell_{\text{coeff}}. \tag{196}$$

We shall refer to this as the **high-frequency approximation**. We can then consider the coefficients of the PDE as approximately constant. The resulting Fourier transform is then $L(i\boldsymbol{k})\hat{u}(\boldsymbol{k})$.

Furthermore, because the solution is rapidly varying, this is dominated by $L^p(i\boldsymbol{k})\hat{u}(\boldsymbol{k})$, because when $|\boldsymbol{k}|$ is large, a higher power of it is much larger.

This motivates a new definition, where a PDE is elliptic *at a point* x if it is elliptic taking into account only the highest derivatives, and "freezing" the coefficients of the PDE at their values at x. Hence we have the following two definitions.

Definition 3.14. (Generalises and replaces Def. 3.11) The linear PDE (now with variable coefficients) $L(\boldsymbol{x}, \nabla)u = 0$ is called **elliptic at the point** $\boldsymbol{x}$ if and only if

$$L^p(\boldsymbol{x}, i\boldsymbol{k}) \neq 0 \qquad \forall \boldsymbol{k} \in \mathbb{R}^n \neq 0.$$

Definition 3.15. (Generalises and replaces Def. 3.13.) The linear PDE system $L(\boldsymbol{x}, \nabla)\boldsymbol{u} = 0$ is called **elliptic at the point** $\boldsymbol{x}$ if and only if

$$\det L^p(\boldsymbol{x}, i\boldsymbol{k}) \neq 0 \qquad \forall \boldsymbol{k} \in \mathbb{R}^n \neq 0. \tag{197}$$

To save repetition, in the following subsections we will define hyperbolicity and parabolicity (at a point) immediately for PDEs and PDE systems with variable coefficients.

3.4 A model parabolic PDE

Consider now the diffusion equation in one dimension

$$u_{,t} = u_{,xx}. \tag{198}$$

One elementary solution is

$$u(x, y) = \sin kx e^{-k^2 t}, \tag{199}$$

for any real constant k. This solution decays with t for all real k. For larger k (more rapidly changing initial data), it just decays faster. Essentially for this reason, the Cauchy problem is in fact well-posed for $t > 0$, as we have seen in Sec. 2.4.1. But the Cauchy problem would be ill-posed for $t < 0$ (the backwards heat equation), because then the solution grows with k without bound at fixed t.

The Dirichlet problem on a rectangle in the xt-plane is also ill-posed, simply because the solution is already determined by data on, for example, $t = 0$, $x = 0$ and $x = \pi$: there is no freedom to specify boundary data at any $t > 0$.

Remark 3.16. *The key property of a parabolic equation determining all this is that every solution decays in one and the same direction (time).*

We will not give a definition of parabolicity for arbitrary PDEs and PDE systems, but we will give one for second-order scalar PDEs, in Sec. 3.7.2 below.

3.5 Hyperbolicity

3.5.1 A model hyperbolic PDE

Consider now the wave equation in one space dimension,

$$u_{,tt} = u_{,xx}. \tag{200}$$

For any real k, one elementary solution is

$$u(x, t) = \sin kx(A \cos kt + B \sin kt) \tag{201}$$

We saw in Sec. 2.2.1 that the Dirichlet problem on the rectangle $0 \leq x \leq \pi$, $0 \leq t \leq \pi$ is ill-posed because it does not have unique solutions. This is because $\sin kt \sin kt$ vanishes on the four sides of this rectangle for $k = 1, 2, \ldots$. Intuitively, this is possible because this solution oscillates, with repeating zeros, in both the x and t-direction. We saw that if we choose a generic rectangle, rather than a square, there is a unique solution, but this can then become arbitrarily large compared to the boundary date. So then the problem is ill-posed because we lack continuous dependence on the initial data. Either way, the Dirichlet problem is ill-posed.

On the other hand, we get continuous dependence on initial data for the Cauchy problem because the solution oscillates but does not grow. We showed this more formally in Sec. 2.4.2.

We also saw from the d'Alembert solution that any Cauchy data $u(x, 0) = f(x)$ and $u_{,t}(x, 0) = g(x)$ give rise to a unique solution. The solution (201), for example, has Cauchy data $f(x) = A \sin kx$ and $g(x) = Bk \sin kx$. We have also seen that we can reassemble the initial data $f(x)$ and $g(x)$ into a right-moving wave $F(x - t)$ and a left-moving wave $G(x + t)$.

Remark 3.17. *The key property of a (strongly or strictly hyperbolic) PDE is that there are enough linearly independent solutions oscillating in all directions (space and time) to split Cauchy data into "waves".*

3.5.2 Strict hyperbolicity

Definition 3.18. The linear PDE $L(\boldsymbol{x}, \nabla)u = 0$ in n independent variables is called **strictly hyperbolic** in the time direction $\boldsymbol{n}$ at the point $\boldsymbol{x}$ if and only if

$$L^p(\boldsymbol{x}, i\boldsymbol{n}) \neq 0, \tag{202}$$

and all roots ω of the polynomial equation

$$L^p(\boldsymbol{x}, i\boldsymbol{k} + i\omega\boldsymbol{n}) = 0 \tag{203}$$

are real and distinct for all $\boldsymbol{k} \in \mathbb{R}^n$ that are not zero or a multiple of $\boldsymbol{n}$.

This is complicated definition, and we look at it piece by piece.

Remark 3.19. By the same argument as before, for a PDE with constant coefficients,

$$u(\boldsymbol{x}) = e^{(i\boldsymbol{k}+i\omega\boldsymbol{n})\cdot\boldsymbol{x}} \tag{204}$$

is a solution of $L^p u = 0$ if and only if (203) holds. Moreover, it is an approximate solution of $Lu = 0$ in the high-frequency approximation.

Remark 3.20. Often, the time direction $\boldsymbol{n}$ we want to consider is just along one of the coordinate axes, say $\boldsymbol{x} =: (t, \boldsymbol{y})$ with $\boldsymbol{n} = (1, \boldsymbol{0})$. Without loss of generality, we can then also restrict $\boldsymbol{k}$ to $\boldsymbol{k} =: (0, \boldsymbol{l})$. (See Example 3.26 for an explanation why this is possible.) Here $\boldsymbol{l} \in \mathbb{R}^{n-1}$ and $\boldsymbol{y} \in \mathbb{R}^{n-1}$. In this special case, (204) becomes

$$u(t, \boldsymbol{y}) = e^{i\boldsymbol{l}\cdot\boldsymbol{y}+i\omega t} = e^{i|\boldsymbol{l}|\left(\hat{\boldsymbol{l}}\cdot\boldsymbol{y}-vt\right)}, \tag{205}$$

where

$$\boldsymbol{l}\cdot\boldsymbol{y} := \sum_{\alpha=1}^{n-1} l_\alpha y^\alpha, \qquad \hat{\boldsymbol{l}} := \frac{\boldsymbol{l}}{|\boldsymbol{l}|}, \qquad v := -\frac{\omega}{|\boldsymbol{l}|}. \tag{206}$$

We see that this corresponds to a plane wave travelling with velocity v in the spatial direction given by the unit vector $\hat{\boldsymbol{l}}$.

Remark 3.21. Why the condition that all eigenvalues ω be real? As the coefficients of L^p are real, if ω is a solution of (203), so is its complex conjugate ω^*. Hence unless ω is real, either ω or ω^* has positive imaginary part, and so (205) grows exponentially with time for one of them. Furthermore, if L is a differential operator of order m, $L^p(i\boldsymbol{k})$ is a homogeneous polynomial of order m, or $L^p(\lambda i\boldsymbol{k}) = \lambda^m L^p(i\boldsymbol{k})$. Hence if the pair $(\boldsymbol{k}, \omega)$ is a solution of (203), so is any multiple $(\lambda\boldsymbol{k}, \lambda\omega)$. Hence if there are complex ω, $Lu = 0$ has solutions (205) that grow the more rapidly the faster they oscillate in space. Hence there cannot be an estimate of the form (134) because no $K(t)$ would grow fast enough, and so the Cauchy problem cannot be well-posed. (This is precisely what happened in our example of trying to solve the Cauchy problem for the Laplace equation, Sec. 2.2.3.)

Remark 3.22. Why the condition that all eigenvalues ω be distinct? Consider a Fourier transform in $\boldsymbol{y}$, but not in t, that is $\hat{u}(\boldsymbol{l}, t)$. The condition that all eigenvalues ω be distinct then guarantees that we can obtain $\hat{u}(\boldsymbol{l}, t)$ (the Fourier transform of the solution) from the the Fourier transform of the Cauchy data $\hat{u}(\boldsymbol{l}, 0)$, $\hat{u}_{,t}(\boldsymbol{l}, 0)$, $\hat{u}_{,tt}(\boldsymbol{l}, 0)$, and so on, up to $k-1$ time derivatives for a k-th order in time PDE.

For a system of PDEs, we again look for plane waves of the form

$$\boldsymbol{u}(\boldsymbol{x}) = e^{(i\boldsymbol{k}+i\omega\boldsymbol{n})\cdot\boldsymbol{x}}\boldsymbol{r}, \tag{207}$$

where $\boldsymbol{r}$ is a constant vector in state space that obeys the matrix equation

$$L(i\boldsymbol{k} + i\omega\boldsymbol{n})\boldsymbol{r} = 0. \tag{208}$$

A necessary condition for such an $\boldsymbol{r} \neq 0$ to exist is that the determinant of this matrix vanishes. We will not spell out all the details, but arguments similar to the ones we gave in the scalar case then motivate the following definition.

Definition 3.23. The linear PDE system $L(\boldsymbol{x}, \nabla)\boldsymbol{u} = 0$ in n independent variables is called **strictly hyperbolic** in the time direction $\boldsymbol{n}$ at the point $\boldsymbol{x}$ if

$$\det L^p(\boldsymbol{x}, i\boldsymbol{n}) \neq 0, \tag{209}$$

and all roots ω of the polynomial equation

$$\det L^p(\boldsymbol{x}, i\boldsymbol{k} + i\omega\boldsymbol{n}) = 0 \tag{210}$$

are real and distinct for all $\boldsymbol{k} \in \mathbb{R}^n$ that are not zero or a multiple of $\boldsymbol{n}$.

There are other definitions of hyperbolicity, notably strong hyperbolicity. We do not mention them here to keep things simple.

3.6 Examples

Example 3.24. The second-order scalar PDE

$$u_{,xx} + u_{,yy} = 0 \tag{211}$$

(Laplace equation) is elliptic. Proof: Write $\boldsymbol{k} := (\xi, \eta)$. Then

$$L(i\boldsymbol{k}) = -(\xi^2 + \eta^2) = -|\boldsymbol{k}|^2. \tag{212}$$

But this is positive definite in $\boldsymbol{k}$, so the only real solution of $|\boldsymbol{k}|^2 = 0$ is $\boldsymbol{k} = 0$.

Example 3.25. The first-order PDE system

$$u_{,y} + v_{,x} = 0, \tag{213}$$

$$u_{,x} - v_{,y} = 0 \tag{214}$$

(the **Cauchy-Riemann equations**) is elliptic. Proof: This system can be written as

$$\begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{215}$$

Hence

$$L(i\boldsymbol{k}) = i \begin{pmatrix} \eta & \xi \\ \xi & -\eta \end{pmatrix} \quad \Rightarrow \quad \det L(i\boldsymbol{k}) = \xi^2 + \eta^2. \tag{216}$$

Note that by taking a derivative this first-order system implies $u_{,xx} + u_{,yy} = 0$ and $v_{,xx} + v_{,yy} = 0$, both of which are also elliptic.

Example 3.26. The second-order scalar PDE

$$-u_{,xx} + u_{,yy} + u_{,zz} = 0 \tag{217}$$

(the wave equation in two space dimensions) is strictly hyperbolic in the x-direction.

Proof: With $\boldsymbol{k} := (\xi, \eta, \zeta)$, the principal symbol (which is equal to the full symbol) is

$$L(i\boldsymbol{k}) = \xi^2 - \eta^2 - \zeta^2 \tag{218}$$

With $\boldsymbol{n}$ the x-direction, we write $\boldsymbol{n} = (1, 0, 0)$. Hence $L(i\boldsymbol{n}) = 1 \neq 0$, and

$$L(i\omega\boldsymbol{n} + i\boldsymbol{k}) = L\left[i(\omega + \xi, \eta, \zeta)\right] = (\omega + \xi)^2 - \eta^2 - \zeta^2. \tag{219}$$

The two solutions of $L(i\omega\boldsymbol{n} + i\boldsymbol{k}) = 0$ are therefore

$$\omega_\pm = -\xi \pm \sqrt{\eta^2 + \zeta^2}. \tag{220}$$

These are real for all real $\boldsymbol{k}$, and they are distinct unless $\eta = \zeta = 0$. But we can exclude that case as then $\boldsymbol{k}$ would be a multiple of $\boldsymbol{n}$.

Note that we could set ξ to zero by absorbing it into ω. Therefore, without loss of generality we can restrict to $\boldsymbol{k} = (0, \boldsymbol{l}) = (0, \eta, \zeta)$ for simplicity. We then have

$$L(i\omega\boldsymbol{n} + i\boldsymbol{k}) = L\left[i(\omega, \eta, \zeta)\right] = \omega^2 - |\boldsymbol{l}|^2, \qquad |\boldsymbol{l}| = \sqrt{\eta^2 + \zeta^2}. \tag{221}$$

The solutions of $L(i\omega\boldsymbol{n} + i\boldsymbol{k}) = 0$ are then

$$\omega_\pm = \pm|\boldsymbol{l}|. \tag{222}$$

Example 3.27. The second-order scalar PDE (217) is in fact strictly hyperbolic in all directions

$$\boldsymbol{n} = (1, \boldsymbol{v}) := (1, v_y, v_z) \quad \text{with} \quad |\boldsymbol{v}|^2 := v_x^2 + v_y^2 < 1, \tag{223}$$

that is in all directions inside a cone around $(1, 0, 0)$.

Beginning of proof: With $\boldsymbol{k} =: (\xi, \eta, \zeta) := (\xi, \boldsymbol{l})$, we have $L(i\boldsymbol{k}) = \xi^2 - |\boldsymbol{l}|^2$. Also, $\omega\boldsymbol{n} + \boldsymbol{k} = (\omega + \xi, \omega\boldsymbol{v} + \boldsymbol{l})$. Hence

$$L(i\omega\boldsymbol{n}+i\boldsymbol{k}) = (\omega+\xi)^2-|\omega\boldsymbol{v}+\boldsymbol{l}|^2 = (1-|\boldsymbol{v}|^2)\omega^2+2\omega(\xi-\boldsymbol{l}\cdot\boldsymbol{v})+(\xi^2-|\boldsymbol{l}|^2) =: A\omega^2 + 2B\omega + C. \tag{224}$$

Setting this to zero has two distinct real solutions if and only if $\Delta := B^2 - AC > 0$. To finish the proof, we need to show that in fact $\Delta > 0$ for all ξ if and only if $|\boldsymbol{v}|^2 < 1$. (We do this first for $\xi = 0$, and then for $\xi \neq 0$.)

Example 3.28. The first-order PDE system

$$\begin{aligned} -u_{,x} + v_{,y} + w_{,z} &= 0, & (225)\\ -v_{,x} + u_{,y} &= 0, & (226)\\ -w_{,x} + u_{,z} &= 0 & (227) \end{aligned}$$

is a) strictly hyperbolic for the same directions $\boldsymbol{n}$ as in Example 3.27, and b) actually equivalent to (217).

Example 3.29. The first-order PDE system

$$\begin{aligned} u_{,x} + \mu u_{,y} &= 0, & (228)\\ v_{,x} + \nu v_{,y} + \lambda u_{,y} &= 0, & (229) \end{aligned}$$

where μ, ν and λ are real constants, is strictly hyperbolic in the x-direction if and only if $\mu \neq \nu$. Beginning of proof: Let $\boldsymbol{n} = (1, 0)$ and $\boldsymbol{k} := (\xi, \eta)$. Then

$$L(i\omega\boldsymbol{n} + i\boldsymbol{k}) = i \begin{pmatrix} \omega + \xi + \mu\eta & 0 \\ \lambda\eta & \omega + \xi + \nu\eta \end{pmatrix}. \tag{230}$$

The result follows by explicitly calculating the eigenvalues and eigenvectors of this matrix.

3.7 Special cases

3.7.1 First order scalar PDEs

Remark 3.30. Any *scalar* first-order linear PDE in n independent variables

$$\sum_{i=1}^{n} a^i(\boldsymbol{x}) u_{,i} + c(\boldsymbol{x})u + d(\boldsymbol{x}) = 0 \tag{231}$$

is strictly hyperbolic in the sense of Def. 3.18 for all $\boldsymbol{n}$ that are not normal to $\boldsymbol{a}$. Hence all such PDEs are of the same type.

Remark 3.31. This is not so for first-order *systems*, which can be hyperbolic, elliptic, or neither.

Remark 3.32. Solving a scalar, first-order PDE is equivalent to first solving a system of first-order ODEs. Intuitively the principal part of (231) can be thought of as just a directional derivative along the vector field $a^i(u, \boldsymbol{x})$. u can then be obtained from initial data by integrating along this vector field.

3.7.2 Second order scalar PDEs

For *scalar, second-order* linear PDEs in n independent variables, there is a complete classification into four types, as follows.

Definition 3.33. Consider the scalar linear second-order PDE for the unknown u in n independent variables $\boldsymbol{x}$,

$$L(\boldsymbol{x}, \nabla)u = \sum_{i,j=1}^{n} a^{ij}(\boldsymbol{x})u_{,ij} + \sum_{i=1}^{n} b^{i}(\boldsymbol{x})u_{,i} + c(\boldsymbol{x})u + d(\boldsymbol{x}) = 0. \tag{232}$$

Hence the principal symbol is

$$L^{p}(\boldsymbol{x}, i\boldsymbol{k}) = -\sum_{i,j=1}^{n} a^{ij}(\boldsymbol{x})k_i k_j. \tag{233}$$

Now consider the coefficients $a^{ij}(\boldsymbol{x})$ of the principal symbol as a symmetric $n \times n$ matrix. Then we call the PDE **parabolic** at $\boldsymbol{x}$ if a^{ij} is singular (so one or more eigenvalues are zero). Conversely, if all eigenvalues are nonzero, we call the PDE **elliptic** at $\boldsymbol{x}$ if all eigenvalues of a^{ij} have the same sign. We call it **hyperbolic** at $\boldsymbol{x}$ if one eigenvalue has the opposite sign from the others. The remaining case, where there is more than one eigenvalue of each sign (but none zero), is called **ultrahyperbolic** at $\boldsymbol{x}$. The PDE is simply called parabolic if it is parabolic at every $\boldsymbol{x}$, and so on.

Proposition 3.34. This definition of ellipticity agrees with Def. 3.14, and this definition of hyperbolicity agrees with strict hyperbolicity for at least one choice of $\boldsymbol{n}$ defined in Def. 3.18 (namely the eigenvector corresponding to the one eigenvalue that has opposite sign from all the others).

Proof. If all eigenvalues of the matrix a^{ij} have the same sign, then without loss of generality we can assume they are all positive. Hence the matrix is positive definite, which is defined as

$$\sum_{i,j} a^{ij}k_i k_j \geq 0 \quad \forall \boldsymbol{k} \in \mathbb{R}^n, \quad \text{with} \quad \sum_{i,j} a^{ij}k_i k_j = 0 \quad \Leftrightarrow \quad \boldsymbol{k} = 0. \tag{234}$$

Hence we have ellipticity.

Now assume that one eigenvalue is negative and all others are positive. Let the corresponding eigenvector be $\boldsymbol{n}$, so that

$$\sum_{i,j} a^{ij}n_i n_j < 0. \tag{235}$$

We can uniquely decompose any other vector $\boldsymbol{k}$ as a part in the direction of $\boldsymbol{n}$ and a part "orthogonal" to it, in the sense that

$$\boldsymbol{k} = \xi\boldsymbol{n} + \boldsymbol{l}, \quad \text{with} \quad \sum_{i,j} a^{ij}n_i l_j = 0. \tag{236}$$

(Here, "orthogonal" means orthogonal with respect to the inner product defined by the positive definite matrix A.) Moreover,

$$\sum_{i,j} a^{ij}l_i l_j \geq 0, \tag{237}$$

with

$$\sum_{i,j} a^{ij}l_i l_j = 0 \quad \Leftrightarrow \quad \boldsymbol{l} = 0 \tag{238}$$

We then have

$$L(i\omega\boldsymbol{n}+i\boldsymbol{k}) = L\left[i(\omega+\xi)\boldsymbol{n} + \boldsymbol{l}\right] = \sum_{i,j} a^{ij}[(\omega+\xi)n_i + l_i][(\omega+\xi)n_j + l_j] = (\omega+\xi)^2 \sum_{i,j} a^{ij}n_i n_j + \sum_{i,j} a^{ij}l_i l_j. \tag{239}$$

Therefore the equation $L(i\omega\boldsymbol{n} + i\boldsymbol{k}) = 0$ has two distinct real solutions

$$\omega = -\xi \pm \sqrt{-\frac{\sum_{i,j} a^{ij}l_i l_j}{\sum_{i,j} a^{ij}n_i n_j}}, \tag{240}$$

for any $\boldsymbol{k}$ that is not a multiple of $\boldsymbol{n}$ (so that $\boldsymbol{l} \neq 0$). □

Remark 3.35. The only type of parabolic second-order PDE we will consider in this course are those with precisely one zero eigenvalue. The corresponding eigenvector $\boldsymbol{n}$ is the direction of time.

We will not say anything about ultrahyperbolic PDEs, which arise less naturally in physics and engineering than elliptic, pararabolic and hyperbolic ones.

3.8 Nonlinear PDEs and systems

Recall the definition of a linear PDE in Sec. 1.1.4. Some PDEs and systems in engineering and physics are linear at a deep physical level, for example the Poisson equation for the gravitational field, the Maxwell equations, or the Schrödinger equation.

Other linear PDEs arise as an approximation when we consider small perturbations of an equilibrium solution. For example, the equations of acoustics are valid for small pressure and velocity perturbations of the Euler equations. For larger perturbations, for example in an explosion, the full nonlinear Euler equations are needed, and interesting things such as shock formation will happen. In another example, the heat equation is linear because the heat flux is approximately proportional to the temperature gradient, but this is only an approximation.

We can write any PDE system in the abstract form

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}, \nabla) = 0, \tag{241}$$

meaning some (vector-valued) function $\boldsymbol{F}$ of $\boldsymbol{x}$, $\boldsymbol{u}$ and the partial derivatives of u with respect to $\boldsymbol{x}$ vanishes.

If one wants to look, for example, at continuous dependence on the data, or at uniqueness, one is in effect asking how the solution changes if there is a small change in the boundary or initial data. Hence consider a solution $\boldsymbol{u}$ that consists of a known solution $\boldsymbol{u}_0$ plus a small change, $\boldsymbol{u} = \boldsymbol{u}_0 + \epsilon\,\delta\boldsymbol{u}$, where $\epsilon > 0$ is a small number. We now expand $\boldsymbol{F}(\boldsymbol{x}, \nabla, \boldsymbol{u})$ in a Taylor series about $\epsilon = 0$ as

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_0 + \epsilon\,\delta\boldsymbol{u}, \nabla) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_0, \nabla) + \epsilon\, L(\boldsymbol{x}, \boldsymbol{u}_0, \nabla)\,\delta\boldsymbol{u} + O(\epsilon^2), \tag{242}$$

where the linear differential operator is defined by

$$L(\boldsymbol{x}, \boldsymbol{u}_0, \nabla)\,\delta\boldsymbol{u} := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}_0 + \epsilon\delta\boldsymbol{u}, \nabla) \tag{243}$$

This is just $F(\epsilon) = F(0) + \epsilon F'(0) + O(\epsilon^2)$ with knobs on. Now $dF(\epsilon\,\nabla\delta\boldsymbol{u})/d\epsilon = F'(\dots)\nabla\delta\boldsymbol{u}$ from the chain rule, and similarly for higher derivatives. This shows that the coefficient of ϵ is linear in $\delta\boldsymbol{u}$ and its derivatives, and can indeed be written in the form $L\delta\boldsymbol{u}$.

Definition 3.36. The **linearisation** $L\delta\boldsymbol{u} = 0$ of a system of PDEs of the form (241) about a solution $\boldsymbol{u}_0(\boldsymbol{x})$ is defined by (243).

Example 3.37. Consider the nonlinear PDE

$$F(u, \nabla) = u(u_{,xx} + u_{,yy}) + u_{,x}^2 + u_{,y}^2 = 0. \tag{244}$$

We linearise about a solution $u_0(x, t)$ by computing

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \boldsymbol{F}(u_0 + \epsilon\delta u, \nabla) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \big[(u_0 + \epsilon\delta u)(u_{0,xx} + \epsilon\delta u_{,xx}) + \dots + (u_{0,x} + \epsilon\delta u_{,x})^2 + \dots\big] \\
&= u_0(\delta u_{,xx} + \delta u_{,xx}) + \delta u(u_{0,xx} + u_{0,yy}) + 2(u_{0,x}\delta u_{,x} + u_{0,y}\delta u_{,y}) \\
&= L(u_0, \nabla)\,\delta u,
\end{aligned} \tag{245}$$

where the dots denote the obvious terms with y-derivatives. We read off

$$L(u_0, \nabla) = u_0\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right) + 2\left(u_{0,x}\frac{\partial}{\partial x} + u_{0,y}\frac{\partial}{\partial y}\right) + (u_{0,xx} + u_{0,yy}). \tag{246}$$

The fact that well-posedness of nonlinear perturbations is about small perturbations motivates the following definition.

Definition 3.38. A nonlinear PDE or PDE system is called **strictly hyperbolic**, **hyperbolic**, **elliptic** or **parabolic**, and so on, *about a solution* $\boldsymbol{u_0}(\boldsymbol{x})$ if its linearisation about $\boldsymbol{u_0}(\boldsymbol{x})$ has the appropriate property. In other words, we look at algebraic properties of the principal symbol of the *linearised* system, $L^p(\boldsymbol{u}_0, \boldsymbol{x}, i\boldsymbol{k})$, where $L(u_0, \boldsymbol{x}, \nabla)$ is defined by (243).

Remark 3.39. Well-posedness of the linearisation of a PDE problem is necessary for well-posedness of the full nonlinear system, but it is not sufficient. (And finding sufficient conditions is an art.)

Definition 3.40. A system of PDEs is called **quasilinear** if derivatives of the principal order occur only linearly. (Their coefficients may depend nonlinearly on the lower derivatives and the independent coordinates).

As an example, the PDE (2) is quasilinear if we allow a, b, c and f to depend only on x, y, u, $u_{,x}$ and $u_{,y}$ (but not on $u_{,xx}$, $u_{,yy}$ and $u_{,xy}$, or any higher derivatives). The systems of conservation laws we look at later (for example the Euler equation) are quasilinear. So are many or other PDEs of interest in physics and engineering. The PDE (244) is also quasilinear.

Remark 3.41. The key property of quasilinear PDE or PDE system is that it and its linearisation have essentially the same principal part. That means we can save the trouble of linearising the non-principal part, and simply consider $L^p(\boldsymbol{u}_0, \boldsymbol{x}, \nabla)$. More precisely, in the principal of the quasilinear PDE, replace the highest derivatives of u by the corresponding derivatives of δu, and replace all lower derivatives of u (including u undifferentiated) by those of u_0.

Example 3.42. Consider the quasilinear first-order PDE

$$u_{,t} + f'(u)u_{,x} = 0 \quad \Leftrightarrow \quad \left(\frac{\partial}{\partial t} + f'(u)\frac{\partial}{\partial x}\right) u = 0. \tag{247}$$

Its linearisation about a solution $u_0(x,t)$ is

$$\delta u_{,t} + f'(u_0)\,\delta u_{,x} + f''(u_0)\,\delta u\, u_{0,x} = 0. \tag{248}$$

Hence

$$L(u_0, x, \nabla) = \frac{\partial}{\partial t} + f'(u_0)\frac{\partial}{\partial x} + f''(u_0)\,u_{0,x} \quad \Rightarrow \quad L^p(u_0, x, \nabla) = \frac{\partial}{\partial t} + f'(u_0)\frac{\partial}{\partial x}. \tag{249}$$

Note the similarity between the last part of (247) and the last part of (249).

Example 3.43. The principal part of the quasilinear PDE (244) is

$$u(u_{,xx} + u_{,yy}) + \cdots = 0, \tag{250}$$

where the dots denote non-principal terms. The principal part of its linearisation $L\delta u = 0$, with L given by (246), is

$$u_0(\delta u_{,xx} + \delta u_{,yy}) + \cdots = 0, \tag{251}$$

where again the dots denote non-principal terms. Hence the principal symbol of the linearisation is

$$L^p(\boldsymbol{x}, i\boldsymbol{k}) = -u_0(\boldsymbol{x})(\xi^2 + \eta^2), \tag{252}$$

where the $\boldsymbol{x}$-dependence of the symbol comes from the x-dependence of u_0.

3.9 Exercises

19. **Homework 8:** (Simple example for the lecture) For the linear PDE with constant coefficients

$$Lu = 0 \quad \Leftrightarrow \quad u_{,xx} + 7u_{,xy} + 3u_{,x} = 0, \tag{253}$$

find $L(\nabla)$ and $L(i\boldsymbol{k})$. Show explicitly that $\widehat{L(\nabla)u} = L(i\boldsymbol{k})\hat{u}(k)$.

20. **Homework 9:** (A fourth-order elliptic PDE, and practice with separation of variables) pdfla) Using separation of variables, find the general solution the PDE

$$u_{,xxxx} + u_{,yyyy} = 0 \tag{254}$$

on the domain $0 \leq x \leq \pi$ with boundary conditions

$$u(0,y) = 0, \qquad u(\pi,y) = 0, \quad u_{,xx}(0,y) = 0, \quad u_{,xx}(\pi,y) = 0 \tag{255}$$

for all y. [Hint: this is not a complete PDE problem yet. We have not specified the domain in y and the related boundary conditions, so your solution should contain four free constants.]

b) Based on this solution, does it look as if the PDE (254) has the key property of an elliptic equation, namely that every solution grows in at least one direction?

21. (A variation on the previous problem: this looks like a fourth-order wave equation but do you think it is hyperbolic?) Using separation of variables, find the general solution of the PDE

$$u_{,xxxx} - u_{,yyyy} = 0 \tag{256}$$

on the domain $0 \leq x \leq \pi$ with boundary conditions

$$u(0,y) = 0, \quad u(\pi,y) = 0, \quad u_{,xx}(0,y) = 0, \quad u_{,xx}(\pi,y) = 0 \tag{257}$$

for all y. [Hint: this is not a complete PDE problem yet. We have not specified the domain in y and the related boundary conditions, so your solution should contain four sets of free constants.] Based on this solution, does it look as if the PDE (256) has any of the following properties: 1) every solution grows in at least one direction, 2) every solution oscillates in all directions, 3) every solution decays in one and the same direction?

22. (Continuation of the previous problem, practice with ill-posedness results) Show that the Cauchy problem consisting of (256) on the domain $0 \leq x \leq \pi$, $y \geq 0$ with boundary conditions (257) and initial data

$$u(x,0) = u_0(x), \quad u_{,y}(x,0) = u_1(x), \quad u_{,yy}(x,0) = u_2(x), \quad u_{,yyy}(x,0) = u_3(x) \tag{258}$$

has a unique solution. Show that this Cauchy problem is nevertheless ill-posed because the solution cannot depend continuously on the initial data in any function norm.

23. (Begun in lecture) Show that

$$-u_{,xx} + u_{,yy} + u_{,zz} + \cdots = 0 \tag{259}$$

is strictly hyperbolic in all directions

$$\boldsymbol{n} = (1, v^y, v^z, \dots) =: (1, \boldsymbol{v}) \tag{260}$$

with $v < 1$, where $v := |\boldsymbol{v}| := \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$.

24. **Homework 10:** (Practice with notation) Show, using the formal definitions, that a) the PDE (254) is elliptic, and that b) the PDE (256) is neither elliptic, nor strictly hyperbolic in the y-direction.

25. **Homework 11:** Show that the system of first-order PDEs

$$\begin{aligned} -u_{,x} + v_{,y} + w_{,z} &= 0, & (261)\\ -v_{,x} + u_{,y} &= 0, & (262)\\ -w_{,x} + u_{,z} &= 0 & (263) \end{aligned}$$

is (a) strictly hyperbolic for suitable $\boldsymbol{n}$ and (b) equivalent to the second-order wave equation $-u_{,xx} + u_{,yy} + u_{,zz} = 0$.

26. **Homework 12:** a) Show that the PDE

$$-4u_{,xy} + u_{,zz} = 0 \tag{264}$$

for $u(x, y, z)$ is hyperbolic, using the criterion for second-order scalar PDEs. b) Show that neither the x, y or z-directions are good time directions for this PDE. In other words, show that the PDE is not strictly hyperbolic in the directions $\mathbf{n} = (1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$. c) Show that the direction $\mathbf{n} = (1, 1, 0)$ is a good time direction. d) Change variables from (x, y, z) to (t, s, Z), where $t = (x+y)/2$, $s = (x-y)/2$ and $Z = z$, and show that you get the standard form of the wave equation. [This is consistent with the fact that we have already shown that $\nabla t = (1/2, 1/2, 0)$ is a good time direction.]

27. The compressible Euler equations in one space dimension are, in *conservation law form*,

$$\rho_{,t} + (\rho v)_{,x} = 0, \tag{265}$$

$$(\rho v)_{,t} + (\rho v^2 + P)_{,x} = 0, \tag{266}$$

$$\left(e + \frac{1}{2}\rho v^2\right)_{,t} + \left[v\left(e + \frac{1}{2}\rho v^2 + P\right)\right]_{,x} = 0, \tag{267}$$

where $P = P(\rho, e)$ is a given function with $P_{,\rho} > 0$ and $P_{,e} > 0$, is strictly hyperbolic . Here $\rho > 0$ is mass/volume, v is velocity, $e > 0$ is internal (heat) energy per volume and $P > 0$ is pressure. (And as we are in one space dimension, "volume" means length.)

a) Write this system of PDEs in the *quasilinear form*

$$A^t \boldsymbol{u}_{,t} + A^x \boldsymbol{u}_{,x} = 0, \tag{268}$$

where $\boldsymbol{u} := (\rho, v, e)$, and A^t and A^x are 3×3 matrices that you need to find, and which depend only on the variables $\boldsymbol{u}$. [Hint: This part is fairly straightforward. Start by expanding $(\rho v)_{,t} = \rho v_{,t} + \rho_{,t} v$ and similarly for the other derivatives. Use the chain rule to write $P_{,x} = P_{,\rho}\rho_{,x} + P_{,e}e_{,x}$. You now have the equations in the quasilinear form where every term in the PDE is one of the derivatives $\rho_{,t}$, $v_{,t}$, $e_{,t}$, $\rho_{,x}$, $v_{,x}$ or $e_{,x}$, times a coefficient that depends only on ρ, v and e. Then read off the matrices.]

b) Write down the conditions for this system to be strictly hyperbolic in the t-direction, in terms of the matrices A^t and A^x. Show that the first of these conditions, that $\det L^p(\boldsymbol{x}, i\boldsymbol{n}) \neq 0$, actually holds (as long as $\rho > 0$). [Hint: This is a straight application of lecture examples. Use the notation $\boldsymbol{n} = (1, 0)$ and $\boldsymbol{k} = (\tau, \xi)$. We have seen that for classification purposes we can treat a quasilinear first-order system as if it is linear. We have to check if the system is hyperbolic at the point x because ρ, v and e in the background solution all depend on x. But that x-dependence makes no practical difference to your calculation.]

c) Show that the three roots ω of $\det L^p(\boldsymbol{x}, i\omega\boldsymbol{n} + i\boldsymbol{k}) = 0$ are given by $\omega = -\xi\omega_0 - \tau$, where ω_0 are the three eigenvalues of the 3×3 matrix $\tilde{A}^x := (A^t)^{-1}A^x$. Hence conclude that all three values of ω are real and distinct if the three eigenvalues ω_0 of $\tilde{A}^x$ are real and distinct. [Hint: this is not covered by the lecture notes. But it helps with the final part, which involves the heavy algebra.]

d) Use a computer algebra program such as Maple or Mathematica to show that the three eigenvalues ω_0 of $\tilde{A}^x$ are v, $v + c$ and $v - c$, where

$$c := \sqrt{P_{,\rho} + \frac{P + e}{\rho}P_{,e}} \tag{269}$$

is the sound speed. (These are real and distinct for $c > 0$, as required.) [Hint: you could just about do this by hand, but it would be tedious and error-prone.]

4 Conservation laws

4.1 Integral and differential form

4.1.1 One space dimension

To give the simplest example of a conservation law, consider the flow of mass through a pipe, so we have a problem in one space dimension x and time t. Let $u(x,t)$ be the density of mass, measured in units of mass/length, at position x and time t.

Let $f(x,t)$ be the mass flux through the pipe, measured in mass/time, again at position x and time t. We count a flux going in the direction of increasing x as positive. Consider a segment of pipe $a \le x \le b$. The mass in that segment at time t is

$$m(t) = \int_a^b u(x,t)\,dx \tag{270}$$

This mass changes with time because of the fluxes through the ends $x = a$ and $x = b$, as follows:

$$\frac{dm}{dt} = f(a,t) - f(b,t). \tag{271}$$

Note the signs: flux towards increasing x means into the segment at $x = a$ but out of it at $x = b$. Combining the last two equations, we have

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = f(a,t) - f(b,t). \tag{272}$$

We now turn (272) into a PDE. Assuming $u(x,t)$ is once differentiable in t, we can write the left-hand side as

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b \frac{\partial}{\partial t}u(x,t)\,dx \tag{273}$$

Also, if $f(x,t)$ is once differentiable in x, we can write the right-hand side as

$$f(a,t) - f(b,t) = -\int_a^b \frac{\partial}{\partial x}f(x,t)\,dx. \tag{274}$$

Combining these two results and bringing both terms on the same side we obtain

$$\int_a^b \left[u_{,t}(x,t) + f(x,t)_{,x}\right]\,dx = 0. \tag{275}$$

If we want this to hold for any segment of pipe, a and b can take any values. Then the integral can vanish only if the integrand in square brackets vanishes for every x, or

$$u_{,t} + f_{,x} = 0. \tag{276}$$

This first-order PDE is a **conservation law** in **differential form** or **strong form**. A solution of this is called a **strong solution** of the conservation law.

However, we are often interested in the case where u and f are not differentiable, and are in fact discontinuous. (This will lead us to "shocks"). Hence we want to go the other way from (272) and also remove the time derivative. For this, we integrate (272) over a time interval $t_0 \le t \le t_1$ to get

$$\int_a^b u(x,t_1)\,dx - \int_a^b u(x,t_0)\,dx = \int_{t_0}^{t_1} f(a,t)\,dt - \int_{t_0}^{t_1} f(b,t)\,dt \tag{277}$$

This consists of four integrals, one over each side of the rectangle ($a \le x \le b, t_0 \le t \le t_1$). Note that u and f now only need to be integrable, not differentiable.

Definition 4.1. A **weak solution** of the conservation law (276) is a function $u(x,t)$ that obeys (277) *for all* a, b, t_0 and t_1.

In defining this, we use (276) only as a shorthand notation for (277). Be sure you get the signs right. This definition of a weak solution looks different from the weak solutions of the wave equation we defined in Sec. 1.7, but is closely related. (277) itself is called the **integral form** or **weak form** of the conservation law (276). We have just shown by construction that any strong solution is also a weak solution. The reverse is clearly not true, as a weak solution does not have to be once differentiable. In fact, interesting weak solutions are often discontinuous.

Note that until we have given an expression for f in terms of u, the problem has not been completely specified.

4.1.2 Higher space dimensions

It is straightforward to generalise the integral and differential form to any number of spatial dimensions. Consider the rectangle $(a \le x \le b, c \le y \le d)$ in two space dimensions. u is the density of mass, now measured in mass/(length)^2. The mass in the rectangle at time t is

$$m(t) = \int_a^b dx \int_c^d dy\, u(x, y, t). \tag{278}$$

It changes with time because of the fluxes through four sides of the rectangle:

$$\frac{dm}{dt} = \int_c^d [f^x(a, y, t) - f^x(b, y, t)]\, dy + \int_a^b [f^y(x, c, t) - f^y(x, d, t)]\, dx. \tag{279}$$

Here $f^x(x, y, t)$ is the mass flux in x-direction, and $f^y(x, y, t)$ the mass flux in the y-direction. Both are measured in mass/(length·time). Once again, this is not the form we need. The differential form is

$$u_{,t} + f^x_{,x} + f^y_{,y} = 0 \tag{280}$$

and the integral form is

$$\begin{aligned} & \int_a^b dx \int_c^d dy\, [u(x, y, t_1) - u(x, y, t_0)] \\ + & \int_{t_0}^{t_1} dt \int_c^d dy\, [f^x(b, y, t) - f^x(a, yt)] \\ + & \int_{t_0}^{t_1} dt \int_a^b dx\, [f^y(x, d, t) - f^y(x, c, t)] = 0. \end{aligned} \tag{281}$$

The integration is along the six faces of the rectangular box $(t_0 \le t \le t_1, a \le x \le b, c \le y \le d)$. Each of these six faces is itself two-dimensional.

In the the three-dimensional case, u is measured in units of mass/(length)^3 and the three fluxes in units of mass/(length^2·time). The differential form of the conservation law is

$$u_{,t} + f^x_{,x} + f^y_{,y} + f^z_{,z} = 0. \tag{282}$$

We do not write out the corresponding integral form, but it now has eight integrals, each over three of the four coordinates (x, y, z, t).

In n space dimensions and time, the differential form generalises to

$$u_{,t} + \sum_{i=1}^{n} f^i{}_{,i} = 0. \tag{283}$$

In vector calculus notation, the same equation is

$$u_{,t} + \nabla \cdot \boldsymbol{f} = 0. \tag{284}$$

We can also write the mixed form of our conservation law as

$$\frac{d}{dt} \int_V u\, dV + \int_S \boldsymbol{f} \cdot \boldsymbol{n}\, dS = 0, \tag{285}$$

and the fully integral form as

$$\int_V [u(\boldsymbol{x}, t_1) - u(\boldsymbol{x}, t_0)]\, dV + \int_{t_0}^{t_1} dt \int_S \boldsymbol{f} \cdot \boldsymbol{n}\, dS = 0, \tag{286}$$

where S is the boundary of V. Because they use the divergence theorem, these now hold for any volume V with boundary S (not just rectangular boxes). In any conservation law of the form (286), u is called the **conserved quantity** and $\boldsymbol{f}$ the corresponding **flux**.

4.2 Scalar conservation laws in one space dimension

The general form of a **scalar conservation law** in one space dimension is

$$u_{,t} + [f(u, x, t)]_{,x} = 0. \tag{287}$$

As for other PDEs, "scalar" means that there is only one PDE for one dependent variable u, as opposed to a system of conservation laws (such as the Euler equations). We have seen in Remark 3.30 above that this equation is hyperbolic. The appropriate initial data are

$$u(x, 0) = g(x). \tag{288}$$

Often, $f(u, x, t)$ is actually independent of x and t. (Any ODE or PDE whose coefficients are independent of all the independent variables is called **autonomous**.) In this autonomous case, (287) reduces to

$$u_{,t} + [f(u)]_{,x} = 0, \qquad u(x, 0) = g(x). \tag{289}$$

The function $f(u)$ is sometimes called the **flux function** or **flux law**. In the following we consider only the autonomous case, which includes many physical conservation laws.

Using the chain rule, we can also write (289) as

$$u_{,t} + f'(u)u_{,x} = 0, \tag{290}$$

where $f'(u) := df/du$. This form is explicitly quasilinear, but no longer explicitly in conservation law form. The two forms are equivalent if and only if $u(x, t)$ is at least once differentiable. By contrast, (289) is often used as a shorthand for the integral form (277), which is defined for any solution $u(x, t)$ that is integrable.

We define (in the autonomous case)

$$v(u) := \frac{f(u)}{u} \tag{291}$$

or

$$f(u) = u\, v(u). \tag{292}$$

If we can interpret u as the density of something countable, say particles per length of pipe, then v is the velocity with which these particles move, the **particle velocity**.

4.2.1 The advection equation

The simplest case of a scalar conservation law is the one where v is just constant in space and time, $v = v_0$ and hence $f(u) = v_0 u$. This is called the **advection equation** (here in one space dimension). It can be solved in closed form. It is easy to verify that the unique solution of

$$u_{,t} + (uv_0)_{,x} = 0, \qquad u(x, 0) = g(x), \tag{293}$$

is

$$u(x, t) = g(x - v_0 t). \tag{294}$$

The solution means that if $g(x_0) = u_0$, then $u = u_0$ all along the **characteristic curve** $x(t) = x_0 + v_0 t$. (Compare this with the discussion of the d'Alembert solution in Remark 1.12.)

We see that the advection equation just translates the initial data along characteristic curves, or with velocity v_0.

It is natural to also admit weak solutions of (293), of the form (294) but where the initial data $g(x)$ and hence the solution $u(x,t)$ are discontinuous. Weak solutions of a conservation law are not everywhere differentiable, and hence do not obey the differential form of the conservation law, but they obey its corresponding integral form. Note that any discontinuities in $g(x)$ also propagate along characteristic curves.

4.2.2 Method of characteristics

Now consider the generic autonomous scalar conservation law (287,288), where $f(u)$ is some given function. We can derive a solution either in implicit form or graphically using the **method of characteristics**.

Assume u is constant on the curve in the (x,t)-plane given by $x = \xi(x_0,t)$ that starts at x_0 at $t = 0$. In other words

$$u[\xi(x_0,t),t] = u(x_0,0) = g(x_0), \tag{295}$$

for all $t \geq 0$. We can therefore take $\partial/\partial t$ of this equation (at constant x_0) and obtain

$$0 = \left.\frac{\partial}{\partial t}\right|_{x_0} u[\xi(x_0,t),t] = u_{,t} + u_{,x}\frac{\partial \xi}{\partial t} = -f'(u)u_{,x} + u_{,x}\frac{\partial \xi}{\partial t} = u_{,x}\left(\frac{\partial \xi}{\partial t} - f'(u)\right). \tag{296}$$

In the second equality we have used the chain rule of partial derivatives, and in the third equality we have used (290). If we ignored the parameter x_0, we could also write

$$\frac{d}{dt}u[\xi(t),t)] = u_{,t} + u_{,x}\,\xi'(t) \tag{297}$$

to stress that u has two arguments x and t, but that the *value* $\xi(t)$ of the *formal argument* x also depends on t. To give an intuitive interpretation of (297), think of $u(x,t)$ as the temperature of the air, which depends on both position and time. You move around, carrying a thermometer around with you. Your position is given by $x = \xi(t)$. Then (297) is the rate of change with time of your thermometer reading.

Generically $u_{,x} \neq 0$, and hence we must have

$$\frac{\partial}{\partial t}\xi(x_0,t) = f'[u(x_0,0)] = f'[g(x_0)]. \tag{298}$$

We also have the initial condition

$$\xi(x_0,0) = x_0. \tag{299}$$

Although ξ depends on x_0 and t, this differential equation contains no derivative with respect to x_0, and so it is in effect an ODE in t, for each value of the parameter x_0. Moreover, the right-hand side of (298) does not depend on ξ or t, and so we can simply integrate both sides to obtain

$$\xi(x_0,t) = x_0 + f'[g(x_0)]\,t, \tag{300}$$

where the integration constant is fixed by the initial condition (299). So the **characteristic curve**s for the general autonomous scalar conservation law in one space dimension are the straight lines

$$x = x_0 + f'[g(x_0)]\,t. \tag{301}$$

We now have the solution $u(x,t)$ in implicit form. If we can solve the algebraic equation (301) to find $x_0(x,t)$ in closed form [if this is possible depends on $f'(u)$], we can also write the solution in explicit form as

$$u(x,t) = g[x_0(x,t)]. \tag{302}$$

[Above, we have already discussed the special case of the advection equation, where the characteric curves were $x = x_0 + v_0 t$. This can be solved for $x_0(x,t) = x - v_0 t$, and hence we obtain the closed-form solution (294).]

$f'(u)$ is called the **characteristic velocity**. Note that this is different from the particle velocity $v = f(u)/u$, except for the advection equation, where both are equal to v_0.

4.2.3 Propagation of small disturbances

Assume that $u(x,t)$ is at least once differentiable. Then start from the quasilinear form (290) of a scalar conservation law in one space dimension, and find its linearisation as

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} [(u_0+\epsilon\delta u)_{,t} + f'(u_0+\epsilon\delta u)(u_0+\epsilon\delta u)_{,x}] = \delta u_{,t} + f'(u_0)\delta u_{,x} + f''(u_0)u_{0,x}\,\delta u = 0. \quad (303)$$

If $u_0(x,t)$ is constant, the third term vanishes and we obtain

$$\delta u_{,t} + f'(u_0)\delta u_{,x} = 0, \quad (304)$$

But this is just the advection equation for δu, so small perturbations travel at the characteristic velocity $v_0 = f'(u_0)$.

As an approximation, the third term in (303) can be neglected with respect to the second one if and only if

$$\left|\frac{f''(u_0)}{f'(u_0)}u_{0,x}\right| \ll \left|\frac{\delta u_{,x}}{\delta u}\right|. \quad (305)$$

We can also write this as

$$\left|(\ln f'(u_0))_{,x}\right| \ll \left|(\ln \delta u)_{,x}\right|. \quad (306)$$

In other words, $\ln \delta u$ varies much more rapidly than $\ln f'(u_0)$, or put differently again, the relative change in the perturbation δu is much greater than the relative change in the (approximate) advection speed $v_0 = f'(u_0)$. We then obtain again an approximate advection equation, now with a v_0 that depends slowly on x and t. You can think of the equations governing the atmosphere: there will be changes of pressure and density over large scales, corresponding to the weather, and small but rapidly varying changes corresponding to sound waves.

Considering that sound can convey information, and sound is a (very) small perturbation of gas pressure and velocity, it is said (in a non-rigorous sense) that *information* in hyperbolic equations travels at the characteristic velocity or velocities.

4.3 Weak solutions

4.3.1 Shock formation and Riemann problems

For the conservation law (289) with initial data (288), consider initial data $g(x)$ such that the function $f'[g(x)]$ is an increasing function of x. Then the characteristics fan out from $t = 0$ and never intersect. Hence the initial density profile is stretched out. The solution remains smooth for all $t > 0$.

Now consider initial data $g(x)$ such that the characteristic speed, given by $f'[g(x)]$ is a decreasing function of x (at least for some interval in x). From the chain rule, this is the case if

$$\frac{d}{dx}f'[g(x)] = f''[g(x)]\,g'(x) < 0. \quad (307)$$

Then the characteristics converge from $t = 0$. Intuitively, "particles" at the back move faster than particles in front, and hence catch them up. The initial density profile is compressed and becomes steeper until the solution given by following characteristics becomes multivalued, and no longer make. At this point the physical solution has become discontinuous, and hence non-differentiable, and we do not know how to continue. A **shock** has formed. Shocks develop generically in nonlinear hyperbolic PDEs, of which nonlinear conservation laws are an example.

To understand what happens once a discontinuity has formed, or when a discontinuity is already present in the initial data, we consider the **Riemann problem**, which consists of a conservation law (here, a scalar conservation law) with piecewise constant initial data:

$$u_{,t} + [f(u)]_{,x} = 0, \qquad u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases}. \quad (308)$$

4.3.2 Propagating shock solutions

We have seen that the advection equation admits solutions with a travelling discontinuity. This motivates us to look for a solution to the Riemann problem (308) with a discontinuity that propagates with constant velocity:

$$u(x,t) = \begin{cases} u_L, & x < st \\ u_R, & x > st \end{cases}, \tag{309}$$

for $t \geq 0$, where s is a constant **shock velocity**. The **shock location** is $x = st$. (309) cannot be a solution of the differential form $u_{,t} + [f(u)]_{,x} = 0$ of the conservation law because it is not differentiable. Instead, consider the equivalent integral form

$$\int_a^b [u(x,t_1) - u(x,t_0)]\,dx + \int_{t_0}^{t_1} (f[u(b,t)] - f[u(a,t)])\,dt = 0, \tag{310}$$

which must hold for all rectangles $(t_0 \leq t \leq t_1, a \leq x \leq b)$. On a rectangle where u is simply constant, this is trivial. Instead consider a rectangle that is cut diagonally into two triangles by the propagating shock, for example the rectangle $(0 \leq t \leq \Delta t, 0 \leq x \leq \Delta x)$, where $\Delta x := s\Delta t$ (assuming here that $s > 0$). Then (310), after dividing by Δt, immediately gives

$$s(u_L - u_R) = f(u_L) - f(u_R), \tag{311}$$

the **Rankine-Hugoniot condition**. This is often written as

$$s[u] = [f(u)], \tag{312}$$

where the square brackets denote the jump across the shock. It is also called the **jump condition** at the shock.

By making the rectangle under consideration arbitrarily small, it is easy to show that the jump condition still holds when the solution is once differentiable but not necessarily constant on either side of the shock. Intuitively, if we zoom in on an isolated discontinuity, the smooth derivative becomes less and less relevant, and the discontinuity looks like a step function. Hence, what happens in the neighbourhood of the shock should depend only on the value just to the left and right. The shock still moves with velocity s given by (311), but that velocity will in general depend on time as u_L and u_R change.

For a scalar conservation law, we can obtain a shock solution for arbitrary values of u_L and u_R, and simply read off the shock speed as

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R}. \tag{313}$$

If we now let $u_L = u + \Delta u$ and $u_R = u$ in (313) and take the limit $\Delta u \to 0$, we see that the right-hand side is just the formal definition of a derivative as a limit, so we find that $s \to f'(u)$. Hence weak shocks (shocks with an infinitesimally small jump) propagate with the characteristic velocity.

4.3.3 Rarefaction waves

The initial data in a Riemann problem do not single out any particular length scale, and neither does the solution (309), and so the solution must be **scale-invariant**. Dimensional analysis shows that there is no length scale that can be formed from the conservation law itself and the initial data of the Riemann problem. The solution must be a **similarity solution**, which here means that it must be a function of x/t only. Looking back, we see that this does hold for (309).

There is in fact another type of similarity solution of the Riemann problem that is continuous, although still only a weak solution. We look for a solution of the form

$$u(x,t) = \phi(z), \qquad z := \frac{x}{t}, \quad t > 0. \tag{314}$$

Substituting this into the differential form of the conservation law we find

$$u_{,t} + f'(u)u_{,x} = \phi'(z)\left(-\frac{x}{t^2}\right) + f'[\phi(z)]\phi'(z)\left(\frac{1}{t}\right) = 0, \tag{315}$$

and after multiplying by t,

$$f'[\phi(z)]\phi'(z) = z\phi'(z). \tag{316}$$

We can assume $\phi'(z) \neq 0$ (or else the solution we are constructing would be constant) and divide by it to obtain

$$f'[\phi(z)] = z. \tag{317}$$

As $f(u)$ and hence $f'(u)$ is a known function, this is just an algebraic equation that in principle can be solved for $\phi(z) = (f')^{-1}(z)$. Assume there exist z_L and z_R with $z_L < z_R$ such that $\phi(z_L) = u_L$ and $\phi(z_R) = u_R$. Then a solution to the Riemann problem is given by

$$u(x,t) = \begin{cases} u_L, & x < z_L t, \\ \phi(x/t), & z_L t < x < z_R t \\ u_R, & x > z_R t, \end{cases} \tag{318}$$

where $\phi(z_{L,R}) = u_{L,R}$ and (317) give

$$z_{L,R} = f'(u_{L,R}). \tag{319}$$

So this solution can only exist when

$$f'(u_L) < f'(u_R). \tag{320}$$

This solution is called a **rarefaction wave**. Clearly, it is continuous but not differentiable at $x = z_{L,R}t$. One can show that it is a weak solution of the conservation law.

4.3.4 The Lax condition

A Riemann problem with given left and right states u_L and u_R may well admit more than one shock or rarefaction solution. One then needs additional input to decide which of these possible weak solutions is the physically correct one.

In this course, we are solving Riemann problems only for scalar conservation laws. Then a shock solution exists for any u_L and u_R. On the other hand, a rarefaction solution exists if and only if $z_L < z_R$, that is for

$$f'(u_L) < f'(u_R). \tag{321}$$

So for values of u_L and u_R that obey this condition we have both a shock solution and a rarefaction solution. Which one is correct?

A possible criterion for selecting the correct solution is to demand continuous dependence on the initial data. If we take initial data for the Riemann problem that obey (321), but smooth out the jump from u_L to u_R in the initial data over a distance ℓ, then in the resulting solution the characteristics diverge everywhere, so a shock never forms. In fact, after a time proportional to ℓ the solution looks very similar to the rarefaction solution of the (unsmoothed) original Riemann problem, but very different from the shock solution. So for $f'(u_L) < f'(u_R)$ the correct solution of the Riemann problem is the rarefaction wave, as smoothing it out on an arbitrarily small scale ℓ changes the solution by arbitrarily little. On the other hand, for

$$f'(u_L) > f'(u_R), \tag{322}$$

we have only the shock solution. Suppose we again smooth out the initial data over a length scale l. Then we initially have a smooth solution, but the characteristics converge, and so after a short time proportional to ℓ a shock forms anyway. So here it is the shock solution that is stable against small perturbations and so is the correct one.

Expressing the same statement more geometrically, shocks are the correct solution to the Riemann problem only if characteristics run into them from both sides. This is called the **Lax shock**

condition. Using our expressions for the shock velocities and the characteristic velocities to the left and right, this is

$$f'(u_L) > \frac{f(u_L) - f(u_R)}{u_L - u_R} > f'(u_R). \tag{323}$$

Of course, (323) is necessary for (323) to holds.

We note for completeness that, conversely, (322) implies (323) if and only if either $f''(u) < 0$ or $f''(u) > 0$ for all u. We say the flux function is **convex**. Draw a graph of $f(u)$ to understand this.

4.3.5 A few words on systems and higher dimensions

A system of N conservation laws for N unknowns $\boldsymbol{u} := (u_1, u_2, \dots u_N)$ in one space dimension can be written as

$$\boldsymbol{u}_{,t} + (\boldsymbol{f}(\boldsymbol{u}))_{,x} = 0, \tag{324}$$

where now both $\boldsymbol{u}$ and $\boldsymbol{f}$ are vectors in state space $\mathbb{R}^N$, or if we write out the N components,

$$\boldsymbol{u}_{\alpha,t} + (f_\alpha(\boldsymbol{u}))_{,x} = 0, \quad \alpha = 1 \dots n. \tag{325}$$

Afer using the chain rule of partial derivatives, we obtain the quasilinear form

$$\boldsymbol{u}_{,t} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{u}) \cdot \boldsymbol{u}_{,x} = 0, \tag{326}$$

where now $\partial \boldsymbol{f}/\partial \boldsymbol{u}$ is an $N \times N$ matrix that depends on $\boldsymbol{u}$, and the dot denotes multiplying the column vector $\boldsymbol{u}$ by this matrix. Writing this out in components, we have

$$\boldsymbol{u}_{\alpha,t} + \sum_{\beta=1}^{N} \frac{\partial f_\alpha}{\partial u_\beta}(\boldsymbol{u})\, u_{\beta,x} = 0, \quad \alpha = 1 \dots n. \tag{327}$$

For a system of N conservation laws in one space dimension, the Rankine-Hugoniot condition can be derived in the same way, and is

$$s(\boldsymbol{u}_L - \boldsymbol{u}_R) = \boldsymbol{f}(\boldsymbol{u}_L) - \boldsymbol{f}(\boldsymbol{u}_R), \tag{328}$$

or in components

$$s\,(u_{\alpha,L} - u_{\alpha,R}) = f_\alpha(u_L) - f_\alpha(u_R), \quad \alpha = 1 \dots n. \tag{329}$$

These are now N equations. We can interpret them as $N-1$ constraints between the $2N$ components of $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$, as well as one equation for the shock speed s, so we cannot choose $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$ freely. Yet in a different way, we can think of $\boldsymbol{u}_L$ (say) as fixed, and interpret the N components of the Rankine-Hugoniot condition as one equation for the shock speed s, and $N-1$ constraints on the N components of $\boldsymbol{u}_R$. This means that the possible values of $\boldsymbol{u}_R$, given $\boldsymbol{u}_L$, form a one-dimensional set, namely a continuous curve through the point $\boldsymbol{u}_R = \boldsymbol{u}_L$ (the trivial shock). In fact, because the equations are nonlinear, through every point $\boldsymbol{u}_L$ in state space there are not just one but N curves of possible values of $\boldsymbol{u}_R$ that obey the Rankine-Hugoniot condition. A similar statement holds for rarefaction waves in a system.

Hence constructing weak solutions for systems is more complicated. Roughly speaking, if we look for the solution of a Riemann problem with arbitrarily given left and right states $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$ in a system of N conservation laws, the solution will consist of N "waves" sandwiched between the left and right state and $N-1$ intermediate states. Here a "wave" means a shock, rarefaction wave, or a third kind of similarity solution called a contact discontinuity.

The statement that a weak shock propagates at a characteristic speed is still true for a system, except that a system of N conservation laws now has N characteristic speeds (the eigenvalues of the matrix $\partial \boldsymbol{f}/\partial \boldsymbol{u}$.)

In more space dimensions, say three, an isolated shock between constant states is planar, so we can simply orient our coordinate system so that the shock propagates in the x-direction, and nothing depends on y and z. A similar statement holds for rarefaction waves. So there is nothing fundamentally different from one space dimension, but of course solutions can become extremely complicated in practice.

4.4 Example: Traffic flow

Look again at (292) but interpret it as the conservation of cars on a road, where u is now measured in cars/length, and v is the velocity of the traffic flow, measured of course in length/time. In this context, (292) is called the **traffic flow equation**.

Now let us look a particular velocity law for traffic flow. There is maximum density $u_{\max}$ when cars are bumper to bumper, and there is a maximum velocity $v_{\max}$ given by the speed limit. On an open road, cars will go at the speed limit, but in dense traffic they will slow down until they reach the maximum density at zero speed – a traffic jam. For simplicity, we assume $v(u)$ to be the linear function defined by these two points, namely

$$v(u) = v_{\max}\left(1 - \frac{u}{u_{\max}}\right), \tag{330}$$

and hence

$$f(u) = uv_{\max}\left(1 - \frac{u}{u_{\max}}\right). \tag{331}$$

The characteristic velocities are therefore

$$f'(u) = v_{\max}\left(1 - 2\frac{u}{u_{\max}}\right). \tag{332}$$

Hence $f'(u)$ is a decreasing function. Finally, the shock velocity is given by

$$\begin{aligned} s &= \frac{f(u_L) - f(u_R)}{u_L - u_R} = v_{\max}\frac{u_L\left(1 - \frac{u_L}{u_{\max}}\right) - u_R\left(1 - \frac{u_R}{u_{\max}}\right)}{u_L - u_R} \\ &= v_{\max}\frac{(u_L - u_R) - \frac{u_L^2 - u_R^2}{u_{\max}}}{u_L - u_R} = v_{\max}\left(1 - \frac{u_L + u_R}{u_{\max}}\right). \end{aligned} \tag{333}$$

Consider now the evolution of two kinds of initial data:

1) Assume that $g(x)$ is a decreasing function. From (332) we see that $f'(u)$ is a decreasing function of u. Hence $f'(g(x))$ is an increasing function of x. In other words, the characteristic velocity in the initial data increase with x. Hence the characteristics fan out from $t = 0$ and never intersect. Physically, the cars in front are in less dense traffic and hence move faster. Hence the initial density profile is stretched out. The solution remains smooth for all $t > 0$.

2) $g(x)$ is an increasing function. Then the characteristics converge from $t = 0$. Physically, the cars in front are in denser traffic and hence move more slowly, allowing the cars behind to catch up. Hence the initial density profile is compressed and becomes steeper, until a moving shock forms in the traffic flow at which each driver suddenly hits the brakes.

4.5 Exercises

28. **Homework 13:** (Very short) In n space dimensions, the general form of a scalar conservation law is

$$u_{,t} + \sum_{i=1}^{n} \left[f^i(u)\right]_{,i} = 0, \tag{334}$$

and the total mass is $m = \int u\, d^n x$. What are the dimensions of u and f^i?

29. **Homework 14:** a) Use separation of variables to solve the Cauchy problem for the advection equation on the line,

$$\begin{aligned} u_{,t} + v_0 u_{,x} &= 0, \quad -\infty < x < \infty, \quad t \geq 0, && (335)\\ u &\to 0 \quad \text{as} \quad x \to \pm\infty, && (336)\\ u(x,0) &= g(x) && (337) \end{aligned}$$

and hence show that the solution $u(x,t) = g(x - v_0 t)$. (Hint: you need a Fourier transform).

b) Now do the same using the method of characteristics.

30. **Homework 15:** Use the method of characteristics to show that the solution of the Burgers equation with linear initial data,

$$u_{,t} + \left(\frac{1}{2}u^2\right)_{,x} = 0, \tag{338}$$

$$u(x,0) = ax, \tag{339}$$

is

$$u(x,t) = \frac{ax}{1+at}. \tag{340}$$

For what a does a shock form at some $t > 0$, and what value of t is that?

31. **Homework 16:** (Very short) Find the shock solution of the Riemann problem to the Burgers equation

$$u_{,t} + \left(\frac{u^2}{2}\right)_{,x} = 0, \qquad u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases}. \tag{341}$$

32. **Homework 17:** (Very short) Find the rarefaction wave solution to the Riemann problem for the Burgers equation, and find when it actually exists. Also find when the shock solution is the correct solution to the Riemann problem instead.

33. **Homework 18:** (Longish, but important) For the traffic flow Riemann problem

$$u_{,t} + \left(v_{\max}\left(u - \frac{u^2}{u_{\max}}\right)\right)_{,x} = 0, \qquad u(x,0) = \begin{cases} u_L = au_{\max}, & x < 0 \\ u_R = bu_{\max}, & x > 0 \end{cases}, \tag{342}$$

find the shock or rarefaction solution, as appropriate, and in either case the characteristics $\xi(x_0,t)$. Let $v_{\max} = 60$mph, and consider the two cases where $a = 0.3$, $b = 0.9$, and $a = 0.9$, $b = 0.3$.

34. For the solution of the traffic flow example, also find the car trajectories $\tilde{\xi}(x_0,t)$.

35. Consider the PDE problem

$$u_{,t} + [f(u)]_{,x} = 0, \quad -\infty \le x \le \infty, \quad t \ge 0, \tag{343}$$

$$u(x,0) = g(x) \tag{344}$$

Determine if these data will form a shock, and if so compute the time t_s when the shock first forms.

Hint: a) Recall the method of characteristics for solving this problem graphically, and recall that a shock forms when two characteristics cross. b) By drawing a picture, or otherwise, convince yourself that the first characteristics that cross will be two neighbouring characteristics. In other words, you cannot have characteristics starting at x_{01} and x_{02} crossing without some characteristics starting from intermediate values of x_0 crossing first, or at the same time. c) Now look at characteristics starting from x_0 and $x_0 + h$, and find out where and when they cross, working to leading order in h. You will find that to leading order the crossing time $t_s = t_s(x_0, h)$ does not depend on h. d) Now find the smallest value of $t_s(x_0)$ and you are done.

36. Solve the PDE problem

$$u_{,t} + \left(\frac{1}{2}u^2\right)_{,x} = 0, \quad -\infty \le x \le \infty, \quad t \ge 0, \tag{345}$$

$$u(x,0) = \begin{cases} u_L, & x < 0, \\ u_L + \frac{u_R - u_L}{L}x, & 0 < x < L, \\ u_R, & x > L. \end{cases} \tag{346}$$

(Burgers' equation with continuous, piecewise linear initial data.)

Hint: You will need to solve this problem separately for the two cases $u_L < u_R$ and $u_L > u_R$, as the solutions are qualitatively different. Sketch the initial data. Sketch some characteristics in the (t, r)-plane. Recall shocks and rarefaction waves. Recall the solution of another homework problem where $u(x, 0) = ax$ for $-\infty \leq x$ (what is a here?). Try to glue together a solution from these ingredients in different regions of the (x, t)-plane.

37. In this system we consider the system of two conservation laws for the two variables $\boldsymbol{u} := (u, v)$,

$$u_{,t} - v_{,x} = 0, \tag{347}$$
$$v_{,t} + (P(u))_{,x} = 0 \tag{348}$$

where $P(u)$ is a given function that obeys $P'(u) < 0$. We first find the *subset* of all Riemann problems that admit a solution with a single shock. We then use this to construct a solution with two shocks for the *general* Riemann problem. (We ignore the existence of solutions with two rarefaction waves, or one shock and one rarefaction wave, and the question which of these is the correct solution.)

a) In this part of the question, we find all the single-shock solutions

$$\boldsymbol{u}(x,t) = \begin{cases} \boldsymbol{u}_L, & x < st, \\ \boldsymbol{u}_R, & x > st, \end{cases} \tag{349}$$

for restricted values of $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$. Write down the two components of the Rankine-Hugoniot condition in terms of the left and right states $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$ and shock speed s. Show that $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$ must be obey the restriction

$$(v_L - v_R) = \mp\sqrt{-\frac{P(u_L) - P(u_R)}{u_L - u_R}}(u_L - u_R), \tag{350}$$

for either the upper or lower sign and find the corresponding two shock speeds $s_\pm$ in terms of u_L and u_R. Show that what is inside the square root in (350) is strictly positive for any u_L and u_R, and find the limit of the shock speeds for small shocks, that is, as $\boldsymbol{u}_L \to \boldsymbol{u}_R$.

b) We now find a solution with two shocks of the general Riemann problem, that is

$$\boldsymbol{u}(x,t) = \begin{cases} \boldsymbol{u}_L, & x < s_L t, \\ \boldsymbol{u}_*, & s_L t < x < s_R t, \\ \boldsymbol{u}_R, & x > s_R t, \end{cases} \tag{351}$$

for arbitrary $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$. Sketch the regions in the (x, t)-plane where $\boldsymbol{u}$ takes each value. For given $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$, write down four equations (from using the Rankine-Hugoniot condition separately for each shock) that can be used in principle to determine u_*, v_*, s_L and s_R, to actually find a solution of the conservation law. Explain your choice of signs, and check that your sketch has the correct signs. Then find a single equation for u_* in terms of $\boldsymbol{u}_L$ and $\boldsymbol{u}_R$ only (you do not need to solve this equation yet), and explain how you would complete the solution.

c) Assume now the specific flux function

$$P(u) = -4u, \tag{352}$$

and the specific numerical values

$$\boldsymbol{u}_L := (u_L, v_L) = (1, 1), \quad \boldsymbol{u}_R := (u_R, v_R) = (4, 3). \tag{353}$$

Plot accurately(!), in the (u, v)-plane, the two curves that link our $\boldsymbol{u}_L$ to all possible $\boldsymbol{u}_*$, and the two curves that link our $\boldsymbol{u}_R$ to all possible $\boldsymbol{u}_*$. Then find the correct $\boldsymbol{u}_*$ graphically, as the intersection of one curve from each pair. Find the corresponding s_L and s_R. There are

two intersections, but only one is the correct intermediate state $\boldsymbol{u}_*$ for solving the Riemann problem: explain your choice. For what other Riemann problem would we need the other intersection $\boldsymbol{u}_*$?

Verify your graphical solution by solving the Rankine-Hugoniot condition for u_*, v_*, s_L and s_R algebraically, as you planned in part b).

d) Repeat part c) for the same left and right state, but now for the flux function

$$P(u) = -e^u. \tag{354}$$

Use Mathematica, Maple, python, Matlab, or some other software to plot the four shock curves (in a single figure, chosen so the two intersections are clearly visible, and with axes clearly labelled). [Hint: use parametric plotting with parameter u_*.] Compute (numerically) the intermediate state $\boldsymbol{u}_*$ and two shock speeds s_L and s_R. [Hint: use FindRoot in Mathematica or something similar in other languages, or write your own Newton solver. Check against the plot that your numbers make sense.]

(This question is adapted from problem 13.7 of R. J. LeVeque, "Finite volume methods for hyperbolic problems".)

5 Elementary generalised functions

5.1 Test functions

Definition 5.1. A function $\phi(x)$ is a **test function** if it has the following properties:

- $\phi(x)$ and all its derivatives exist and are continuous at all points $-\infty < x < \infty$ (ϕ is smooth);
- the integrals $\int_{-\infty}^{\infty}$ of $\phi(x)$ and all its derivatives exist and are finite.

Note that since the integrals of a test function and all its derivatives exist, the test function and all its derivatives must vanish as $x \to \pm\infty$.

A simple example of a test function is

$$\phi(x) = e^{-x^2}, \tag{355}$$

which is analytic. Another example is the function

$$f(x) = \begin{cases} 0, & x \le a \\ e^{-\frac{1}{(x-a)(b-x)}}, & a < x < b \\ 0, & x \ge b \end{cases} \tag{356}$$

which is smooth but vanishes outside the interval (a, b) (and is not analytic at $x = a$ and $x = b$).

5.2 The δ-function

Definition 5.2. A **generalised function** G (also called a **distribution**) is a linear map from test functions to real numbers. That is, it assigns to each test function ϕ a number $G[\phi]$. Two generalised functions F and G are equal if $F[\phi] = G[\phi]$ for all test functions ϕ.

This may seem a little abstract, but in practice we can think of generalised functions as "functions" that are only defined when integrated over a test function. We can then write

$$G[\phi] := \int_{-\infty}^{\infty} G(x)\phi(x)\,dx. \tag{357}$$

In the following, we only use this integral notation, and never $G[\phi]$.

The most important generalised function is the δ-function, which in spite of its name is not a function, but a generalised function. Intuitively, one can think of it as a "function" that is only defined under an integral.

Definition 5.3. The **δ-function** is the generalised function defined by

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx := \phi(0) \tag{358}$$

for every test function $\phi(x)$.

Remark 5.4. Intuitively, we can think of the δ-function as an ordinary function with the properties

$$\delta(x) = 0 \ \text{ if } \ x \neq 0, \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = 1. \tag{359}$$

Note that we do not, and cannot, assign a value to $\delta(0)$.

Remark 5.5. We can also define the δ-function as the limit of various sequences of regular functions that obey

$$\lim_{\epsilon\to 0} f_\epsilon(x) = 0, \quad x \neq 0 \tag{360}$$

and

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f_\epsilon(x) = 1, \tag{361}$$

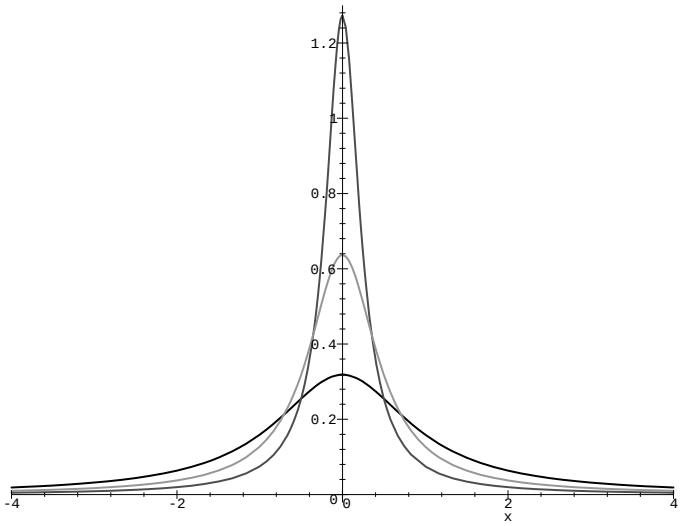

Figure 2: Sequence of functions converging to the δ-function.

An example is

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}, \tag{362}$$

see Fig. 2. This definition of the δ-function as a limit has a physical interpretation in terms of a very large force acting over a very short time, while conveying a finite momentum. In applications the δ-function can be used to represent an impulse, eg. when a string is hit with a hammer.

Remark 5.6. More generally, if $f(x)$ is any function such that $\int_{-\infty}^{\infty} f(x)dx = 1$, which implies that $f(x) \to 0$ as $|x| \to 0$, then

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) = \delta(x). \tag{363}$$

Say $\phi(x)$ is a test function. Then, with the change of variable $y = x/\epsilon$,

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) \phi(x) dx = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f(y)\phi(\epsilon y) dy = \int_{-\infty}^{\infty} f(y)\phi(0) dy = \phi(0) \int_{-\infty}^{\infty} f(y) dy = \phi(0). \tag{364}$$

5.3 The Heaviside and signum functions

Definition 5.7. The **Heaviside function** is the generalised function defined by

$$\int_{-\infty}^{\infty} H(x)\phi(x)\, dx := \int_{0}^{\infty} \phi(x)\, dx \tag{365}$$

for all test functions $\phi(x)$.

Remark 5.8. It is easy to see that this is equivalent to

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \tag{366}$$

but note that we do not need to assign a value to $H(0)$.

Remark 5.9. We have

$$\int_{-\infty}^{x} \delta(y)\, dy = H(x). \tag{367}$$

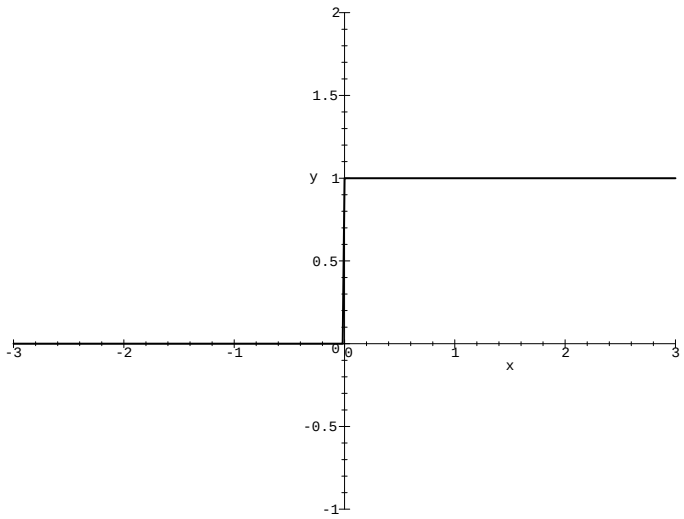

Figure 3: The Heaviside function.

Because $\delta(x) = 0$ for $x \neq 0$, it only matters if $y = 0$ is part of the integration domain $-\infty < y < x$. Note also that the left-hand side of this equation is not defined for $x = 0$, and hence for consistency we cannot give $H(x)$ a value at $x = 0$ either.

In applications the Heaviside function is often used as a "switch". For example, it can be used to mathematically model an electrical circuit where the power is switched on at a specific moment in time.

Definition 5.10. The **signum function** $\text{sgn}(x)$ is defined by

$$\text{sgn}(x) = H(x) - H(-x). \tag{368}$$

Hence we have

$$\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}. \tag{369}$$

Clearly $\text{sgn}(x)$ is just the sign of the number x, but $\text{sgn}(0)$ is not defined. Considering $\text{sgn}(x)$ as a generalised function allows us to show that

$$\text{sgn}(x) = \frac{d\,|x|}{dx} \tag{370}$$

and that

$$\frac{d}{dx}\text{sgn}x = 2\delta(x). \tag{371}$$

5.4 Generalised functions and derivatives

We are going to use generalised functions to solve inhomogeneous differential equations. In manipulations we will often need the derivative, $G'(x)$, of a generalised function, $G(x)$.

Definition 5.11. The derivative $G'(x)$ of the generalised function $G(x)$ is defined by

$$\int_{-\infty}^{\infty} G'(x)\phi(x)\,dx := -\int_{-\infty}^{\infty} G(x)\phi'(x)\,dx \tag{372}$$

for all test functions $\phi(x)$.

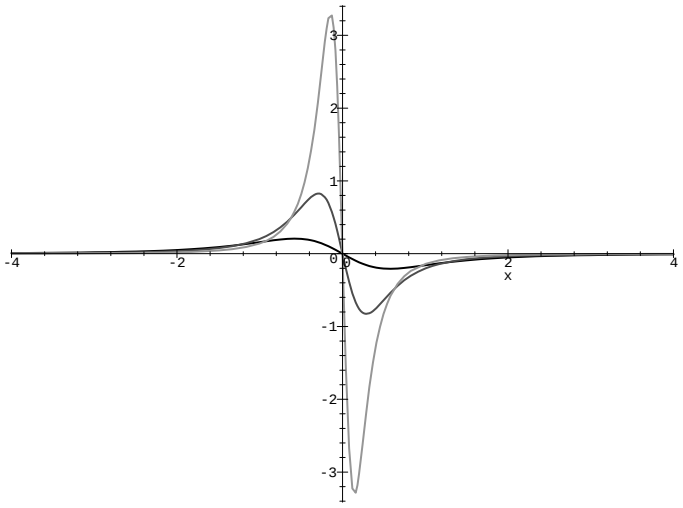

Figure 4: Sequence of functions converging to the derivative of the δ-function.

Remark 5.12. This definition is essentially integration by parts (since test functions vanish at $\pm\infty$). The right-hand-side of this equation is always known since that is how $G(x)$ is defined, and if ϕ is a test function so is ϕ'.

Example 5.13. In this sense we have

$$H'(x) = \delta(x), \tag{373}$$

because

$$\int_{-\infty}^{\infty} H'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} H(x)\phi'(x)\,dx = -\int_{0}^{\infty} \phi'(x)\,dx = -\phi(x)|_0^{\infty} = \phi(0)\ . \tag{374}$$

Remark 5.14. The derivative of the δ-function, $\delta'(x)$, is defined by

$$\int_{-\infty}^{\infty} \delta'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} \delta(x)\phi'(x)\,dx = -\phi'(0) \tag{375}$$

for all test functions $\phi(x)$.

We can also define δ' as the limit of a sequence of functions such as

$$\delta'(x) = \lim_{\epsilon\to 0} \frac{d}{dx}\frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + x^2} = \lim_{\epsilon\to 0} -\frac{2}{\pi}\frac{\epsilon x}{\left(\epsilon^2 + x^2\right)^2} \tag{376}$$

as shown in Fig. 4.

5.5 Properties of the δ-function

All of the following properties are a consequence of the definition of the δ-function and the standard properties of integrals. Recall that to prove $F(x) = G(x)$ for two generalised functions F and G, we really have to prove that $\int_{-\infty}^{\infty} F(x)\phi(x)\,dx = \int_{-\infty}^{\infty} G(x)\phi(x)\,dx$ for any test function $\phi(x)$. Typically, this will involve a change of variable, and working from both sides towards a number.

1. $\int_{-\infty}^{\infty} \phi(x)\delta(x)\,dx = \phi(0)$
2. $\int_{a}^{b} \phi(x)\delta(x)\,dx = \phi(0)$ for $a < 0 < b$

3. $\int_{-\infty}^{\infty} \phi(x)\delta(x-a)\,dx = \phi(a)$
4. $\delta(-x) = \delta(x)$
5. $\delta(ax) = \delta(x)/\left|a\right|$
6. $\delta(a^2 - x^2) = \left[\delta(x-a) + \delta(x+a)\right]/(2|a|)$
7. $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x^i)|}$, where the x^i are the simple zeros of $f(x)$
8. $x\delta(x) = 0$
9. $g(x)\delta(x) = g(0)\delta(x)$ provided $g(x)$ is continuous and $g(0)$ exists.
10. $H'(x) = \delta(x)$
11. $H(x) = \int_{-\infty}^{x} \delta(y)\,dy$
12. $\int_{-\infty}^{\infty} \delta'(x)\phi(x)\,dx = -\phi'(0)$
13. $(g(x)\delta(x))' = g(0)\delta'(x)$ (the naive product rule does *not* apply)

Example 5.15. Let us take a closer look at property 6. Consider $\delta(a^2 - x^2)$ as a generalised function and let it operate on a test function $\phi(x)$, i.e. evaluate

$$I = \int_{-\infty}^{\infty} \phi(x)\delta(a^2 - x^2)dx. \tag{377}$$

Change variables to $y = a^2 - x^2$ and use

$$\begin{aligned} x &= \sqrt{a^2 - y} \quad \text{for } x > 0, &(378)\\ x &= -\sqrt{a^2 - y} \quad \text{for } x < 0. &(379) \end{aligned}$$

Then we get

$$\begin{aligned} I &= \int_{x=0}^{x=\infty} \phi(x)\delta(y)\left[-\frac{dy}{2\sqrt{a^2-y}}\right] + \int_{x=-\infty}^{x=0} \phi(x)\delta(y)\left[\frac{dy}{2\sqrt{a^2-y}}\right] = \\ &= \int_{-\infty}^{a^2} \frac{\phi(\sqrt{a^2-y})\delta(y)dy}{2\sqrt{a^2-y}} + \int_{-\infty}^{a^2} \frac{\phi(-\sqrt{a^2-y})\delta(y)dy}{2\sqrt{a^2-y}} = \\ &= \frac{1}{2|a|}\left[\phi(|a|) + \phi(-|a|)\right] = \frac{1}{2|a|}\int_{-\infty}^{\infty} \left[\delta(x-a) + \delta(x+a)\right]\phi(x)dx. \end{aligned} \tag{380}$$

Example 5.16. To show property 11, let $\Phi(x)$ be any function such that $\Phi'(x) = \phi(x)$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(x)\left(\int_{-\infty^x} \delta(y)\,dy\right)dx &= \left[\Phi(x)\int_{-\infty^x} \delta(y)\,dy\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Phi(x)\delta(x)\,dx \\ &= \Phi(\infty)\int_{-\infty}^{\infty} \delta(y)\,dy - \Phi(-\infty)\int_{-\infty}^{-\infty} \delta(y)\,dy - \Phi(0) \\ &= \Phi(\infty) - \Phi(0) = \int_0^{\infty} \phi(x)\,dx = -\int_{-\infty}^{\infty} \phi(x)H(x)\,dx, \end{aligned} \tag{381}$$

for all test functions ϕ.

5.6 Exercises

38. **Homework 19:** Show that $\lim_{\epsilon \to 0} f_\epsilon(x) = \delta(x)$, where

$$f_\epsilon(x) := \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}. \tag{382}$$

39. Prove that, in the sense of generalised functions, $\delta(-x) = \delta(x)$.

40. Prove that

$$\int_{-\infty}^{\infty} \delta(x - y) f(x)\, dx = f(y). \tag{383}$$

41. Show that $\int_a^b \phi(x)\delta(x)\, dx = \phi(0)$ for $a < 0 < b$, $\int_a^b \phi(x)\delta(x)\, dx = -\phi(0)$ for $b < 0 < a$, otherwise zero, or undefined if $a = 0$ or $b = 0$

6 Green's functions for ODEs

In this Section we introduce the concept of Green's function, which can be used to solve inhomogeneous differential equations. Since the technique readily generalises from ODEs to PDEs we will first consider the simpler case of ODEs. We begin with the general definition, and then consider some examples.

Definition 6.1. A Green's function for the inhomogeneous linear ODE

$$L\,y(t) = f(t), \tag{384}$$

where $L = L(t, d/dt)$ is any homogeneous linear ordinary differential operator, is a generalised function $G(t, s)$ of two variables that satisfies

$$L_t\,G(t, s) = \delta(t - s), \tag{385}$$

where L_t signifies that the differential operator L acts on the variable t, not s, that is $L_t := L(t, \partial/\partial t)$.

Remark 6.2. The reason for this definition is this: if we put

$$y(t) = \int_{-\infty}^{\infty} G(t, s) f(s)\, ds, \tag{386}$$

we find that

$$Ly(t) = \int_{-\infty}^{\infty} L_t G(t, s) f(s)\, ds = \int_{-\infty}^{\infty} \delta(t - s) f(s)\, ds = f(t), \tag{387}$$

so that (386) is a solution of (384).

Remark 6.3. The above definition of a Green's function becomes unique only when we complement it with appropriate boundary conditions, either on the ODE, or equivalently on the Greens's function. Clearly, two different Green's functions for the same problem must differ by a solution of the homogeneous problem $L_t G = 0$.

6.1 A simple example: first-order linear ODE with constant coefficients

Example 6.4. A Green's function for the first-order linear ODE with constant coefficients,

$$\dot{y} + ay = f(t) \tag{388}$$

is a generalised function $G(t, s)$ of two variables t and s that satisfies

$$G_{,t} + aG = \delta(t - s). \tag{389}$$

To find the Green's function $G(t, s)$ for (388), we note that for $t \neq s$ we have

$$G_{,t} + aG = 0, \tag{390}$$

and hence we have

$$G(t, s) = \begin{cases} A(s)e^{-at} & \text{for } t < s \\ B(s)e^{-at} & \text{for } t > s, \end{cases} \tag{391}$$

where $A(s)$ and $B(s)$ are integration constants. (They are functions of s, rather than true constants because, although $G(t, s)$ obeys an ODE in the single variable t, it also depends on the parameter s, and so these integration "constants" can depend on s. You may remember seeing something similar when you learned how to solve exact PDEs by integration.)

We now recall that a δ-function is the derivative of the Heaviside function, i.e. a jump discontinuity of unit size. Thus it will be enough to make $G(t, s)$ jump by one as we go across $t = s$. Thus we want

$$\lim_{t \to s_+} G(t, s) = \lim_{t \to s_-} G(t, s) + 1, \tag{392}$$

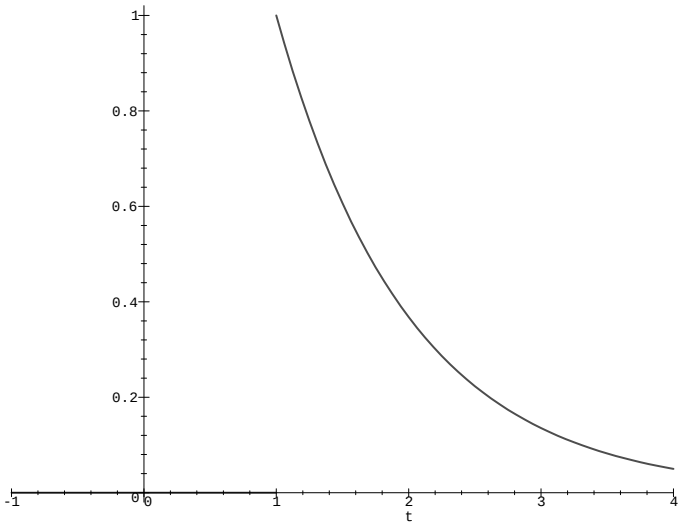

Figure 5: The Green's function $H(t-1)e^{-(1-t)}$.

or

$$B(s)e^{-as} = A(s)e^{-as} + 1, \tag{393}$$

from which it follows that

$$B(s) = A(s) + e^{as}. \tag{394}$$

Thus we have

$$G(t,s) = \begin{cases} A(s)e^{-at} & t < s \\ A(s)e^{-at} + e^{a(s-t)} & t > s, \end{cases} \tag{395}$$

which can be written as

$$G(t,s) = A(s)e^{-at} + H(t-s)e^{-a(t-s)}. \tag{396}$$

Then we see that (386) implies

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} G(t,s)f(s)\,ds \\ &= e^{-at}\int_{-\infty}^{\infty} A(s)f(s)\,ds + \int_{-\infty}^{\infty} H(t-s)e^{-a(t-s)}f(s)\,ds \\ &= Ce^{-at} + \int_{-\infty}^{t} e^{-a(t-s)}f(s)\,ds. \end{aligned} \tag{397}$$

The term Ce^{-at} is of course the general solution of (388) with $f(t) = 0$, or in other words, the complementary function. Like an ordinary solution to this ODE, the Green's function needs boundary conditions to be uniquely defined.

If we choose $A(s) = 0$, and hence $C = 0$, then

$$G(t,s) = H(t-s)e^{-a(t-s)} = \begin{cases} 0, & t < s \\ e^{-a(t-s)}, & t > s \end{cases}. \tag{398}$$

This Green's function represents the reaction of the system to a unit impulse at time $t = s$ where the system is at rest prior to the impulse; see Figs. 5 and 6. It is called the **causal Green's function**, defined by the property that

$$G(t,s) = 0 \quad \text{for} \quad t < s. \tag{399}$$

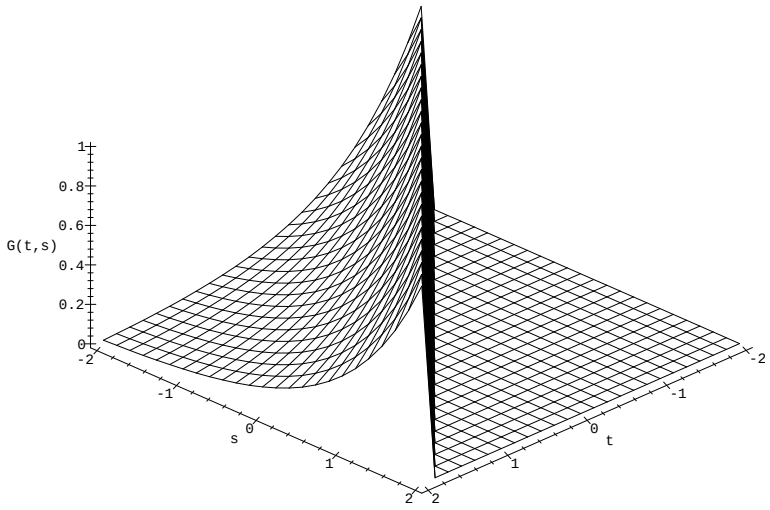

Figure 6: The Green's function $H(t-s)e^{-(t-s)}$ for $-2 < s < 2$, $-2 < t < 2$.

The solution (386) with (398) is

$$y(t) = \int_{-\infty}^{t} e^{-a(t-s)} f(s)\, ds. \tag{400}$$

It is called the **causal solution** because it depends only on $f(s)$ for $s < t$, that is, it depends only on the values of $f(s)$ before the present time. Thus this solution can be said to be *caused* by the driving force $f(t)$. This is in contrast to the general case (386), where $y(t)$ can depend on the values of $f(s)$ for both $s < t$ (i.e., values that have already occurred) and $t < s$ (i.e., values that have yet to happen). One is typically mainly interested in the causal solution to a physical problem. In general, we obtain such solutions by imposing $G(t,s) = 0$ for $t < s$. The factor $H(t-s)$ in (398) obviously makes sure of that.

6.2 Another example: the harmonic oscillator

Definition 6.5. The causal Green's function for the harmonic oscillator problem

$$\ddot{y} + \omega^2 y = f(t) \tag{401}$$

is a function $G(t,s)$ that satisfies

$$G_{,tt} + \omega^2 G = \delta(t-s). \tag{402}$$

and $G = 0$ for $t < s$.

We take $G(t,s) = 0$ for $t < s$. For $t > s$ we have

$$G_{,tt} + \omega^2 G = 0, \tag{403}$$

so that

$$G(t,s) = A(s)e^{i\omega t} + B(s)e^{-i\omega t}. \tag{404}$$

The δ-function in (402) must come from the term $G_{,tt}$, which implies that $G_{,t}$ must jump by one as we go across $t = s$. This implies that $G(t,s)$ must be continuous across $t = s$. If $G(t,s)$ was discontinuous then $G_{,t}$ would have a δ-function and $G_{,tt}$ would contain $\delta'(t)$.

This can be derived more formally in the following way. Assume that the Green's function is continuous, but that its derivative may have discontinuities. Then integrate (402) from $s-\epsilon$ to $s+\epsilon$. This gives

$$\int_{s-\epsilon}^{s+\epsilon} \left[G_{,tt} + \omega^2 G\right]\, dt = G_{,t}|_{s+\epsilon} - G_{,t}|_{s-\epsilon} = \int_{s-\epsilon}^{s+\epsilon} \delta(t-s)\, dt = 1. \tag{405}$$

(In the limit $\epsilon \to 0$, the integral over $\omega^2 G$ vanishes). This method can be used also for more complicated equations.

As we let $t \to s_+$ we have

$$\lim_{t \to s_+} G(t,s) = A(s)e^{i\omega s} + B(s)e^{-i\omega s}, \tag{406}$$

which must be zero since $G(t,s) = 0$ for $t < s$ and we want $G(t,s)$ to be continuous. Thus

$$A(s)e^{i\omega s} + B(s)e^{-i\omega s} = 0. \tag{407}$$

We have $G_{,t} = 0$ for $t < s$ and we want $G_{,t}$ to jump by one as we go across $t = s$, so we must have

$$\lim_{t \to s_+} G_{,t} = i\omega A(s)e^{i\omega s} - i\omega B(s)e^{-i\omega s} = 1. \tag{408}$$

Solving for $A(s)$ and $B(s)$ we find that

$$A(s) = \frac{1}{2\omega i}e^{-i\omega s}, \quad B(s) = -\frac{1}{2\omega i}e^{i\omega s}, \tag{409}$$

and hence (see Fig. 7)

$$\begin{aligned} G(t,s) &= \begin{cases} 0, & t < s \\ \frac{1}{\omega}\sin\omega(t-s), & t > s \end{cases} \\ &= \frac{1}{\omega}H(t-s)\sin\omega(t-s). \end{aligned} \tag{410}$$

Thus, the causal solution of (401) is

$$y(t) = \frac{1}{\omega}\int_{-\infty}^{t} f(s)\sin\omega(t-s)\,ds. \tag{411}$$

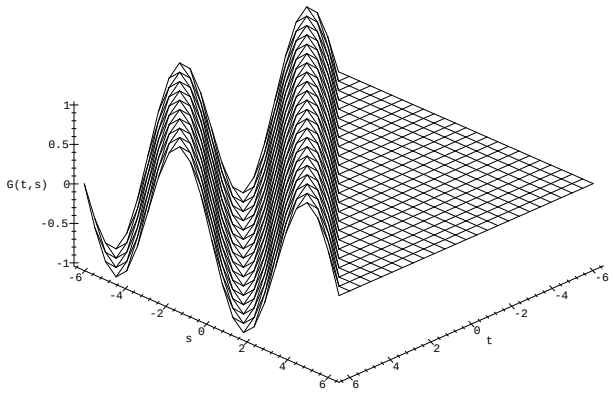

Figure 7: The Green's function $H(t-s)e^{-(t-st)}$ as a function of s and t.

6.3 The general second order linear ODE with constant coefficients

The procedure from the previous Section works for more general equations. Take the general linear second order ODE with constant coefficients,

$$\ddot{y} + \alpha\dot{y} + \beta y = f(t). \tag{412}$$

The causal Green's function is the solution of

$$G_{,tt} + \alpha G_{,t} + \beta G = \delta(t-s), \tag{413}$$

which satisfies $G(t,s) = 0$ for $t < s$. Finding $G(t,s)$ involves solving

$$G_{,tt} + \alpha G_{,t} + \beta G = 0 \tag{414}$$

for $t > s$, and the solution is

$$G(t,s) = A(s)e^{\lambda_1 t} + B(s)e^{\lambda_2 t}, \tag{415}$$

where λ_1 and λ_2 are the roots of the quadratic equation

$$\lambda^2 + \alpha\lambda + \beta = 0. \tag{416}$$

Thus $G(t,s)$ may be written as

$$G(t,s) = \begin{cases} 0 & t < s \\ A(s)e^{\lambda_1 t} + B(s)e^{\lambda_2 t} & t > s. \end{cases} \tag{417}$$

(Here we have assumed that $\lambda_1 \neq \lambda_2$.) In order to obtain the term $\delta(t-s)$ we need $G(t,s)$ to be continuous at $t = s$ and $G_{,t}(t,s)$ to have a jump of magnitude one. Thus we find that

$$A(s)e^{\lambda_1 s} + B(s)e^{\lambda_2 s} = 0 \tag{418}$$

from the continuity of $G(t,s)$ at $t = s$, and that

$$\lambda_1 A(s)e^{\lambda_1 s} + \lambda_2 B(s)e^{\lambda_2 s} = 1 \tag{419}$$

from the jump in $G_{,t}$. These are two equations for the functions $A(s)$ and $B(s)$ which we can solve to find them. Once we have done this, the Green's function is

$$G(t,s) = H(t-s)\Big[A(s)e^{\lambda_1 t} + B(s)e^{\lambda_2 t}\Big]. \tag{420}$$

6.4 Initial-value problems

Rather than starting at $t = -\infty$ and looking for the causal solution, we often want to solve the initial value problem. We will illustrate this with the example of the general second-order linear ODE (not restricting to constant coefficients). It should be clear afterwards how to apply the same method to a first-order or higher-order linear ODE.

So consider the initial value problem

$$\ddot{y} + \alpha(t)\dot{y} + \beta(t)y = f(t), \qquad y(0) = A, \qquad \dot{y}(0) = B \tag{421}$$

for $t > 0$. We can extend the Green's function methods described above to deal with this situation. We define a new function

$$z(t) = H(t)y(t), \tag{422}$$

so that $z(t)$ equals $y(t)$ for $t > 0$ and is zero for $t < 0$. Then

$$\dot{z}(t) = y(0)\delta(t) + \dot{y}(t)H(t) = A\delta(t) + \dot{y}(t)H(t) \tag{423}$$

and

$$\ddot{z}(t) = A\delta'(t) + \dot{y}(t)\delta(t) + \ddot{y}(t)H(t) = A\delta'(t) + B\delta(t) + \ddot{y}(t)H(t), \tag{424}$$

and thus

$$\ddot{z} + \alpha(t)\dot{z} + \beta(t)z = A\delta'(t) + [\alpha(0)A + B]\delta(t) + H(t)f(t) =: \tilde{f}(t). \tag{425}$$

Now assume that $G(t,s)$ is the causal Green's function, so that the causal solution of (425) is

$$\begin{aligned} z(t) &= \int_{-\infty}^{t} G(t,s)\tilde{f}(s)\,ds \\ &= \int_{-\infty}^{t} G(t,s)\left\{A\delta'(s) + [\alpha(0)A + B]\delta(s) + H(s)f(s)\right\}\,ds \\ &= H(t)\Big\{-AG_{,s}(t,0) + [\alpha(0)A + B]G(t,0) + \int_0^t G(t,s)f(s)\,ds\Big\}. \end{aligned} \tag{426}$$

The factor $H(t)$ multiplies the first two terms because $\delta(s)$ and $\delta'(s)$ both vanish for $s \neq 0$, and hence for $t < 0$ they vanish everywhere inside the integration range. Similarly, $H(t)$ also multiplies the third term because $H(s) = 0$ for $s < 0$, and so if $t < 0$, it vanishes everywhere inside the integration range. Finally, if $t > 0$, $H(s) = 1$ still only for $s > 0$, and so we can remove it if we limit the integration range to $s > 0$ (and $s < t$).

For $t > 0$ we have $y(t) = z(t)$ and hence

$$y(t) = -AG_{,s}(t,0) + [\alpha(0)A + B]G(t,0) + \int_0^t G(t,s)f(s)\,ds. \tag{427}$$

$G(t,0)$ obeys the homogeneous equation $y'' + \alpha y' + \beta y = 0$ for $t > 0$ because $G(t,s)$ does for $t > s$ and $s = 0$ here. Similarly, $G_{,s}(t,0)$ also obeys the homogeneous equation. The third term obeys the inhomogeneous equation $y'' + \alpha y' + \beta y = f$ for $t > 0$ by construction. Therefore the first two terms can be considered as the "complementary function" and the third term as the "particular integral".

(427) obeys the initial conditions by construction, but it is instructive to check this explicitly. Note that at $t = s$, $G(t,s)$ is continuous, and $G_{,t}(t,s)$ jumps by 1. This implies that

$$G(t,s) = H(t-s)\left[(t-s) + (t-s)^2 g(t,s)\right] \tag{428}$$

where $g(t,s)$ is some regular function of t and s. But from this we have that

$$G(0_+,0) = 0, \quad G_{,s}(0_+,0) = -1, \quad G_{,t}(0_+,0) = 1, \quad G_{,ts}(0_+,0) = -2g(0,0) = -G_{,tt}. \tag{429}$$

Furthermore, $G(t,s)$ obeys the homogenous ODE $G_{,tt} + \alpha G_{,t} + \beta G = 0$ for $t > s$. Therefore

$$G_{,tt}(0_+,0) = -\alpha(0)G_{,t}(0_+,0) - \beta(0)G(0_+,0) = -\alpha(0). \tag{430}$$

Using these results, we have

$$y(0_+) = -AG_{,s}(0_+,0) + (\alpha(0)A + B)G(0_+,0) = A, \tag{431}$$

$$\begin{aligned} y'(0_+) &= -AG_{,st}(0_+,0) + (\alpha(0)A + B)G_{,t}(0_+,0) \\ &= AG_{,tt}(0_+,0) + (\alpha(0)A + B)G_{,t}(0_+,0) = B. \end{aligned} \tag{432}$$

6.5 Exercises

42. **Homework 20:** (Very short) Check by differentiation that

$$G(t,s) = A(s)e^{-at} + H(t-s)e^{-a(t-s)} \tag{433}$$

obeys

$$G_{,t} + aG = \delta(t-s). \tag{434}$$

43. **Homework 21:** (Very short) Check by differentiation that

$$G(t,s) = H(t-s)\frac{1}{\omega}\sin\omega(t-s) \tag{435}$$

obeys

$$G_{,tt} + \omega^2 G = \delta(t-s). \tag{436}$$

44. (Standard exam question) a) Find the causal Green's function for $dy/dt + y/t = f(t)$. b) Use it to solve $dy/dt + y/t = f(t)$ for $t > 1$, with $y(1) = A$.

45. Show that the Green's function $G(t,s) = \frac{1}{\omega}H(t-s)\sin\omega(t-s)$ satisfies

$$G_{,tt} + \omega^2 G = \delta(t-s) \tag{437}$$

in the sense that

$$\int_{-\infty}^{\infty} G(t,s)\left(\ddot{\phi}(t) + \omega^2\phi(t)\right)\,dt = \phi(s) \tag{438}$$

for every test function $\phi(t)$.

46. (Standard exam questions) a) Find the Green's function and causal solutions of

$$\ddot{y} + 5\dot{y} + 6y = f(t). \tag{439}$$

Find the particular causal solutions when b) $f(t) = e^{-t}$ and c) $f(t) = H(t)e^{-t}$.

47. (All standard exam questions) Use the method of Green's functions to find the causal solutions of:

(a) $\ddot{y} + 4\dot{y} + 4y = H(t) + H(-t)e^{2t}$;

(b) $\ddot{y} + 4\dot{y} + 4y = H(-t)\sin t$;

(c) $\ddot{y} + 4\dot{y} + 3y = 2\sin 5t$;

(d) $\ddot{y} + 6\dot{y} + 9y = t^2$;

(e) $\ddot{y} + 3\dot{y} + 2y = \dot{\delta}(t)$;

(f) $y''' + y' = f(x)$

Hint for the last one: you can either solve $\partial^3 G/\partial x^3 + \partial G/\partial x = \delta(x-y)$ with $G = 0$ for $x < y$, or put $f(x) = F'(x)$ and integrate the equation once.

7 Green's functions for the Poisson and Helmholtz equations

After warming up with inhomogeneous linear ODEs, we now turn to inhomogeneous linear PDEs. In this Section we consider

$$Lu = f(\boldsymbol{x}) \tag{440}$$

where L is the Laplacian Δ or the Helmholtz operator $\Delta + k^2$ and $f(\boldsymbol{x})$ is a given function. The nature of the solution depends on the boundary conditions and in this Section we assume that solutions are required in unbounded space.

7.1 Three-dimensional δ-function

Definition 7.1. The three-dimensional δ-function can be defined by

$$\delta(\boldsymbol{x}) = \delta(x_1)\delta(x_2)\delta(x_3), \tag{441}$$

or by the properties

$$\delta(\boldsymbol{x}) = 0 \text{ for } \boldsymbol{x} \neq \boldsymbol{0}, \text{ and } \int \delta(\boldsymbol{x})\, d^3x = 1, \tag{442}$$

or by the property

$$\int \delta(\boldsymbol{x})\phi(\boldsymbol{x})\, d^3x = \phi(\boldsymbol{0}) \tag{443}$$

for all test functions $\phi(\boldsymbol{x})$.

Proposition 7.2. In the spherical polar coordinates (30), with $r^2 = x_1^2 + x_2^2 + x_3^2$, we have

$$\delta(\boldsymbol{x}) = \frac{\delta(r)}{4\pi r^2}. \tag{444}$$

Proof. We will show that (444) obeys (443). It makes sense to carry out the integral $\int d^3x$ in spherical polar coordinates. The integration range is $0 \leq r < \infty$, and we face the problem that $\int_0^\infty \delta(r)\, dr$ is not defined. But $\int_0^\infty \delta(r-\epsilon)\, dr = 1$ for all $\epsilon > 0$ as now $r - \epsilon = 0$ occurs inside the integration range.

So instead of (444) we consider

$$F_\epsilon(\boldsymbol{x}) = \frac{\delta(r-\epsilon)}{4\pi r^2} \tag{445}$$

for $\epsilon > 0$, and then take $\epsilon \to 0_+$ at the end. Now

$$\int F_\epsilon(\boldsymbol{x})\phi(\boldsymbol{x})\, d^3x = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\delta(r-\epsilon)}{4\pi r^2}\ \phi(r,\theta,\varphi)\, r^2\, dr\ \sin\theta\, d\theta\, d\varphi \tag{446}$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \phi(\epsilon,\theta,\varphi)\ \sin\theta\, d\theta\, d\varphi, \tag{447}$$

where the two factors of r^2 have cancelled, and we have carried out the integration over r. The remaining integral over θ and φ is the average of ϕ over a sphere of radius ϵ. As $\epsilon \to 0$, and the sphere shrinks to a point, this average becomes simply $\phi(\boldsymbol{0})$. We have shown that $\lim_{\epsilon\to 0} F_\epsilon(\mathbf{x}) = \delta(\mathbf{x})$. We shall use this trick of "protecting" $\delta(r)$ again. □

Remark 7.3. Although we have said that

$$g(x)\delta(x) = g(0)\delta(x), \tag{448}$$

this only applies to functions $g(x)$ which are continuous at $x = 0$. An expression of the form $g(x)\delta(x)$ when $g(x)$ is not continuous at $x = 0$ must be interpreted as a generalised function in its own right, that is, under an integral. Thus $\delta(r)/r^2$ is a generalised function and it is certainly not equal to $\delta(r)/0^2$.

7.2 Free space Green's function for the Poisson equation

Definition 7.4. The **free space problem** for the Poisson equation is

$$\Delta u = f(\boldsymbol{x}) \tag{449}$$

with the boundary condition at infinity

$$u(\boldsymbol{x}) \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{450}$$

We assume that $f(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$ so that there are no sources at infinity.

That is, we consider the effect of a source distribution that falls off far away from these sources, and require a solution that also falls off. This **fall-off condition** takes the place of the Dirichlet, Neumann or Robin boundary conditions. We can solve this problem in much the same way as we found causal solutions for ODEs.

Definition 7.5. The free space Green's function $G(\boldsymbol{x}, \boldsymbol{y})$ for the Laplace equation satisfies

$$\Delta_x G(\boldsymbol{x}, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}), \quad G(\boldsymbol{x}, \boldsymbol{y}) \to 0 \text{ as } |\boldsymbol{x} - \boldsymbol{y}| \to \infty. \tag{451}$$

Here Δ_x indicates differentiation with respect to $\boldsymbol{x} = (x_1, x_2, x_3)$ and not with respect to $\boldsymbol{y} = (y_1, y_2, y_3)$. If we put

$$u(\boldsymbol{x}) = \int G(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y})\, d^3y, \tag{452}$$

then, since the integral is with respect to $\boldsymbol{y}$ and the derivatives are with respect to $\boldsymbol{x}$,

$$\Delta_x u = \int \Delta_x G(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y})\, d^3y = \int \delta(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, d^3y = f(\boldsymbol{x}). \tag{453}$$

Because the boundary conditions are at infinity, we can shift the source $f(\boldsymbol{y})$ and the solution $u(\boldsymbol{x})$ about together without making a difference. Hence the Green's function $G(\boldsymbol{x}, \boldsymbol{y})$ depends only on the relative position of $\boldsymbol{x}$ and $\boldsymbol{y}$ and so $G(\boldsymbol{x}, \boldsymbol{y}) = G(\boldsymbol{x} - \boldsymbol{y}) = G(\boldsymbol{x'})$. Then the problem for G becomes

$$\Delta_{x'} G = \delta(\boldsymbol{x'}). \tag{454}$$

Because the problem is invariant under rotations, we expect G to be spherically symmetric, since $\Delta_{x'}$ and $\delta(\boldsymbol{x})$ are, so we look for a generalised function

$$G(\boldsymbol{x'}) = G(|\boldsymbol{x'}|) =: G(r). \tag{455}$$

Then, using (31) and (444), (454) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) = \frac{1}{4\pi}\frac{\delta(r)}{r^2}. \tag{456}$$

For $r > 0$, $\delta(r) = 0$, so we find that

$$G = \frac{A}{r} + B \quad (\text{for } r > 0), \tag{457}$$

where A and B are constants. Since $G \to 0$ as $r \to \infty$, $B = 0$.

To find A we argue as follows. Since $\Delta_{x'} G = \delta(\boldsymbol{x'})$, we must have

$$\int_V \Delta_{x'} G\, d^3x' = 1 \tag{458}$$

for any volume V that includes the origin. By the divergence theorem we also have

$$\int_V \Delta_{x'} G\, d^3x = \int_S G_{,n}\, d^2\boldsymbol{x'}, \tag{459}$$

where S is the surface of V.

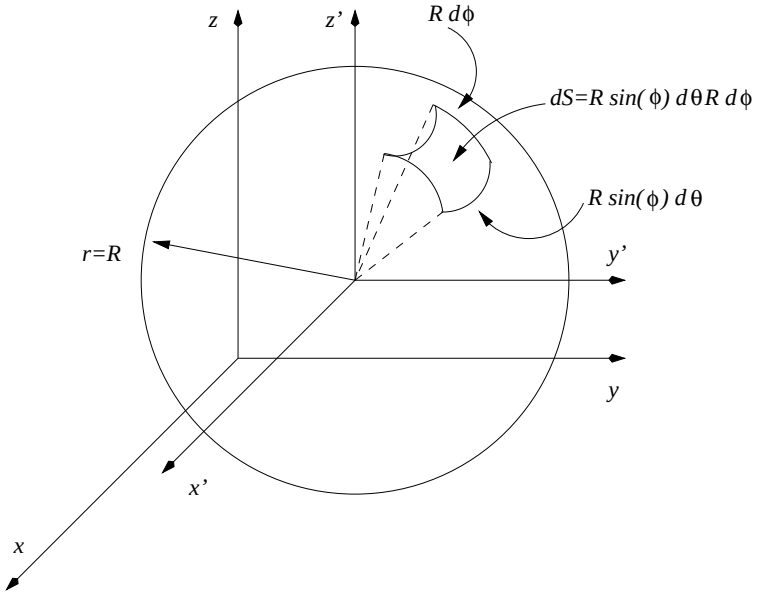

Figure 8: Sphere of radius R

Now, choose V to be a sphere of radius R (see Fig. 8). The outward normal to the surface is just the unit vector pointing from the origin to the point on the surface and so

$$G_{,n} = G_{,r} = -\frac{A}{r^2} = -\frac{A}{R^2} \tag{460}$$

on the surface $r = R$ of the sphere. Also, on the surface $r = R$, the surface element d^2x' is given by $R^2 \sin\theta \, d\theta \, d\varphi$, so

$$\int_0^{\pi} \int_0^{2\pi} \left. \frac{\partial G}{\partial r} \right|_{r=R} R^2 \sin\theta \, d\theta \, d\varphi = 1. \tag{461}$$

Thus

$$-A \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\theta \, d\varphi = 1, \tag{462}$$

and so

$$A = -\frac{1}{4\pi}. \tag{463}$$

Thus

$$G = -\frac{1}{4\pi \, r} = -\frac{1}{4\pi} \frac{1}{|\boldsymbol{x}'|} = -\frac{1}{4\pi} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} = -\frac{1}{4\pi} \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}}, \tag{464}$$

and the solution of the problem

$$\Delta u = f(\boldsymbol{x}), \quad u \to 0 \text{ as } |\boldsymbol{x}| \to \infty \tag{465}$$

is

$$u(\boldsymbol{x}) = -\frac{1}{4\pi} \int \frac{f(\boldsymbol{y}) \, d^3y}{|\boldsymbol{x} - \boldsymbol{y}|}, \tag{466}$$

or

$$u(x_1, x_2, x_3) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(y_1, y_2, y_3) \, dy_1 \, dy_2 \, dy_3}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}}. \tag{467}$$

Note that $G(\boldsymbol{x}, \boldsymbol{y})$ is not defined at $\boldsymbol{x} = \boldsymbol{y}$. Rather, it is a generalised function. One useful interpretation (see Section 7.4) is

$$G(\boldsymbol{x}, \boldsymbol{y}) = \lim_{\epsilon \to 0} -\frac{1}{4\pi} \frac{H(|\boldsymbol{x} - \boldsymbol{y}| - \epsilon)}{|\boldsymbol{x} - \boldsymbol{y}|} \tag{468}$$

as shown in Fig. 9.

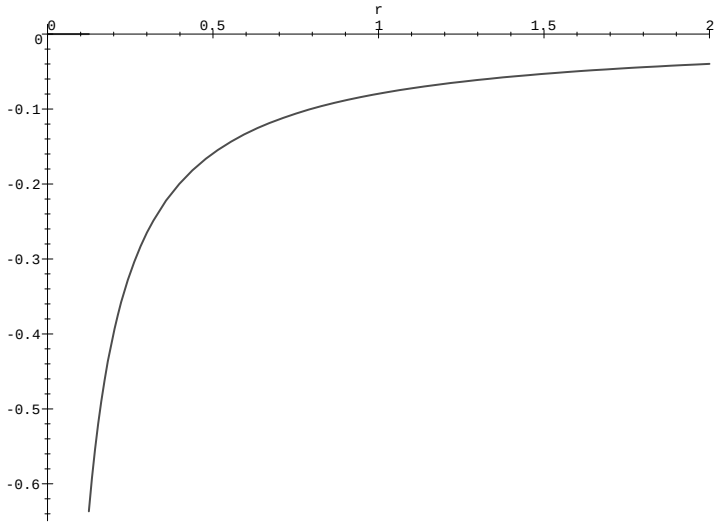

Figure 9: Generalised function interpretation of Green's function. We consider $r = |\boldsymbol{x} - \boldsymbol{y}|$ and illustrate $H(r - \epsilon)/r$ for a small non-zero ϵ.

7.3 Free space Green's function for the Helmholtz equation

Recall the derivation of the Helmholtz equation for time-periodic solutions with angular frequency ω of the wave equation in Remark 1.13.

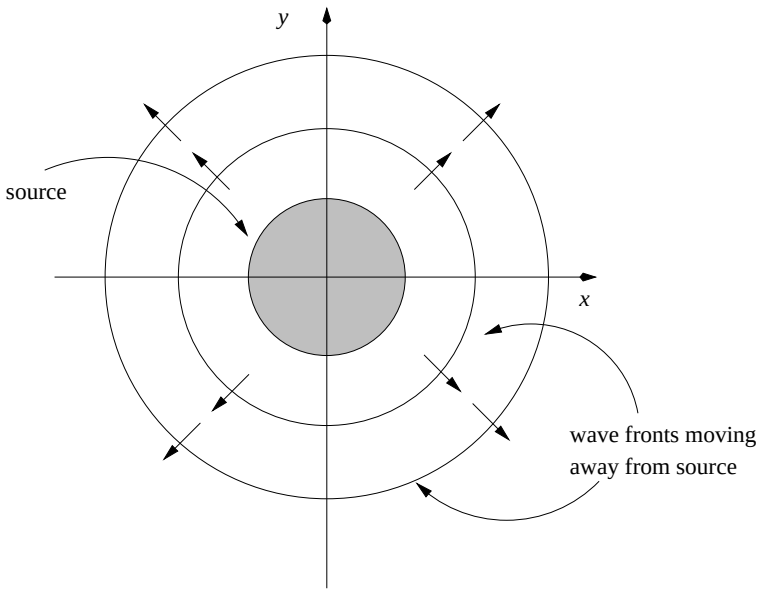

Figure 10: Radiation condition; waves move away from the source

The Green's function for the Helmholtz equation satisfies

$$(\Delta_x + k^2)G(\boldsymbol{x}, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{469}$$

Hence

$$u(\boldsymbol{x}) = \int G(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y})\, d^3y \tag{470}$$

is a solution of (57). However, this solution depends on the boundary conditions, which we have not yet specified. For the Helmholtz equation, it is natural to impose this boundary condition directly on the Green's function. Physically, we expect waves to propagate away from the disturbance generating them and not towards it. This gives us a **radiation boundary condition**, which replaces the fall-off condition that $u \to 0$ as $|\boldsymbol{x}| \to \infty$ used for Poisson's equation. We shall see shortly what form this radiation condition takes for the Green's function.

As before, it is convenient to introduce $\boldsymbol{x'} = \boldsymbol{x} - \boldsymbol{y}$, in which case the problem becomes

$$(\Delta_{x'} + k^2)G = \delta(\boldsymbol{x'}), \tag{471}$$

which clearly has spherical symmetry. So, we look for a solution with $G(\boldsymbol{x'}) = G(r)$, and the problem is then

$$\frac{1}{r}\left[\frac{\partial^2}{\partial r^2}(rG) + k^2(rG)\right] = \frac{\delta(r)}{4\pi r^2}, \tag{472}$$

since

$$\Delta_3 f(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}f\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) \tag{473}$$

when acting on a **spherically symmetric** function $f(\boldsymbol{x}) = f(|\boldsymbol{x}|) = f(r)$. (The second equality is a useful identity to remember.) So for $r > 0$ we have

$$\frac{\partial^2}{\partial r^2}(rG) + k^2(rG) = 0, \tag{474}$$

which implies that $rG = \alpha e^{ikr} + \beta e^{-ikr}$ or

$$G = \frac{A}{r}e^{ikr} + \frac{B}{r}e^{-ikr}. \tag{475}$$

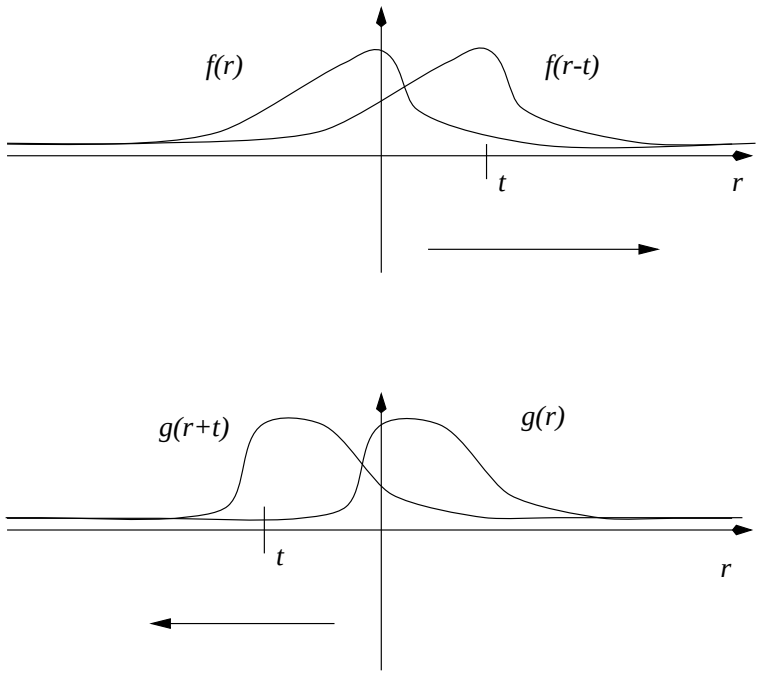

Figure 11: Radiation condition; waves move away from the source

If we consider Ge^{-ikct}, which is a solution of the wave equation if G is a solution of the Helmholtz equation (remember Remark 1.13) we have

$$Ge^{-ikct} = \frac{A}{r}e^{ik(r-ct)} + \frac{B}{r}e^{-ik(r+ct)}. \tag{476}$$

Now any function $f(r-ct)$ represents a wave moving away from $r=0$ towards $r\to\infty$ with speed c as t increases, because f is constant on lines $r-ct=C$. Fig. 11). On the other hand a function $g(r+ct)$ represents a wave moving inwards, see Fig. 11. The δ-function in the problem for G represents a disturbance at the origin. Physically we expect waves to propagate outward away from this disturbance and not inward from infinity towards the disturbance. So the radiation condition tells us that $B=0$. Hence

$$G=\frac{A}{r}e^{ikr},\quad r>0. \tag{477}$$

To find A in a handwaving way, we note that the δ-function at r in $(\Delta+k^2)G=\delta$ must be coming from the $1/r$ term, as e^{ikr} is a regular function, that takes the value one there. Put differently, at $\boldsymbol{x}=0$ we can neglect k^2 compared to $\delta(\boldsymbol{x})$. Hence by analogy with $\Delta G=\delta$ we set $A=-1/(4\pi)$.

We obtain

$$G(\boldsymbol{x}')=-\frac{1}{4\pi r}e^{ikr}=-\frac{1}{4\pi\left|\boldsymbol{x}'\right|}e^{ik|\boldsymbol{x}'|}, \tag{478}$$

or

$$\begin{aligned} G(\boldsymbol{x},\boldsymbol{y}) &= -\frac{1}{4\pi\left|\boldsymbol{x}-\boldsymbol{y}\right|}e^{ik|\boldsymbol{x}-\boldsymbol{y}|} \\ &= -\frac{1}{4\pi}\frac{\exp\left(ik\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}\right)}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}} \end{aligned} \tag{479}$$

Note that as $k\to 0$ we recover the Green's function for the Poisson equation.

To summarize: The solution of the inhomogeneous Helmholtz problem

$$(\Delta+k^2)u = f(\boldsymbol{x}), \tag{480}$$

$$f(\boldsymbol{x}) \to 0, \qquad |\boldsymbol{x}|\to\infty \tag{481}$$

that satisfies the outgoing radiation boundary condition is given by

$$u(\boldsymbol{x})=-\frac{1}{4\pi}\int\frac{f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}e^{ik|\boldsymbol{x}-\boldsymbol{y}|}\,d^3y. \tag{482}$$

This represents the (spatial part of) an outgoing train of waves caused by a disturbance in the region where $f(\boldsymbol{x})\neq 0$.

7.4 An alternative derivation

It is instructive to check by direct differentiation that the free space Green's functions we have derived for the Poisson and Helmholtz equations actually obey the inhomogeneous PDEs they are supposed to.

Let $G(r)$ be the (generalised) function defined by

$$G(r):=\lim_{\epsilon\to 0}G_\epsilon(r),\quad G_\epsilon(r):=-\frac{H(r-\epsilon)}{4\pi r}f(r), \tag{483}$$

where $f(r)$ smooth and bounded (so that $f(r)$ times any test function is again a test function). This is similar to the trick we have used in Sec. 7.1, where we replaced $\delta(r)$ by $\delta(r-\epsilon)$ and took $\epsilon\to 0$ at the end.

We now use the product rule to find

$$\Delta G_\epsilon = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{H(r-\epsilon)}{4\pi r}f(r)\right)\right] \tag{484}$$

$$= -\frac{1}{4\pi r^2}\frac{\partial}{\partial r}\left[rf(r)\delta(r-\epsilon)+(rf'(r)-f(r))H(r-\epsilon)\right]. \tag{485}$$

To be sure that we evaluate the derivative of the term with the δ-function correctly, we integrate it over a test function $\phi(r)$:

$$-\int \frac{1}{4\pi r^2}[rf(r)\delta(r-\epsilon)]'\phi(r)4\pi r^2\,dr = \int rf(r)\delta(r-\epsilon)\phi'(r)dr \tag{486}$$

$$= \quad \epsilon f(\epsilon)\phi'(\epsilon) = -\int \frac{1}{4\pi r^2}\epsilon f(\epsilon)\delta'(r-\epsilon)\phi(r)4\pi r^2\,dr. \tag{487}$$

The important point here is that the $1/r^2$ term in (484) is cancelled by the factor of r^2 in the volume element $4\pi r^2\,dr$ in the integral (486), and so is not differentiated during the integration by parts in r, but it comes back when we write (487) as a volume integral over $4\pi r^2\,dr$ again. To be quite sure, we could have carried out the entire calculation under a volume integral, but this is the one step where this really matters. Putting this term together with the other, unproblematic, terms in (485), we have

$$\Delta G_\epsilon = -\frac{1}{4\pi r^2}\left[\epsilon f(\epsilon)\delta'(r-\epsilon) + (\epsilon f'(\epsilon) - f(\epsilon))\delta(r-\epsilon) + (rf''(r) + f'(r) - f'(r))H(r-\epsilon)\right]. \tag{488}$$

Again, note that the $1/r^2$ in front has not become a $1/\epsilon^2$. In the limit $\epsilon \to 0$

$$\Delta\left(-\frac{f(r)}{4\pi r}\right) = -\frac{f''(r)}{4\pi r} + \frac{f(0)}{4\pi r^2}\delta(r), \tag{489}$$

Hence we have proved that

$$\Delta G = -\frac{1}{4\pi r}f''(r) + f(0)\delta(\boldsymbol{x}). \tag{490}$$

Then for $f(r) = 1$ we have

$$\Delta\left(-\frac{1}{4\pi r}\right) = \delta(\boldsymbol{x}), \tag{491}$$

which confirms our result for the Poisson equation, and for $f(r) = e^{ikr}$ we have

$$(\Delta + k^2)\left(-\frac{e^{ikr}}{4\pi r}\right) = \delta(\boldsymbol{x}), \tag{492}$$

which confirms our result for the Helmholtz equation.

7.5 The large distance and long wavelength approximations

If the source distribution is nonzero only in a bounded region, or if it falls off sufficiently rapidly with distance, one can deduce the approximate behaviour of the solution at large distance from the source without solving the full problem.

It is important that two related but separate approximations are necessary here. For the Poisson equation, the point where we evaluate the solution must be much further away from the source than the size of the source (large distance approximation). For the Helmholtz equation, we need the large distance approximation *and* a separate approximation, namely that the size of the source is much smaller than the wavelength (long wavelength approximation). (Recall that the Helmholtz equation is about waves of a specific frequency and hence wavelength).

We study these two approximations separately.

Example 7.6. Consider the free-space Poisson problem

$$\Delta u = e^{-|\boldsymbol{x}|^2/\ell^2} =: f(\boldsymbol{x}), \qquad u(\boldsymbol{x}) \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{493}$$

What approximation can we make for $u(\boldsymbol{x})$ as $|\boldsymbol{x}| \gg \ell$?

The exact solution is

$$u(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{y\in\mathbb{R}^3} \frac{f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\,d^3y. \tag{494}$$

But let us assume we cannot or do not want to evaluate this exactly. The key observation is that in this example, although the source $f(\boldsymbol{x})$ is nowhere zero, it falls off very rapidly with $|\boldsymbol{x}|$, and we can approximate it as zero for $|\boldsymbol{x}| > R$, where R is the "size" of the source region, chosen to be a few ℓ. [Of course if $f(\boldsymbol{x})$ is nonzero only on a bounded region, say $f(\boldsymbol{x}) = H(R - |\boldsymbol{x}|)$, the size of the source is defined unambiguously]. So we approximate

$$u(\boldsymbol{x}) \simeq -\frac{1}{4\pi} \int_{|\boldsymbol{y}|<R} \frac{f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d^3y. \tag{495}$$

For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and the Euclidean norm $|\cdot|$, we have the triangle equalities

$$\Big||\boldsymbol{x}| - |\boldsymbol{y}|\Big| \le |\boldsymbol{x}-\boldsymbol{y}| \le |\boldsymbol{x}| + |\boldsymbol{y}|. \tag{496}$$

The outer bracket in the first expression is just the absolute value of a real number. (To just convince yourself that it is true, draw some vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^2$. Prove it as an Exercise (for any norm).)

Now we consider the limit $|\boldsymbol{x}| \gg R$ (and hence in particular $|\boldsymbol{x}| > R$). In the integral (495), $|\boldsymbol{y}| < R$. Hence we have

$$|\boldsymbol{x}| - R \le |\boldsymbol{x}-\boldsymbol{y}| \le |\boldsymbol{x}| + R. \tag{497}$$

We can write this as

$$|\boldsymbol{x}-\boldsymbol{y}| = |\boldsymbol{x}| + O(R) \quad \text{as} \quad |\boldsymbol{x}| \to \infty. \tag{498}$$

The symbol $O(\cdot)$ is pronounced "order of" or "big-O of". Its formal definition is that $f(x) = O[g(x)]$ as $x \to \infty$ if $cg(x) < |f(x)| < Cg(x)$ for two constants $0 < c < C$ as $x \to \infty$. Its intuitive meaning is "grows or decays like". Note that a limit is always part of the definition of O.

We now rewrite the estimate (498) of the absolute error in $|\boldsymbol{x}-\boldsymbol{y}|$ as a relative error,

$$|\boldsymbol{x}-\boldsymbol{y}| = |\boldsymbol{x}| \left[1 + O\left(\frac{R}{|\boldsymbol{x}|}\right)\right] \tag{499}$$

We can approximate this by $|\boldsymbol{x}|$ if and only if $R/|\boldsymbol{x}| \ll 1$, that is when the *relative* error in $|\boldsymbol{x}-\boldsymbol{y}|$ is small. Clearly, it is the relative error (in percent) that matters here, not the absolute error (in meters).

We can now continue from (495) as

$$\begin{aligned} u(\boldsymbol{x}) &\simeq -\frac{1}{4\pi} \int_{|\boldsymbol{y}|<R} \frac{f(\boldsymbol{y})}{|\boldsymbol{x}|}\, d^3y \\ &= -\frac{1}{4\pi|\boldsymbol{x}|} \int_{|\boldsymbol{y}|<R} f(\boldsymbol{y})\, d^3y \\ &\simeq -\frac{1}{4\pi|\boldsymbol{x}|} \int_{\boldsymbol{y}\in\mathbb{R}^3} f(\boldsymbol{y})\, d^3y \\ &= -\frac{A}{4\pi|\boldsymbol{x}|}, \end{aligned} \tag{500}$$

where

$$A := \int f(\boldsymbol{y})\, d^3y \tag{501}$$

is the *magnitude* of the source. This is the **large distance approximation**: the source term is approximated as a *point source* of magnitude A located at the origin.

Example 7.7. Consider the free-space Helmholtz problem

$$(\Delta + k^2)u = e^{-|\boldsymbol{x}|^2/\ell^2} =: f(\boldsymbol{x}), \quad \text{outgoing wave BC as} \quad |\boldsymbol{x}| \to \infty. \tag{502}$$

What approximation can we now make for $u(\boldsymbol{x})$ as $|\boldsymbol{x}| \gg \ell$?

The exact solution is now

$$u(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{y\in\mathbb{R}^3} \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|} f(\boldsymbol{y})\, d^3y. \tag{503}$$

We now want to approximate $|\boldsymbol{x}-\boldsymbol{y}|$ by $|\boldsymbol{x}|$ in two different places, namely in the amplitude $1/|\boldsymbol{x}-\boldsymbol{y}|$ and in the complex phase $\exp ik|\boldsymbol{x}-\boldsymbol{y}|$. For the amplitude, we have once again (499), where it matters that the *relative error in amplitude* (in percent) is small. So, as in the Poisson example, we need the **large distance condition**

$$\frac{R}{|\boldsymbol{x}|} \ll 1. \tag{504}$$

But the complex phase, measured in radians, is

$$k|\boldsymbol{x}-\boldsymbol{y}| = k|\boldsymbol{x}|\left[1+O\left(\frac{R}{|\boldsymbol{x}|}\right)\right] = k|\boldsymbol{x}| + O(kR). \tag{505}$$

We are clearly completely out of phase when the error in the phase approaches 2π. The relative error in the phase (first equality above) is completely irrelevant, what matters is the absolute phase error (second equality). Hence we can approximate

$$e^{ik|\boldsymbol{x}-\boldsymbol{y}|} \simeq e^{ik|\boldsymbol{x}|} \tag{506}$$

if and only if

$$kR \ll 2\pi. \tag{507}$$

(You will also find $kR \ll 1$ in the literature, which is equally good within the approximation implied by $\ll$.) As $k = 2\pi/\lambda$, this means that the wavelength of the waves is much larger than the size of the source,

$$\frac{R}{\lambda} \ll 1, \tag{508}$$

the **long wavelength condition**.

As a real world example, consider the loudspeaker of a radio, with a diameter of 10cm. Hence the large distance approximation holds if we look at the sound field at distances much larger than 10cm from the loud speaker, and the long wavelength approximation holds for wavelengths much larger than 10cm, or frequencies much lower than (340m/s)/(10cm)=3400Hz.

7.6 Uniqueness of the solution to the free-space problem

Given the Green's function we deduced in Sec. 5.3, we know that the solution of the free space Poisson problem is

$$u(\boldsymbol{x}) = -\frac{1}{4\pi}\int \frac{f(\boldsymbol{y})\, d^3y}{|\boldsymbol{x}-\boldsymbol{y}|}. \tag{509}$$

Theorem 7.8. The solution (509) of the free space Poisson problem (449,450) is unique.

Proof. Assume there are two solutions u_1 and u_2, and let $u := u_1 - u_2$. The proof is then identical to the proof for a bounded volume V given in Sec. 2.4.3 up to Eq. (151).

Now consider S in that equation to be a sphere of radius R, and let $R\to\infty$. On the sphere

$$u \sim \frac{1}{R}, \quad u_{,n} = \left.\frac{\partial u}{\partial r}\right|_{r=R} \sim \frac{1}{R^2}, \quad d^2x = R^2\sin\theta\, d\theta d\varphi, \tag{510}$$

and so

$$\int_S u u_{,n}\, d^2x \sim \int_0^\pi \int_0^{2\pi} \frac{1}{R}\sin\theta\, d\varphi d\theta = \frac{4\pi}{R} \to 0. \tag{511}$$

Thus, in the limit $R\to\infty$, we get

$$\int |\nabla u|^2\, d^3x = 0, \tag{512}$$

so that $|\nabla u|^2 = 0$ at all points. Hence u must be constant. Now note that since $u\to 0$ as $|\boldsymbol{x}|\to\infty$, this constant must be zero. Hence $u_1 = u_2$, and the solution is unique. □

7.7 Exercises

48. **Homework 22:** (Short and easy) Check that

$$\Delta f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} f\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) \tag{513}$$

when acting on a spherically symmetric function $f(\boldsymbol{x}) = f(r)$.

49. Verify that the 2-dimensional δ-function

$$\delta(x, y) := \delta(x)\delta(y) \tag{514}$$

can also be defined as either of the limits a)

$$\lim_{\epsilon \to 0} \begin{cases} 1/4\epsilon^2 & |x| < \epsilon \text{ and } |y| < \epsilon \\ 0 & |x| > \epsilon \text{ or } |y| > \epsilon \end{cases} \tag{515}$$

or b)

$$\lim_{\epsilon \to 0} \frac{\epsilon}{2\pi(x^2 + y^2 + \epsilon^2)^{3/2}} \tag{516}$$

Hint: use polar coordinates and recall that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = \int_0^{\infty}\int_0^{2\pi} f(r, \theta)\, d\theta\, r dr. \tag{517}$$

50. Show by differentiation, and using $\Delta f = r^{-1}(rf)''$ for $f(\boldsymbol{x}) = f(r)$, that

$$\Delta \frac{f(r)}{4\pi r} = -\frac{\delta(r) f(0)}{4\pi r^2} + \frac{f''(r)}{4\pi r}. \tag{518}$$

51. **Homework 23:** (The version of the triangle inequality used in the lecture) We originally met the triangle inequality as

$$||X + Z|| \leq ||X|| + ||Z||. \tag{519}$$

Make two appropriate choices of Z to obtain the equivalent triangle inequalities

$$\Big|\,||X|| - ||Y||\,\Big| \leq ||X - Y|| \leq ||X|| + ||Y||. \tag{520}$$

52. By integration over a suitable Green's function, show that the solution of the Poisson equation (in three space dimensions) with free space boundary conditions and a spherically symmetric source,

$$\Delta u = f(r) \text{ where } r := |\boldsymbol{x}|, \tag{521}$$

$$u \to 0 \text{ as } |\boldsymbol{x}| \to 0 \tag{522}$$

is

$$u(r) = -\frac{1}{r}\int_0^r f(\rho)\rho^2\, d\rho - \int_r^{\infty} f(\rho)\rho\, d\rho. \tag{523}$$

Hint: use the appropriate Green's function, and do the integration in spherical polar coordinates (ρ, θ, φ). Without loss of generality you can assume that the point $\boldsymbol{x}$ is on the x_3-axis. Further hint: consider the substitution $z = -\cos\theta$.

53. Show that (523) is a solution (it is in fact the unique solution) of (521,522) by checking the PDE explicitly by differentiating, and the boundary condition by taking a limit.

54. If we approximate the earth as a sphere of radius r_0 and constant mass density ρ_0, then its gravitational potential $\Phi(\boldsymbol{x})$ obeys

$$\Delta\Phi = \begin{cases} 4\pi G\rho_0, & 0 < |\boldsymbol{x}| < r_0, \\ 0, & |\boldsymbol{x}| > r_0, \end{cases} \tag{524}$$

$$\Phi \to 0 \text{ as } |\boldsymbol{x}| \to 0, \tag{525}$$

where G is Newton's gravitational constant.

a) Using the formula (523), find $\Phi(r)$ where $r := |\boldsymbol{x}|$. Hint: calculate separately for $0 < r < r_0$ and $r > r_0$.

b) The gravitational acceleration of a freely falling body is given by

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\nabla\Phi \tag{526}$$

You build a rapid transit system by drilling a straight hole from the north pole to the south pole through the centre of the earth. You drop a capsule (like a lift cabin) from the north pole. It accelerates down, reaches maximum velocity at the centre of the earth, and comes to a stop just as it reaches the south pole. Find the ODE obeyed by its position $z(t)$, and solve it with initial conditions $z(0) = r_0$ and $dz/dt(0) = 0$, where $z = r_0$ is the north pole and $z = -r_0$ the south pole. (We assume the tunnel goes along the earth's rotation axis so we do not have to worry about forces due to the rotation of the earth). Look up numbers for r_0, ρ_0 and G and calculate the time (in minutes) the one-way trip takes.

c) You convert your tunnel into a gigantic cannon and shoot the capsule (now a spaceship) straight up from the north pole into the sky. Find the ODE obeyed by its position $z(t)$. This cannot be solved in closed form. However, show that the total energy (per mass) of the capsule

$$E(t) := \frac{1}{2}\left(\frac{dz}{dt}\right)^2 - \frac{M}{z} \tag{527}$$

is conserved, that is $dE/dt = 0$. Hence find the maximum height $z_{\max}$ reached for a given initial velocity v_0. Hence find the escape velocity v_{escape} for which the capsule never comes back. Find the numerical value for the Earth's escape velocity (in kilometers per second).

8 Green's functions for bounded regions

So far we have only considered free-space problems. However, in many situations of interest we are looking for a solution that satisfies both the PDE and given boundary conditions. This Section discusses how we can find a Green's function solution that satisfies the relevant boundary conditions. We focus on the Helmholtz equation

$$(\Delta + k^2)u = f(\boldsymbol{x}), \tag{528}$$

but the method is readily generalised to other problems. For example, results for the Poisson equation follow by taking the limit $k \to 0$.

Our main mathematical tool will be the Kirchhoff-Helmholtz formula. To derive it, we need two major ingredients, Green's theorem and the reciprocal theorem.

8.1 Green's theorem

Theorem 8.1. Suppose that $G(\boldsymbol{x})$ and $u(\boldsymbol{x})$ are functions with continuous second derivatives on a region V with surface S. Then

$$\int_V (G\Delta u - u\Delta G)\, d^3x = \int_S (Gu_{,n} - uG_{,n})\, d^2x. \tag{529}$$

Proof. Recalling that

$$\Delta u = \nabla \cdot (\nabla u) \tag{530}$$

and likewise for G, and that

$$\nabla \cdot (G\nabla u) = G\Delta u + (\nabla G) \cdot (\nabla u), \tag{531}$$
$$\nabla \cdot (u\nabla G) = u\Delta G + (\nabla u) \cdot (\nabla G), \tag{532}$$

we find that

$$G\Delta u - u\Delta G = \nabla \cdot (G\nabla u - u\nabla G). \tag{533}$$

Thus, integrating over V we have

$$\int_V (G\Delta u - u\Delta G)\, d^3x = \int_V \nabla \cdot (G\nabla u - u\nabla G)\, d^3x. \tag{534}$$

By the divergence theorem

$$\int_V \nabla \cdot (G\nabla u - u\nabla G)\, d^3x = \int_S (G\nabla u - u\nabla G) \cdot \boldsymbol{n}\, d^2x = \int_S (Gu_{,n} - uG_{,n})\, d^2x, \tag{535}$$

which establishes Green's theorem. □

8.2 The reciprocal theorem

In all of the problems in this Section, the Green's function is symmetric, that is

$$G(\boldsymbol{x}, \boldsymbol{y}) = G(\boldsymbol{y}, \boldsymbol{x}). \tag{536}$$

The physical meaning of this symmetry is that hitting the system at $\boldsymbol{x}$ produces the same effect at $\boldsymbol{y}$ as the other way around. Mathematically speaking, Green's functions are symmetric if they correspond to self-adjoint differential operators with homogeneous (that is, zero) boundary conditions. Not all physical systems have this property. Note also that the causal Green's function for a time evolution problem (or for an ODE) does not have this symmetry under the interchange of t and s.

Theorem 8.2. If $G(\boldsymbol{x},\boldsymbol{y})$ is the solution of

$$\begin{aligned}(\Delta_x + k^2)G(\boldsymbol{x},\boldsymbol{y}) &= \delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{for } \boldsymbol{x},\boldsymbol{y} \text{ in } V, &(537)\\ \alpha G(\boldsymbol{x},\boldsymbol{y}) + \beta G_{\boldsymbol{n}_x}(\boldsymbol{x},\boldsymbol{y}) &= 0 \quad \text{for } \boldsymbol{x} \text{ on } S, &(538)\end{aligned}$$

then

$$G(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{y},\boldsymbol{x}). \tag{539}$$

The theorem also holds for the Poisson equation, with $k = 0$.

Proof. Consider the two PDEs

$$(\Delta_x + k^2)G(\boldsymbol{x},\boldsymbol{y}_1) = \delta(\boldsymbol{x}-\boldsymbol{y}_1), \qquad (\Delta_x + k^2)G(\boldsymbol{x},\boldsymbol{y}_2) = \delta(\boldsymbol{x}-\boldsymbol{y}_2), \tag{540}$$

and multiply the first equation by $G(\boldsymbol{x},\boldsymbol{y}_2)$, multiply the second equation by $G(\boldsymbol{x},\boldsymbol{y}_1)$, subtract and integrate over V:

$$\begin{aligned}&\int_V \Big[G(\boldsymbol{x},\boldsymbol{y}_2)\Delta_x G(\boldsymbol{x},\boldsymbol{y}_1) - G(\boldsymbol{x},\boldsymbol{y}_1)\Delta_x G(\boldsymbol{x},\boldsymbol{y}_2)\Big]\, d^3x\\ = &\int_V \Big[G(\boldsymbol{x},\boldsymbol{y}_2)\delta(\boldsymbol{x}-\boldsymbol{y}_1) - G(\boldsymbol{x},\boldsymbol{y}_1)\delta(\boldsymbol{x}-\boldsymbol{y}_2)\Big]\, d^3x. &(541)\end{aligned}$$

(The terms containing k^2 have cancelled). Now apply Green's theorem to the left-hand side and integrate out the δ-function on the right-hand side. This gives

$$\int_S \left[G(\boldsymbol{x},\boldsymbol{y}_2)G_{,n_x}(\boldsymbol{x},\boldsymbol{y}_1) - G(\boldsymbol{x},\boldsymbol{y}_1)G_{,n_x}(\boldsymbol{x},\boldsymbol{y}_2)\right] d^2x = G(\boldsymbol{y}_1,\boldsymbol{y}_2) - G(\boldsymbol{y}_2,\boldsymbol{y}_1). \tag{542}$$

One can argue from the boundary condition (538) that this surface integral vanishes (see problem). We conclude that

$$G(\boldsymbol{y}_2,\boldsymbol{y}_1) - G(\boldsymbol{y}_1,\boldsymbol{y}_2) = 0. \tag{543}$$

This proves the theorem since $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ can be any points inside V. □

The reciprocal theorem is of physical interest in its own right, but for the derivation of the Kirchhoff-Helmholtz formula that follows we need the following

Corollary 8.3. If G obeys (537) and (538) then it also obeys

$$\begin{aligned}(\Delta_y + k^2)G(\boldsymbol{x},\boldsymbol{y}) &= \delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{for } \boldsymbol{x},\boldsymbol{y} \text{ in } V, &(544)\\ \alpha G(\boldsymbol{x},\boldsymbol{y}) + \beta G_{\boldsymbol{n}_y}(\boldsymbol{x},\boldsymbol{y}) &= 0 \quad \text{for } \boldsymbol{y} \text{ on } S, &(545)\end{aligned}$$

Idea of proof: Because G is symmetric under interchange of its two arguments, we can write

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\left(G(\boldsymbol{x},\boldsymbol{y}) + G(\boldsymbol{y},\boldsymbol{x})\right). \tag{546}$$

Using suffixes 1 and 2 to denote partial derivatives with respect to the first and second argument of the abstract function $G(\cdot,\cdot)$, this gives us

$$\nabla_x G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\left(\nabla_1 G(\boldsymbol{x},\boldsymbol{y}) + \nabla_2 G(\boldsymbol{y},\boldsymbol{x})\right). \tag{547}$$

Now write out (537) and (538) in this notation, interchange the labels $\boldsymbol{x}$ and $\boldsymbol{y}$, use the reciprocal theorem $G(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{y},\boldsymbol{x})$ and the identity $\delta(\boldsymbol{x}-\boldsymbol{y}) = \delta(\boldsymbol{y}-\boldsymbol{x})$, and so obtain (544) and (545).

8.3 The Kirchhoff-Helmholtz formula

We want to derive a formula for the solution $u(\boldsymbol{x})$ of a Helmholtz problem

$$(\Delta + k^2)u(\boldsymbol{x}) = f(\boldsymbol{x}) \tag{548}$$

on a bounded domain V, with some boundary conditions on $S = \partial V$. We leave these boundary conditions unspecified for now.

To obtain the desired formula, consider

$$\begin{aligned}
& \int_V G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, d^3y - u(\boldsymbol{x}) && (549)\\
= & \int_V \left[G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y}) - u(\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{y}) \right] d^3y && (550)\\
= & \int_V \Big[G(\boldsymbol{x},\boldsymbol{y})(\Delta + k^2)u(\boldsymbol{y}) - u(\boldsymbol{y})(\Delta_y + k^2)G(\boldsymbol{x},\boldsymbol{y}) \Big] d^3y && (551)\\
= & \int_V \Big[G(\boldsymbol{x},\boldsymbol{y})\Delta u(\boldsymbol{y}) - u(\boldsymbol{y})\Delta_y G(\boldsymbol{x},\boldsymbol{y}) \Big] d^3y && (552)\\
= & \int_S \left[G(\boldsymbol{x},\boldsymbol{y}) u_{,n}(\boldsymbol{y}) - u(\boldsymbol{y}) G_{,n_y}(\boldsymbol{x},\boldsymbol{y}) \right] d^2y && (553)
\end{aligned}$$

In (550) we have used the definition of the δ-function. In (551), we have used (548), but with $\boldsymbol{x}$ renamed to $\boldsymbol{y}$, to replace $f(\boldsymbol{y})$, and we have used (544) to replace the δ-function. In (552), we have cancelled the term proportional to k^2. In (553), we have used Green's theorem in the variable $\boldsymbol{y}$. Here $u_{,n_y}$ denotes the normal derivative with respect to the $\boldsymbol{y}$ variables, i.e.,

$$G_{,n_y} := \boldsymbol{n} \cdot (\nabla_y G). \tag{554}$$

Combining (549) and (553) and rearranging, we obtain the **Kirchhoff-Helmholtz representation**

$$u(\boldsymbol{x}) = \int_V G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, d^3y + \int_S \left[G_{,n_y}(\boldsymbol{x},\boldsymbol{y}) u(\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y}) u_{,n}(\boldsymbol{y}) \right] d^2y. \tag{555}$$

This formula gives the value of $u(\boldsymbol{x})$ inside the region V in terms of the source distribution $f(\boldsymbol{x})$ in V and the values of u and $u_{,n}$ on the surface S. It is true for *any* $G(\boldsymbol{x},\boldsymbol{y})$ that is symmetric.

Remark 8.4. When attempting to solve (548) analytically, we must choose G so that we minimise the amount of information we need to know about u and $u_{,n}$ on the boundary. For example, if we are given a Dirichlet problem, where u is prescribed on the boundary, we try to find G so that $G(\boldsymbol{x},\boldsymbol{y}) = 0$ when $\boldsymbol{y}$ is on the boundary. This eliminates the unknown $u_{,n}$ and allows us to calculate u in terms of known quantities. Similarly, for the Neumann problem where we know $u_{,n}$ on the boundary we try to find G so that $G_{,n_y}(\boldsymbol{x},\boldsymbol{y}) = 0$ when $\boldsymbol{y}$ is on the boundary. This eliminates the unknown u from the integral over the surface.

Remark 8.5. (555) can also be used numerically. We choose a simple G, say the free space Green's function, and this gives us an integral equation to solve numerically. For example if we are given a Neumann problem with $u_{,n}$ specified on the boundary (but where we do not know u on the boundary) then by choosing $\boldsymbol{x}$ to be a point on the boundary, (548) becomes an integral equation for the unknown $u(\boldsymbol{x})$ on the boundary. This is solved numerically, and once we know $u(\boldsymbol{x})$ on the boundary, then (548) tells us the value of $u(\boldsymbol{x})$ at all points inside the boundary. This is the essence of boundary integral methods. Note that the integral equation is 2-dimensional whereas the original problem is 3-dimensional. This reduction in dimensionality is why boundary integral methods are useful.

8.4 Problems on bounded regions

8.4.1 The Dirichlet problem

This is the problem of finding u in V given that

$$\begin{aligned}
(\Delta + k^2)u(\boldsymbol{x}) &= f(\boldsymbol{x}) \quad \text{in } V,\\
u(\boldsymbol{x}) &= g(\boldsymbol{x}) \quad \text{on } S.
\end{aligned} \tag{556}$$

We solve this in terms of the Kirchhoff–Helmholtz representation by eliminating the unknown $u_{,n}$ from the integral, that is, we attempt to find a Green's function such that

$$\begin{aligned}(\Delta_x + k^2)G(\boldsymbol{x},\boldsymbol{y}) &= \delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{in } V,\\ G(\boldsymbol{x},\boldsymbol{y}) &= 0 \quad \text{when } \boldsymbol{y} \text{ on } S.\end{aligned} \tag{557}$$

In practice it may be difficult to find such a G, but assuming G is known the solution is then, from the Kirchhoff-Helmholtz representation,

$$u(\boldsymbol{x}) = \int_V G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y}) d^3y + \int_S G_{,n_y}(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{y})\, d^2y. \tag{558}$$

8.4.2 The Neumann problem for the Helmholtz equation

This is the problem of finding u in V given that

$$(\Delta + k^2)u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{in } V, \qquad u_{,n}(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \text{on } S. \tag{559}$$

We solve this in terms of the Kirchhoff–Helmholtz representation by eliminating the unknown u from the integral, that is, we attempt to find a Green's function such that

$$(\Delta_x + k^2)G(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{in } V, \qquad G_{,n}(\boldsymbol{x},\boldsymbol{y}) = 0 \quad \text{when } \boldsymbol{y} \text{ on } S. \tag{560}$$

Assuming G can be found, the solution is then

$$u(\boldsymbol{x}) = \int_V G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, d^3y - \int_S G(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{y})\, d^2y. \tag{561}$$

8.4.3 The Neumann problem for the Poisson equation

As we have already discussed in Remark 1.7, the Neumann problem for the Poisson equation ($k = 0$),

$$\begin{aligned}\Delta u(\boldsymbol{x}) &= f(\boldsymbol{x}) \quad \text{in } V,\\ u_{,n}(\boldsymbol{x}) &= g(\boldsymbol{x}) \quad \text{on } S,\end{aligned} \tag{562}$$

is posed consistently only if

$$\int_V f(\boldsymbol{x})\, d^3x = \int_V \Delta u\, d^3x = \int_S u_{,n}\, d^2x = \int_S g(\boldsymbol{x})\, d^2x. \tag{563}$$

That is, $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ must satisfy the **compatibility condition** for the data of the Poisson equation with Neumann boundary conditions,

$$\int_V f(\boldsymbol{x}) d^3x = \int_S g(\boldsymbol{x}) d^2x. \tag{564}$$

By the same reasoning, for $\boldsymbol{x} \in V$ we find another compatibility condition, this time for the Green's function $G(\boldsymbol{x},\boldsymbol{y})$ (and integrating over $\boldsymbol{y}$),

$$\int_S G_{,n_y}(\boldsymbol{x},\boldsymbol{y})\, d^2y = \int_V \Delta_y G(\boldsymbol{x},\boldsymbol{y})\, d^3y = \int_V \delta(\boldsymbol{x}-\boldsymbol{y})\, d^3y = 1. \tag{565}$$

Hence we cannot impose $G_{,n_y}(\boldsymbol{x},\boldsymbol{y}) = 0$ for $\boldsymbol{y}$ on S!

However, it turns out that this is not necessary. Instead, consider the Green's function defined by

$$\Delta_x G(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{in } V, \tag{566}$$

$$G_{,n_y}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{A} \quad \text{for } \boldsymbol{y} \text{ on } S, \tag{567}$$

where

$$A := \int_S d^2y \tag{568}$$

is the area of S. Then the compatibility condition (565) for the Green's function is satisfied. The solution of the Neumann problem is, from the Kirchhoff-Helmholtz representation (555), given by

$$u(\boldsymbol{x}) = \int_V G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, d^3y + \int_S \left(\frac{u(\boldsymbol{y})}{A} - G(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{y}) \right) d^2y. \tag{569}$$

Now in the term

$$\frac{1}{A} \int_S u(\boldsymbol{y})\, d^2y \tag{570}$$

we do not know $u(\boldsymbol{y})$ on the boundary, so we cannot evaluate this integral, but it is just a constant (independent of $\boldsymbol{x}$). On the other hand, the solution to the Neumann problem is unique only up to adding an arbitrary constant anyway. So we can absorb the unknown additive constant (570) into this, and write

$$u(\boldsymbol{x}) = \int_V G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, d^3y - \int_S G(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{y})\, d^2y + C, \tag{571}$$

where C is an arbitrary constant.

The compatibility conditions (564) for the data or (565) for the Green's function only apply if the volume V and hence its boundary S are finite. If all or part of the boundary are at infinity, with the usual condition that $u \to 0$ at infinity, then there is no compatibility condition, and we must have $C = 0$.

8.4.4 Robin boundary conditions

This is the problem of finding u in V given that

$$\begin{aligned} (\Delta + k^2) u(\boldsymbol{x}) &= f(\boldsymbol{x}) \quad \text{in } V, \\ u_{,n}(\boldsymbol{x}) + \lambda(\boldsymbol{x}) u(\boldsymbol{x}) &= g(\boldsymbol{x}) \quad \text{on } S, \end{aligned} \tag{572}$$

where $f(\boldsymbol{x})$, $g(\boldsymbol{x})$ and $\lambda(\boldsymbol{x})$ are all given functions.

We solve this in terms of the Kirchhoff–Helmholtz representation by eliminating the unknown $u_{,n}$ from the problem using the fact that

$$u_{,n}(\boldsymbol{x}) = g(\boldsymbol{x}) - \lambda(\boldsymbol{x}) u(\boldsymbol{x}) \quad \text{on } S. \tag{573}$$

so the Kirchhoff-Helmholtz representation (555) becomes

$$u(\boldsymbol{x}) = \int_V G(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, d^3y + \int_S \left[G_{,n_y}(\boldsymbol{x},\boldsymbol{y}) + \lambda(\boldsymbol{y}) G(\boldsymbol{x},\boldsymbol{y}) \right] u(\boldsymbol{y})\, d^2y - \int_S G(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{y})\, d^2y. \tag{574}$$

Then, as we do not know u on the surface S, we choose G so that this term is eliminated. That is, we choose G to be a solution of

$$\begin{aligned} (\Delta_x + k^2) G(\boldsymbol{x},\boldsymbol{y}) &= \delta(\boldsymbol{x} - \boldsymbol{y}) \quad \text{in } V, \\ G_{,n_y}(\boldsymbol{x},\boldsymbol{y}) + \lambda(\boldsymbol{y}) G(\boldsymbol{x},\boldsymbol{y}) &= 0 \quad \text{when } \boldsymbol{y} \text{ on } S. \end{aligned} \tag{575}$$

Assuming G can be found, the solution is then

$$u(\boldsymbol{x}) = \int_V f(\boldsymbol{y}) G(\boldsymbol{x},\boldsymbol{y})\, d^3y - \int_S g(\boldsymbol{y}) G(\boldsymbol{x},\boldsymbol{y})\, d^2y. \tag{576}$$

8.5 The method of images

8.5.1 Example: Laplace equation with Neumann BCs on a plane

It remains to find a Green's function that obeys the homogeneous version of the boundary condition we want to impose. As an example, consider the half-space Neumann problem (see Fig. 12) for Laplace's equation:

$$\begin{aligned} \Delta u &= 0, \quad x_3 > 0, \\ u_{,x_3} &= v(x_1, x_2) \text{ on } x_3 = 0, \\ u &\to 0 \text{ as } x_3 \to \infty. \end{aligned} \tag{577}$$

As we are given $u_{,x_3} = -u_{,n}$ on the boundary, we choose a Green's function that satisfies

$$G_{,n_y}(\boldsymbol{x}, \boldsymbol{y}) = 0 \text{ on } y_3 = 0. \tag{578}$$

Technically, the surface we need to integrate over also includes $y_3 \to \infty$. However, we can assume that $G \to 0$ and $G_{,n} \to 0$ as $y_3 \to \infty$, hence eliminating the integral over those parts of the surface at ∞.

The Kirchhoff-Helmholtz representation (555) becomes

$$u(\boldsymbol{x}) = -\int_S G(\boldsymbol{x}, \boldsymbol{y}) u_{,n}(\boldsymbol{y})\, d^2y \tag{579}$$

or, with $u_{,n} = -u_{,x_3} = -v$,

$$u(\boldsymbol{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G\left(\boldsymbol{x}, (y_1, y_2, 0)\right) v(y_1, y_2)\, dy_1\, dy_2. \tag{580}$$

We still need to find a Green's function that obeys the homogeneous Neumann condition (578). This can be done by the **method of images**. We start with the free space Green's function

$$G_F(\boldsymbol{x}, \boldsymbol{y}) = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}. \tag{581}$$

This represents the effect at $\boldsymbol{x}$ of a unit source at $\boldsymbol{y}$. Now consider what would happen if there were another point source at the **image point**

$$\boldsymbol{y}' := (y_1, y_2, -y_3). \tag{582}$$

Because the sources are now symmetric under a reflection in the x_3-plane (see Fig. 12), so is the solution $u(\boldsymbol{x})$. Hence its x_3-derivative vanishes on the $x_3 = 0$ plane.

To add in this fictitious image charge, we use the Green's function

$$G(\boldsymbol{x}, \boldsymbol{y}) = G_F(\boldsymbol{x}, \boldsymbol{y}) + G_F(\boldsymbol{x}, \boldsymbol{y}') = -\frac{1}{4\pi}\left(\frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} + \frac{1}{|\boldsymbol{x} - \boldsymbol{y}'|}\right) \tag{583}$$

$$= -\frac{1}{4\pi}\left(\frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} + \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3)^2}}\right). \tag{584}$$

Clearly this obeys

$$G(\boldsymbol{x}, (y_1, y_2, y_3)) = G(\boldsymbol{x}, (y_1, y_2, -y_3)), \tag{585}$$

and so $G_{,y_3} = 0$ on $y_3 = 0$.

When we evaluate $G(\boldsymbol{x}, \boldsymbol{y})$ on $y_3 = 0$, the two terms in (584) become identical, and substituting into (580) we obtain

$$u(x_1, x_2, x_3) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{v(y_1, y_2)\, dy_1\, dy_2}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2}}. \tag{586}$$

One might object that G obeys $\Delta G = \delta(\boldsymbol{x}-\boldsymbol{y}) + \delta(\boldsymbol{x}-\boldsymbol{y}')$. But when $\boldsymbol{y}$ is inside the domain of the PDE then $\boldsymbol{y}'$ is outside and so we do not integrate over the region containing $\boldsymbol{y}'$, and so $\delta(\boldsymbol{x}-\boldsymbol{y}')$ does not make a contribution.

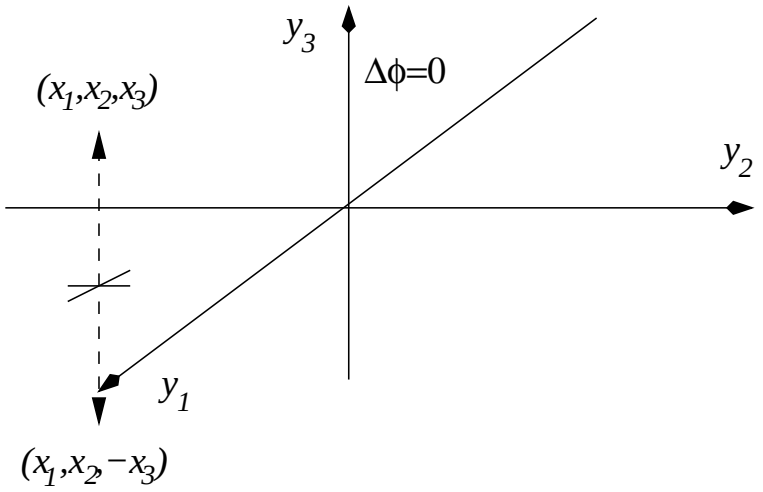

Figure 12: Method of images for the Neumann problem for Laplace's equation.

8.5.2 Example: Helmholtz equation with Dirichlet BCs on a plane

As a second example, consider now the (homogeneous) Helmholtz equation with (inhomogeneous) Dirichlet BCs on a plane,

$$\begin{aligned}(\Delta + k^2)u &= 0, \quad x_3 > 0, &(587)\\ u &= v(x_1, x_2), \quad x_3 = 0, &(588)\\ u &\to 0, \quad |\boldsymbol{x}| \to \infty. &(589)\end{aligned}$$

We need G to obey $G(\boldsymbol{x}, \boldsymbol{y}) = 0$ for $\boldsymbol{y} \in S$, that is for $y_3 = 0$. The required Green's function is given by

$$G(\boldsymbol{x}, \boldsymbol{y}) = G_F(\boldsymbol{x}, \boldsymbol{y}) - G_F(\boldsymbol{x}, \boldsymbol{y}'), \tag{590}$$

where G_F is the Green's function for the free space Helmholtz problem,

$$G_F(\boldsymbol{x}, \boldsymbol{y}) = -\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}, \tag{591}$$

and where the image point $\boldsymbol{y}' = (y_1, y_2, -y_3)$.

We then have

$$u(\boldsymbol{x}) = \int_S G_{,n_y}(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y})\, d^2y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -G_{,y_3}\left(\boldsymbol{x}, (y_1, y_2, 0)\right) v(y_1, y_2)\, dy_1\, dy_2, \tag{592}$$

where the sign comes from $\boldsymbol{n} = (0, 0, -1)$. (The rest of the example is left as an exercise)

8.6 Exercises

55. **Homework 24:** Fill in a missing step in the proof of the reciprocal theorem in the lecture notes by showing that

$$\int_S \left[G(\boldsymbol{x}, \boldsymbol{y}_2) G_{,n_x}(\boldsymbol{x}, \boldsymbol{y}_1) - G(\boldsymbol{x}, \boldsymbol{y}_1) G_{,n_x}(\boldsymbol{x}, \boldsymbol{y}_2)\right] d^2x = 0 \tag{593}$$

if G obeys either Dirichlet, Neumann or Robin boundary conditions for $\boldsymbol{x} \in S$.

56. **Homework 25:** (Begun in lecture, past exam question) Solve the PDE problem

$$\begin{aligned}(\Delta + k^2)u &= 0, \quad x_3 > 0, &(594)\\ u &= v(x_1, x_2), \quad x_3 = 0, &(595)\\ u &\to 0, \quad |\boldsymbol{x}| \to \infty. &(596)\end{aligned}$$

57. Find the solution $u(r)$ of

$$\begin{aligned}\Delta u &= f(r), \quad 0 \le r < R &(597)\\ \alpha u(R) + \beta u_{,r}(R) &= 0 &(598)\end{aligned}$$

(Poisson problem in three space dimensions inside a sphere with spherically symmetric source and spherically symmetric Robin BC) in the form

$$u(r) = \int_0^R G(r, s) f(s)\, ds. \tag{599}$$

Hint: Write the PDE in spherical coordinates. We effectively now have an ODE problem. Proceed from first principles, as we have done for constructing the Green's function for ODEs. It is implicit in the problem that $u(r)$ must obey the boundary condition $u'(0) = 0$. If $u'(0) \neq$, $u(x, y, z)$ would have a conical shape at the origin $r = 0$ (or $x = y = z = 0$), and this would correspond to a δ-function source.

9 The diffusion equation

In the previous Sections we have set out the general principles involved in solving a PDE by means of the Green's function technique. The method readily generates to PDEs other than the ones we have considered so far. The only additional complication is that we need to also consider time-dependent problems, like diffusion or wave equations. These are the topics of this and the following Section.

The diffusion equation (or heat equation) is

$$u_{,t'} - \kappa \Delta u = \hat{f}(\boldsymbol{x}, \hat{t}) \tag{600}$$

where κ is the diffusion constant (with dimension length^2/time). This can be reduced to the form

$$u_{,t} - \Delta u = f(\boldsymbol{x}, t) \tag{601}$$

by making the change of variable

$$t := \kappa \hat{t}, \qquad f(\boldsymbol{x}, t) := \frac{1}{\kappa} \hat{f}(\boldsymbol{x}, \hat{t}). \tag{602}$$

We shall assume that this change of variable has been made and only consider (601) in these notes.

We shall begin with the heat equation in one space dimension. This is interesting in its own right, but as we shall see, it is easy to find the Green's function in n space dimensions from the one in one space dimension.

9.1 The one-dimensional diffusion equation

The one-dimensional diffusion equation is

$$u_{,t} - u_{,xx} = f(x, t), \tag{603}$$

that is, we assume that u depends only on t and x. The associated causal Green's function, $G(x, t; y, \tau)$ satisfies

$$G_{,t} - G_{,xx} = \delta(t - \tau)\delta(x - y), \quad G = 0 \text{ if } t < \tau. \tag{604}$$

The causal solution of (603) is given by

$$u(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} G(x, t; y, \tau) f(y, \tau)\, dy\ d\tau. \tag{605}$$

In order to find $G(x, t; y, \tau)$ we introduce the variables

$$z = x - y, \quad T = t - \tau. \tag{606}$$

In terms of these variables, the problem for $G(x, t; y, \tau) = G(z, T)$ becomes

$$G_{,T} - G_{,zz} = \delta(T)\delta(z), \quad G(z, T) = 0 \text{ if } T < 0. \tag{607}$$

In order to accommodate the condition $G(z, T) = 0$ if $T < 0$ we put

$$G(z, T) = H(T) g(z, T). \tag{608}$$

Recalling that $dH(T)/dT = \delta(T)$ and $\delta(T)g(z, T) = \delta(T)g(z, 0)$, we find that

$$G_{,T} - G_{,zz} = \delta(T) g(z, 0) + H(T) \left(g_{,T} - g_{,zz} \right), \tag{609}$$

so that problem (607) for $G(z, T)$ is satisfied if $g(z, T)$ satisfies

$$g_{,T} = g_{,zz}, \quad g(z, 0) = \delta(z). \tag{610}$$

There are many ways of solving (610) to find $g(z,T)$. One particularly elegant way involves the use of **similarity variables**, leading to a **similarity ansatz**.

The basic idea behind the similarity method is to notice that if $\lambda > 0$ is a constant and we put

$$\bar{z} = \lambda z, \quad \bar{T} = \lambda^2 T, \quad \bar{g}(\bar{z},\bar{T}) = \lambda^{-1} g(z,T), \tag{611}$$

then

$$\bar{g}_{,\bar{T}} = \frac{\lambda^{-1}}{\lambda^2} g_{,T} = \frac{\lambda^{-1}}{\lambda^2} g_{,zz} = \bar{g}_{,\bar{z}\bar{z}}, \tag{612}$$

and

$$\bar{g}(\bar{z},0) = \lambda^{-1} g(z,0) = \lambda^{-1}\delta(z) = \delta(\lambda z) = \delta(\bar{z}), \tag{613}$$

[The third equality is the identity $\delta(ax) = \delta(x)/|a|$.] That is, the problem (610) is invariant under the change of variables (611), in the sense that it is the same in barred variables and unbarred variables. That is, if (in unbarred variables),

$$g_{,T} = g_{,zz}, \quad g(z,0) = \delta(z) \tag{614}$$

then (in barred variables)

$$\bar{g}_{,\bar{T}} = \bar{g}_{,\bar{z}\bar{z}}, \quad \bar{g}(\bar{z},0) = \delta(\bar{z}) \ . \tag{615}$$

Given that the problem is invariant under the transformation (611), it is sensible to look for solutions of the problem in terms of variables which are invariant under the same transformation. It is fairly obvious that both of

$$\psi = \sqrt{T}\, g = \sqrt{\lambda^2 T}\, (\lambda^{-1} g) = \sqrt{\bar{T}}\, \bar{g} \tag{616}$$

and

$$\xi = z/\sqrt{T} = \lambda z/\sqrt{\lambda^2 T} = \bar{z}/\sqrt{\bar{T}} \tag{617}$$

are invariant under the transformation (611). As x and T are independent variables and g is a dependent variable, it is reasonable to look for a solution, in terms of the invariants ψ and ξ, in the form

$$\psi = \psi(\xi). \tag{618}$$

Since $\psi = \sqrt{T}\, g$ and $\xi = z/\sqrt{T}$ this amounts to looking for a solution of the form

$$g(z,T) = \frac{1}{\sqrt{T}}\psi(\xi), \qquad \text{where } \xi = \frac{z}{\sqrt{T}}. \tag{619}$$

If we look for a solution of this form we find that

$$\begin{aligned} g_{,T} &= -\frac{1}{2T^{3/2}}\psi(\xi) + \frac{1}{T^{1/2}}\xi_{,T}\frac{d\psi}{d\xi}(\xi) \\ &= -\frac{1}{2T^{3/2}}\psi(\xi) + \frac{1}{T^{1/2}}\left(-\frac{z}{2T^{3/2}}\right)\psi'(\xi) \\ &= -\frac{1}{2T^{3/2}}[\psi(\xi) + \xi\psi'(\xi)], \end{aligned} \tag{620}$$

and that

$$g_{,zz} = \frac{1}{T^{1/2}}\frac{\partial^2}{\partial z^2}\psi(\xi) = \frac{1}{T^{3/2}}\psi''(\xi), \tag{621}$$

since $\partial/\partial z = (\partial\xi/\partial z)d/d\xi = T^{-1/2}d/d\xi$.

This shows that

$$-\frac{1}{2T^{3/2}}\left[\psi(\xi) + \xi\psi'(\xi)\right] = \frac{1}{T^{3/2}}\psi''(\xi), \tag{622}$$

so that $\psi(\xi)$ satisfies the ordinary differential equation

$$\psi''(\xi) + \tfrac{1}{2}[\xi\psi'(\xi) + \psi(\xi)] = 0, \tag{623}$$

which can be written as

$$\psi''(\xi) + \tfrac{1}{2}[\xi\psi(\xi)]' = 0. \tag{624}$$

This can be integrated once to give

$$\psi'(\xi) + \tfrac{1}{2}\xi\psi(\xi) = C. \tag{625}$$

This is a first order linear ordinary differential equation for $\psi(\xi)$. With the integrating factor $\exp(\xi^2/4)$ it is equivalent to

$$\frac{d}{d\xi}\left(e^{\frac{\xi^2}{4}}\psi\right) = Ce^{\frac{\xi^2}{4}}. \tag{626}$$

This integrates to give

$$\psi(\xi) = Ce^{-\frac{\xi^2}{4}}\int_0^\xi e^{\frac{\eta^2}{4}}\,d\eta + Be^{-\frac{\xi^2}{4}}. \tag{627}$$

In order to get $\psi(\xi)$ to vanish fast enough as $\xi \to \pm\infty$ (that is, in order to get $g(x,0) = \delta(x)$) we have to take $C = 0$. [One way of seeing this is to note from (625) that for $C \neq 0$, $\psi \simeq 2C/\xi$ as $\xi \to \pm\infty$: asymptotically, the last two terms of the ODE balance each other, while the first term becomes negligible. Hence for $C \neq 0$, $g \sim 1/z$, but $\int z^{-1}\,dz$ does not converge.] With $C = 0$, we have

$$\psi(\xi) = Be^{-\frac{\xi^2}{4}}. \tag{628}$$

Subsituting this into (619) gives

$$g(z,T) = \frac{B}{\sqrt{T}}\exp\left(-\frac{z^2}{4T}\right). \tag{629}$$

To determine the constant B we note that

$$\int_{-\infty}^{\infty} g(z,0)\,dz = \int_{-\infty}^{\infty}\delta(z)\,dz = 1. \tag{630}$$

We also have

$$\int_{-\infty}^{\infty} g(z,T)\,dz = \frac{B}{\sqrt{T}}\int_{-\infty}^{\infty}\exp\left(-\frac{z^2}{4T}\right)\,dz \tag{631}$$

and putting $q = z/2\sqrt{T}$ this becomes

$$\int_{-\infty}^{\infty} g(z,T)\,dz = 2B\int_{-\infty}^{\infty} e^{-q^2}dq = 2B\sqrt{\pi}. \tag{632}$$

This is valid for any $T > 0$, so taking the limit $T \to 0$

$$\int_{-\infty}^{\infty} g(z,0)\,dz = 2B\sqrt{\pi} = 1 \tag{633}$$

and hence

$$B = \frac{1}{2\sqrt{\pi}}. \tag{634}$$

Recalling that $G(z,T) = H(T)g(z,T)$, $z = x-y$ and $T = t-\tau$ we conclude that the one dimensional causal Green's function for the heat equation is

$$G(x,t;y,\tau) = \frac{H(t-\tau)}{2\sqrt{\pi(t-\tau)}}\exp\left[-\frac{(x-y)^2}{4(t-\tau)}\right]. \tag{635}$$

9.2 The initial-value problem in one dimension

In practice, we are often interested in finding the solutions of initial value problems, rather than causal solutions. Consider the initial value problem

$$\begin{aligned} u_{,t} &= u_{,xx}, \quad t > 0, \\ u(x,0) &= g(x). \end{aligned} \tag{636}$$

Since we are only interested in $u(x,t)$ for $t > 0$, we use again a trick we have seen before, and write

$$\psi(x,t) = H(t)u(x,t), \tag{637}$$

so $\psi = 0$ for $t < 0$ and $\psi = u$ for $t > 0$. Compare the definition of $z(t)$ in Eq. (422) above. Then

$$\psi_{,t} = \delta(t)u(x,t) + H(t)u_{,t} \tag{638}$$

and since $\delta(t)u(x,t) = \delta(t)u(x,0)$ (this is just $\delta(t)f(t) = \delta(t)f(0)$ – think of x as some fixed parameter), this becomes

$$\psi_{,t} = \delta(t)g(x) + H(t)u_{,t}. \tag{639}$$

As $H(t)$ does not depend on x,

$$\psi_{,xx} = H(t)u_{,xx} \tag{640}$$

and hence

$$\psi_{,t} - \psi_{,xx} = \delta(t)g(x) + H(t)\left(u_{,t} - u_{,xx}\right) \tag{641}$$

so that

$$\psi_{,t} - \psi_{,xx} = \delta(t)g(x). \tag{642}$$

The causal solution of this problem is

$$\psi(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4(t-\tau)}} \delta(\tau)g(y)\,dy\,d\tau. \tag{643}$$

For $t > 0$, the integral $\int_{-\infty}^{t} \ldots \delta(\tau)\,d\tau$ simply picks out the value of the integrand at $\tau = 0$, while for $t < 0$ the point $\tau = 0$ is not in the integration range, so that

$$\psi(x,t) = H(t)\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} g(y)\,dy, \tag{644}$$

and hence for $t > 0$ we have

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} g(y)\,dy. \tag{645}$$

Example 9.1. Consider the particular case $g(x) = H(x)$. Since $H(y) = 0$ for $y < 0$ and $H(y) = 1$ for $y > 0$,

$$u(x,t) = \int_{0}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}}\,dy. \tag{646}$$

Now put $q = (y-x)/2\sqrt{t}$ to obtain

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-x/2\sqrt{t}}^{\infty} e^{-q^2}\,dq. \tag{647}$$

This integral cannot be evaluated in terms of elementary functions, but we can express it in terms of a special function, the complementary error function.

$$u(x,t) = \frac{1}{2}\mathrm{erfc}\left(-\frac{x}{2\sqrt{t}}\right). \tag{648}$$

The **error function** is defined as

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-q^2}\,dq, \tag{649}$$

and it has the properties (see Fig. 13)

- $\mathrm{erf}(-x) = -\mathrm{erf}(x)$,
- $\mathrm{erf}(0) = 0$,

- $\mathrm{erf}(\infty) = 1$,
- $\mathrm{erf}(-\infty) = -1$,
- $\mathrm{erf}(x)$ is a monotonically increasing function of x.

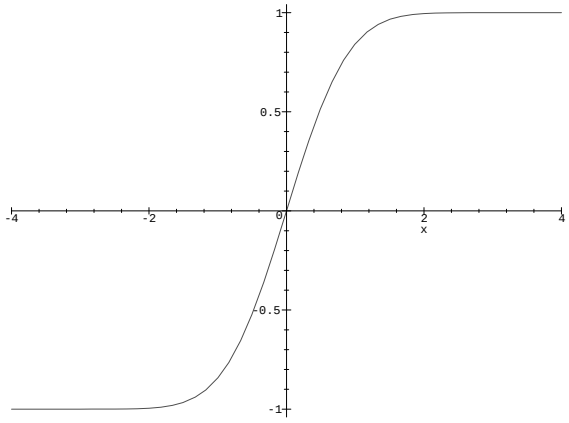

Figure 13: The error function $\mathrm{erf}(x)$.

The **complementary error function** is defined as

$$\mathrm{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-q^2}\, dq. \tag{650}$$

The error and complementary error functions are related by

$$\mathrm{erf}(x) + \mathrm{erfc}(x) = 1, \tag{651}$$

because

$$\frac{2}{\sqrt{\pi}} \left(\int_0^x e^{-q^2}\, dq + \int_x^\infty e^{-q^2}\, dq \right) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-q^2}\, dq = 1. \tag{652}$$

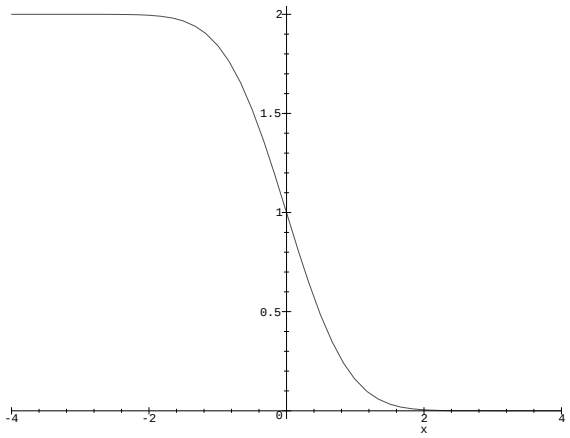

Figure 14: The complementary error function erfc(x).

Example 9.2. Show that if $u(x,t)$ satisfies

$$\begin{aligned} u_{,t} - u_{,xx} &= 0,\ t > 0, \\ u(x,0) &= e^{-\frac{x^2}{\ell^2}} \end{aligned} \tag{653}$$

then

$$u(0,t) = \frac{\ell}{\sqrt{\ell^2 + 4t}}. \tag{654}$$

We have

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} e^{-\frac{y^2}{\ell^2}}\, dy \tag{655}$$

Hence

$$u(0,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-y^2\left(\frac{1}{\ell^2}+\frac{1}{4t}\right)} dy \tag{656}$$

and putting $y\sqrt{1/\ell^2 + 1/4t} =: q$ this becomes

$$u(0,t) = \frac{1}{2\sqrt{\pi t}} \frac{1}{\sqrt{\frac{1}{\ell^2}+\frac{1}{4t}}} \int_{-\infty}^{\infty} e^{-q^2} dq = \frac{\ell}{\sqrt{\ell^2 + 4t}}. \tag{657}$$

9.3 The three-dimensional problem

The (causal) Green's function, $G(\boldsymbol{x}, t; \boldsymbol{y}, \tau)$ for the diffusion equation is defined by

$$G_{,t} - \Delta_x G = \delta(t-\tau)\delta(\boldsymbol{x} - \boldsymbol{y}), \quad G = 0 \text{ if } t < \tau. \tag{658}$$

To find it, let $G(\boldsymbol{x}, t, \boldsymbol{y}, \tau) = G(z, T)$, where $z := |\boldsymbol{x} - \boldsymbol{y}|$ and $T := t - \tau$, and recall that

$$g(z,T) = \frac{1}{2\sqrt{\pi T}} \exp\left(-\frac{z^2}{4T}\right) \tag{659}$$

obeys

$$g_{,T} = g_{,zz}, \quad g(z,0) = \delta(z). \tag{660}$$

Therefore $G(z,T) = H(T)g(z,T)$ obeys

$$G_{,T} - G_{,zz} = \delta(T)g(z,0) + H(T)\left(g_{,T} - g_{,zz}\right) = \delta(T)\delta(z) \tag{661}$$

In three dimensions, try

$$G(\boldsymbol{z},T) = H(T)g(z_1,T)g(z_2,T)g(z_3,T). \tag{662}$$

Clearly this vanishes for $T<0$ as required. Furthermore, it also obeys

$$\begin{aligned} G_{,T} - \Delta G &= G_{,T} - (G_{,z_1z_1} + G_{,z_2z_2} + G_{,z_3z_3}) \\ &= \delta(T)\, g(z_1,0)\, g(z_2,0)\, g(z_3,0) \\ &\quad + H(T)\left\{[g_{,t}(z_1,t) - g_{,z_1z_1}(z_1,t)]\, g(z_2,T)g(z_3,T) + 2 \text{ more terms}\right\} \\ &= \delta(T)\delta(z_1)\delta(z_2)\delta(z_3) = \delta(T)\delta(\boldsymbol{z}). \end{aligned} \tag{663}$$

We have shown that the causal Green's function for the heat equation in three dimensions in free space is

$$G(\boldsymbol{x},t;\boldsymbol{y},\tau) = \frac{H(t-\tau)}{8(\pi(t-\tau))^{3/2}} \exp\left[-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4(t-\tau)}\right]. \tag{664}$$

By a similar argument, the two dimensional Green's function is given by

$$G(\boldsymbol{x},t;\boldsymbol{y},\tau) = \frac{H(t-\tau)}{4\pi(t-\tau)} \exp\left[-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4(t-\tau)}\right] \tag{665}$$

where now $\boldsymbol{x} = (x_1,x_2)$, $\boldsymbol{y} = (y_1,y_2)$. Clearly this method works in any number of space dimensions.

Remark 9.3. Consider the initial value problem for the heat equation in three dimensions,

$$\begin{aligned} u_{,t} - \Delta u &= 0, \\ u(\boldsymbol{x},0) &= f(\boldsymbol{x}). \end{aligned} \tag{666}$$

We turn this initial value problem into a causal problem in the usual way, that is, we assume $t>0$ and write

$$\psi(\boldsymbol{x},t) = H(t)u(\boldsymbol{x},t). \tag{667}$$

Then ψ satisfies

$$\psi_{,t} - \Delta\psi = \delta(t)f(\boldsymbol{x}) \tag{668}$$

Thus

$$\psi(\boldsymbol{x},t) = \int_{-\infty}^{t}\int_{\mathbb{R}^3} G(\boldsymbol{x},t;\boldsymbol{y},\tau)\delta(\tau)f(\boldsymbol{y})\, d^3\boldsymbol{y}\, d\tau. \tag{669}$$

For $t>0$ the $\delta(\tau)$ in the integral $\int_{-\infty}^{t}\ldots d\tau$ simply picks out the value of the integrand at $\tau=0$, and for $t<0$ we get nothing, so

$$\psi(\boldsymbol{x},t) = H(t)\int_{\mathbb{R}^3} G(\boldsymbol{x},t;\boldsymbol{y},0)f(\boldsymbol{y})\, d^3\boldsymbol{y}. \tag{670}$$

Thus for $t>0$ (when $\psi = u$) we have

$$u(\boldsymbol{x},t) = \int_{\mathbb{R}^3} G(\boldsymbol{x},t;\boldsymbol{y},0)f(\boldsymbol{y})\, d^3\boldsymbol{y}. = \int_{\mathbb{R}^3} \frac{1}{8(\pi t)^{3/2}} \exp\left[-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t}\right] f(\boldsymbol{y})\, d^3\boldsymbol{y}. \tag{671}$$

Example 9.4. Suppose that $u(\boldsymbol{x},t)$ satisfies the initial value problem

$$u_{,t} = \Delta u,\ t>0, \quad u(\boldsymbol{x},0) = (1-|\boldsymbol{x}|)H(1-|\boldsymbol{x}|). \tag{672}$$

Show that for $t>0$

$$u(\boldsymbol{0},t) = \operatorname{erf}\left(\frac{1}{2\sqrt{t}}\right) + 4\sqrt{\frac{t}{\pi}}\left(e^{-1/4t}-1\right). \tag{673}$$

The solution of the initial value problem is

$$u(\boldsymbol{x},t)=\frac{1}{8(\pi t)^{3/2}}\int_{\mathbb{R}^3}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t}\right)(1-|\boldsymbol{y}|)H(1-|\boldsymbol{y}|)d^3\boldsymbol{y}. \tag{674}$$

and hence

$$u(\boldsymbol{0},t)=\frac{1}{8(\pi t)^{3/2}}\int_{\mathbb{R}^3}\exp\left(-\frac{|\boldsymbol{y}|^2}{4t}\right)(1-|\boldsymbol{y}|)H(1-|\boldsymbol{y}|)d^3\boldsymbol{y}. \tag{675}$$

As this integral depends only on $r=|\boldsymbol{y}|$ we change to spherical polar coordinates;

$$u(\boldsymbol{0},t)=\frac{1}{8(\pi t)^{3/2}}\int_0^\infty\int_0^{2\pi}\int_0^{\pi}\exp\left(-\frac{r^2}{4t}\right)(1-r)H(1-r)r^2\sin\theta\,d\theta\,d\varphi\,dr. \tag{676}$$

The integrals separate and we have

$$\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\varphi=4\pi, \tag{677}$$

so that

$$u(\boldsymbol{0},t)=\frac{1}{2\sqrt{\pi}(t)^{3/2}}\int_0^\infty\exp\left(-\frac{r^2}{4t}\right)(1-r)H(1-r)r^2\,dr. \tag{678}$$

Since $H(1-r)=0$ for $r>1$ and $H(1-r)=1$ for $r<1$ this becomes

$$u(\boldsymbol{0},t)=\frac{1}{2\sqrt{\pi}(t)^{3/2}}\int_0^1\exp\left(-\frac{r^2}{4t}\right)(1-r)r^2\,dr. \tag{679}$$

If we put $q=r/2\sqrt{t}$ this becomes

$$u(\boldsymbol{0},t)=\frac{4}{\sqrt{\pi}}\int_0^{1/2\sqrt{t}}e^{-q^2}\left(q^2-2\sqrt{t}q^3\right)dq, \tag{680}$$

and hence, integrating by parts, we obtain the desired solution (673).

Note that we can always integrate an integral of the form $\int e^{-q^2}q^n\,dq$ by parts to get either an error function if n is even, or an exponential function if n is odd.

9.4 Exercises

58. **Homework 26:** (A good way of understanding the lecture better) Show that an alternative similarity ansatz to the one in the lectures is $\xi:=z^2/T$ and $\psi:=zg$, and use this to find $g(z,T)$.

59. **Homework 27:** (Exam-style) Find the solution of

$$\begin{aligned} u_{,t} &= u_{,xx}, \qquad t>0, &(681)\\ u &\to 0, \qquad x\to\pm\infty, &(682)\\ u(x,0) &= H(x+1)H(1-x) &(683)\end{aligned}$$

    in terms of error functions.

60. **Homework 28:** (Exam-style) Evaluate

$$u(\boldsymbol{0},t)=\frac{1}{2\sqrt{\pi}t^{3/2}}\int_0^1\exp\left(-\frac{r^2}{4t}\right)(1-r)r^2\,dr. \tag{684}$$

10 The wave equation

10.1 One space dimension

10.1.1 The Green's function in one space dimension

Consider the Green's function for the wave equation in one space dimension,

$$\frac{1}{c^2}G_{,tt} - G_{,xx} = \delta(x-y)\delta(t-\tau). \tag{685}$$

If we put $z = x - y$ and $T = t - \tau$ this becomes

$$\frac{1}{c^2}G_{,TT} - G_{,zz} = \delta(z)\delta(T). \tag{686}$$

As a boundary condition, we want the solution $G(z,T)$ which has outgoing wave behaviour, that is, the solution for which waves move outwards from $z = 0$ towards infinity.

Outside the point $(z = 0, T = 0)$, the d'Alembert solution applies, which can be written as

$$G(z,T) = F(T - z/c) + E(T + z/c). \tag{687}$$

Here $F(T - z/c)$ represents a wave travelling towards increasing z, while $E(T + z/c)$ represents a wave travelling towards decreasing z.

Now, if we want a wave that travels away from its source at $z = 0$, then for $z > 0$ we need the wave moving towards increasing z, but for $z < 0$ we need it to travel towards decreasing z. Consider therefore the ansatz

$$G(z,t) = f\left(T - \frac{|z|}{c}\right). \tag{688}$$

Note that $|z| = z$ for $z > 0$ and $|z| = -z$ for $z < 0$. It is easy to see that (688) has the property of travelling away from $z = 0$ for either $z < 0$ or $z > 0$. However, at $z = 0$ it does not obey the wave equation. But that may be to the good, because $d|z|/dz = \mathrm{sgn}(z)$ and $d\,\mathrm{sgn}(z)/dz = 2\delta(z)$.

Taking two time derivatives of (688), we easily find

$$\frac{1}{c^2}G_{,TT} = \frac{1}{c^2}f''(T - |z|/c). \tag{689}$$

Taking a first space derivative, we find

$$G_{,z} = -\frac{\mathrm{sgn}(z)}{c}f'(T - |z|/c), \tag{690}$$

and hence taking another space derivative we find

$$G_{,zz} = -\frac{2\delta(z)}{c}f'(T - |z|/c) + \left(\frac{-\mathrm{sgn}(z)}{c}\right)^2 f''(T - |z|/c). \tag{691}$$

Since $\mathrm{sgn}(z)^2 = 1$ whatever z is and $\delta(z)f(z) = \delta(z)f(0)$, this becomes

$$G_{,zz} = -\frac{2\delta(z)}{c}f'(T) + \frac{1}{c^2}f''(T - |z|/c). \tag{692}$$

Hence

$$\frac{1}{c^2}G_{,TT} - G_{,zz} = \frac{2\delta(z)}{c}f'(T). \tag{693}$$

Comparing with (686), we have

$$f'(T) = \frac{c}{2}\delta(T) \tag{694}$$

and hence

$$f(T) = \frac{c}{2}H(T) \tag{695}$$

plus a constant, which is seen to be zero from causality: we want $f(t - \tau) = 0$ for $t < \tau$. Thus

$$G(z,T) = f(T - |z|/c) = \frac{c}{2}H(T - |z|/c) \tag{696}$$

so that

$$G(x,t;y,\tau) = \frac{c}{2}H\left[(t-\tau) - |x-y|/c\right]. \tag{697}$$

10.1.2 The initial value problem in one space dimension

Often, instead of wanting the causal solution, we want to solve the initial value problem.

$$\frac{1}{c^2}u_{,tt} - u_{,xx} = 0,\ t > 0, \quad u(x,0) = f(x), \quad u_{,t}(x,0) = g(x). \tag{698}$$

We can turn this into a problem solvable with the causal Green's function by the usual method. That is, put

$$\psi(x,t) = H(t)u(x,t), \tag{699}$$

so that $\psi = u$ for $t > 0$. Then, as usual, we have

$$\psi_{,t} = \delta(t)u(x,0) + H(t)u_{,t}, \tag{700}$$

where we have used that $\delta(t)u(x,t) = \delta(t)u(x,0)$,

$$\psi_{,tt} = \delta'(t)u(x,0) + \delta(t)u_{,t}(x,0) + H(t)u_{,tt}, \tag{701}$$

and note that the naive Leibniz rule does not apply. Also

$$\psi_{,xx} = H(t)u_{,xx}. \tag{702}$$

Thus

$$\frac{1}{c^2}\psi_{,tt} - \psi_{,xx} = \frac{1}{c^2}\left[\delta'(t)f(x) + \delta(t)g(x) + H(t)\left(u_{,tt} - c^2u_{,xx}\right)\right] \tag{703}$$

and hence

$$\frac{1}{c^2}\psi_{,tt} - \psi_{,xx} = \frac{1}{c^2}\left(\delta'(t)f(x) + \delta(t)g(x)\right). \tag{704}$$

The source term vanishes except at $t = 0$. If we solve this with the causal Green's function (697) we therefore get a solution $\psi(x,t)$ that vanishes for $t < 0$, which is precisely what we want, so the causal Green's function is the correct one to use.

For $t > 0$, $\psi = u$ and we have

$$u = \frac{1}{c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,t;y,\tau)\left(\delta'(\tau)f(y) + \delta(\tau)g(y)\right)dy\,d\tau. \tag{705}$$

where G is the one dimensional Greens function. Using the definitions of $\delta(\tau)$ and its derivative $\dot{\delta}(\tau)$, we obtain

$$u(x,t) = \frac{1}{c^2}\int_{-\infty}^{\infty}\left[G(x,t;y,0)g(y) - G_{,\tau}(x,t;y,0)f(y)\right]dy. \tag{706}$$

Now from (697) we find

$$G_{,\tau}(x,t;y,0) = -\frac{c}{2}\delta(t - |x-y|/c), \tag{707}$$

and so

$$u(x,t) = \frac{1}{2c}\left[\int_{-\infty}^{\infty} H(t-|x-y|/c)g(y)\,dy + \int_{-\infty}^{\infty}\delta(t-|x-y|/c)f(y)\,dy\right]. \tag{708}$$

Finally note that

$$t - |x-y|/c < 0 \quad \Leftrightarrow \quad y < x - ct \text{ or } y > x + ct, \tag{709}$$

$$t - |x-y|/c = 0 \quad \Leftrightarrow \quad y = x - ct \text{ or } y = x + ct, \tag{710}$$

$$t - |x-y|/c > 0 \quad \Leftrightarrow \quad x - ct < y < x + ct, \tag{711}$$

so that

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy. \tag{712}$$

It is interesting to write this explicitly in the d'Alembert form. In terms of the primitive function $\hat{g}$ of g, defined by

$$\hat{g}(z) := \frac{1}{c}\int^{z} g(y)\,dy, \tag{713}$$

we can write

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + \hat{g}(x+ct)\right] + \frac{1}{2}\left[f(x-ct) - \hat{g}(x-ct)\right], \tag{714}$$

which is of the form

$$u(x,t) = F(x-ct) + E(x+ct). \tag{715}$$

10.2 The three-dimensional problem

10.2.1 The Green's function in three space dimensions

The three dimensional wave equation problem is

$$\frac{1}{c^2}u_{,tt} - \Delta u = f(\boldsymbol{x}, t), \tag{716}$$

together with a radiation condition. Specifically, waves should travel outwards from points where $f(\boldsymbol{x}, t) \neq 0$ towards infinity rather than travel in from infinity.

The associated Green's function, $G(\boldsymbol{x}, t; \boldsymbol{y}, \tau)$ satisfies

$$\frac{1}{c^2}G_{,tt} - \Delta_x G = \delta(t-\tau)\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{717}$$

together with the radiation condition that disturbances should radiate away from the point of disturbance, $\boldsymbol{x} = \boldsymbol{y}$, rather than towards it. If we introduce $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$ and $T = t - \tau$ then (717) becomes

$$\frac{1}{c^2}G_{,TT} - \Delta_z G = \delta(T)\delta(\boldsymbol{z}) \tag{718}$$

Since $\delta(\boldsymbol{z}) = \delta(r)/4\pi r^2$, where $r = |\boldsymbol{z}|$ it follows that $G = G(r, T)$. Using this, and the identity (473), we find

$$\frac{1}{c^2}G_{,TT} - \frac{1}{r}(rG)_{,rr} = \delta(T)\frac{\delta(r)}{4\pi r^2}, \tag{719}$$

For $r \neq 0$ we have $\delta(r) = 0$, and after multiplying through by r we obtain

$$\frac{1}{c^2}(rG)_{,TT} - (rG)_{,rr} = 0, \tag{720}$$

which is the one-dimensional wave equation for rG. Its general solution can be written as

$$G(r, T) = \frac{F(T - r/c)}{4\pi r} + \frac{E(T + r/c)}{4\pi r} \tag{721}$$

for some functions $F(T - r/c)$ and $E(T + r/c)$. (The factor of $1/4\pi$ is introduced purely for convenience in what follows.)

The radiation condition that disturbances should radiate away from $\boldsymbol{x} = \boldsymbol{y}$ rather than towards it becomes the condition that disturbances should move away from $r = 0$ towards increasing r. Hence we must take

$$G(r, T) = \frac{F(T - r/c)}{4\pi r}. \tag{722}$$

This solution is singular as $r \to 0$, so we define a "generalised version" of it by

$$G(r, T) = \lim_{\epsilon \to 0}\left(H(r-\epsilon)\frac{F(T - r/c)}{4\pi r}\right) \tag{723}$$

where the point $r = 0$ is excluded. We then use the result from Sec. 7.4 that

$$\lim_{\epsilon \to 0}\Delta_z\left(\frac{H(r-\epsilon)f(r)}{4\pi r}\right) = \frac{f''(r)}{4\pi r} - f(0)\delta(\boldsymbol{z}), \tag{724}$$

where $r := |\boldsymbol{z}|$, to find

$$\Delta_z G = \frac{F''(T - r/c)}{4\pi r c^2} - F(T)\delta(\boldsymbol{z}). \tag{725}$$

We also have

$$G_{,TT} = \frac{F''(T - r/c)}{4\pi r}, \tag{726}$$

and putting the two together, we have

$$\frac{1}{c^2}G_{,TT} - \Delta_z G = F(T)\delta(\boldsymbol{z}). \tag{727}$$

Thus we need

$$F(T) = \delta(T) \tag{728}$$

and so

$$G(r,T) = \frac{\delta(T - r/c)}{4\pi r}. \tag{729}$$

Recalling our shorthands $T = t - \tau$, $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$ and $r = |\boldsymbol{z}|$, written out in full this is

$$G(\boldsymbol{x}, t; \boldsymbol{y}, \tau) = \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \delta\left(t - \tau - \frac{1}{c} |\boldsymbol{x} - \boldsymbol{y}|\right). \tag{730}$$

Note that $G = 0$ except where $|\boldsymbol{x} - \boldsymbol{y}| = c(t - \tau)$. The δ-function here is the scalar one, not the three-dimensional one.

10.2.2 Retarded potentials

The causal solution of the problem

$$\frac{1}{c^2} u_{,tt} - \Delta u = f(\boldsymbol{x}, t) \tag{731}$$

in three space dimensions is

$$u(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} f(\boldsymbol{y}, \tau) \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{y}|} \delta\left(t - \tau - \frac{1}{c} |\boldsymbol{x} - \boldsymbol{y}|\right) d^3y \, d\tau. \tag{732}$$

The integral over τ simply picks out the value of the integrand at $\tau = t - |\boldsymbol{x} - \boldsymbol{y}| / c$, and so

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\boldsymbol{y}, t - |\boldsymbol{x} - \boldsymbol{y}| / c)}{|\boldsymbol{x} - \boldsymbol{y}|} \, d^3y. \tag{733}$$

This form of the solution is called a **retarded potential**. Essentially, the solution at point $\boldsymbol{x}$ and time t represents the superposition (in the form of an integral over space) of disturbances at points $\boldsymbol{y}$, and this is attenuated by a factor of 1/distance. This looks very similar to the Greens' function solution of the Poisson equation; hence the name "potential". But this source is evaluated not at time t, but at an earlier time, namely earlier by the time takes for the disturbance to travel in a straight line at speed c; hence the name "retarded".

Example 10.1. Find the field generated by a time harmonic point source at the origin which is switched on at time $t = 0$:

$$\frac{1}{c^2} u_{,tt} - \Delta u = H(t) \delta(\boldsymbol{x}) e^{i\omega t}. \tag{734}$$

Using the retarded potential integral (733) with $f(\boldsymbol{x}, t) = \delta(\boldsymbol{x}) H(t) e^{i\omega t}$, we have

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\delta(\boldsymbol{y}) H(t - |\boldsymbol{x} - \boldsymbol{y}| / c) e^{i\omega(t - |\boldsymbol{x} - \boldsymbol{y}|/c)}}{|\boldsymbol{x} - \boldsymbol{y}|} \, d^3y = \frac{H(t - |\boldsymbol{x}| / c)}{4\pi |\boldsymbol{x}|} e^{i\omega(t - |\boldsymbol{x}|/c)}. \tag{735}$$

The Heaviside function confines the solution to the expanding sphere

$$|\boldsymbol{x}| \leq ct \tag{736}$$

whose radius expands at the wave speed c. The surface $|\boldsymbol{x}| = ct$ of this sphere represents the wave front of the expanding sphere of disturbance. Once the wave front has passed a given point $\boldsymbol{x}$, so that $\boldsymbol{x}$ is inside the sphere, the solution is simply

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi |\boldsymbol{x}|} e^{i\omega(t - |\boldsymbol{x}|/c)} \tag{737}$$

and the phase of the oscillation at $\boldsymbol{x}$ differs from the phase of the oscillation at the source of the disturbance, $\boldsymbol{0}$, by a $\omega |\boldsymbol{x}| / c = k|\boldsymbol{x}|$. This phase difference is simply the time it takes the

disturbance to propagate from $\mathbf{0}$ to $\boldsymbol{x}$, travelling at speed c. The factor of $1/(4\pi\,|\boldsymbol{x}|)$ says that the amplitude decays inversely with the distance from the source.

Discarding the time factor in (737), this has the same form as the Green's function for the Helmholtz equation (but with the opposite sign). This is not surprising since, for $|\boldsymbol{x}| < ct$, we have

$$\frac{1}{c^2}u_{,tt} = -\frac{\omega^2}{c^2}u = -k^2 u \tag{738}$$

and hence

$$\frac{1}{c^2}u_{,tt} - \Delta u = H(t)\delta(\boldsymbol{x})e^{i\omega t} \tag{739}$$

reduces to

$$(\Delta + k^2)u = -H(t)\delta(\boldsymbol{x})e^{i\omega t} \tag{740}$$

and writing $u = H(t)e^{i\omega t}\psi$ this becomes the problem for (minus) the Helmholtz Green's function;

$$(\Delta + k^2)\psi = -\delta(\boldsymbol{x}) \tag{741}$$

10.3 The method of descent

10.3.1 From three to one dimensions

The one dimensional wave equation, with a source term, is

$$\frac{1}{c^2}u_{,tt} - u_{,xx} = f(x_1, t) \tag{742}$$

We can think of this as a three dimensional equation in **planar symmetry**

$$\frac{1}{c^2}u_{,tt} - \Delta u = f(x_1, t) \tag{743}$$

where, since the source term depends only on x_1 and t, we can choose $u(\boldsymbol{x}, t)$ to also depend only on x_1 and t. Thus the solution can be written as

$$u(x_1, t) = \int_{t=-\infty}^{\infty}\int_{y_1=-\infty}^{\infty}\left(\int_{y_2=-\infty}^{\infty}\int_{y_2=-\infty}^{\infty} G_3(\boldsymbol{x}, t; \boldsymbol{y}, \tau)\,dy_2\,dy_3\right) f(y_1, \tau)\,dy_1\,d\tau, \tag{744}$$

where $G_3(\boldsymbol{x}, t; \boldsymbol{y}, \tau)$ is the three dimensional Green's function (730). Because f depends only on y_1 and τ, we can evaluate the term in round brackets first.

But because the solution of the one-dimensional problem is also, by definition, given by

$$u(x_1, t) = \int_{-\infty}^{\infty}\int_{y_1=-\infty}^{\infty} G_1(x_1, t; y_1, \tau) f(y_1, \tau)\,dy_1\;d\tau, \tag{745}$$

we have an expression for the one-dimensional Greens' function G_1 as an integral over the three-dimensional one G_3, namely

$$G_1(x_1, t; y_1, \tau) = \int_{y_2=-\infty}^{\infty}\int_{y_3=-\infty}^{\infty} G_3(\boldsymbol{x}, t; \boldsymbol{y}, \tau)\,dy_2 dy_3 \tag{746}$$

This is called the **method of descent** (in the number of dimensions). It is equally valid for the Poisson, Helmholtz and diffusion equations (although it is rather pointless in the case of the diffusion equation, where we used the one-dimensional Green's function to find the three-dimensional one).

We now evaluate this integral. First we write $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$ and $T = t - \tau$ so that the problem becomes

$$G_1(z_1, T) = \int_{z_2=-\infty}^{\infty}\int_{z_3=-\infty}^{\infty} \frac{1}{4\pi\,|\boldsymbol{z}|}\delta\left(T - |\boldsymbol{z}|/c\right)\,dz_2 dz_3. \tag{747}$$

Now introduce cylindrical polar coordinates in which z_1 is the axial direction and

$$z_2 = r\cos\theta, \quad z_3 = r\sin\theta \tag{748}$$

so that

$$|\boldsymbol{z}| = \sqrt{z_1^2 + z_2^2 + z_3^2} = \sqrt{z_1^2 + r^2} \tag{749}$$

and

$$dz_2 dz_3 = r\, d\theta\, dr. \tag{750}$$

with $-\infty < z_1 < \infty$, $0 \le r < \infty$ and $0 \le \theta < 2\pi$. Then

$$G_1(z_1, T) = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} \frac{1}{4\pi\sqrt{z_1^2 + r^2}}\delta\left(T - \frac{1}{c}\sqrt{z_1^2 + r^2}\right) r\, d\theta\, dr. \tag{751}$$

As the integrand does not depend on θ, the integration over θ simply gives a factor of 2π, or

$$G_1(z_1, T) = \frac{1}{2}\int_0^{\infty} \frac{1}{\sqrt{z_1^2 + r^2}}\delta\left(T - \frac{1}{c}\sqrt{z_1^2 + r^2}\right) r\, dr \tag{752}$$

Now note that

$$\frac{d}{dr}\sqrt{z_1^2 + r^2} = \frac{r}{\sqrt{z_1^2 + r^2}} \tag{753}$$

so that we can make the substitution

$$q = \frac{1}{c}\sqrt{z_1^2 + r^2} \quad \Rightarrow \quad dq = \frac{1}{c}\frac{d}{dr}\left(\sqrt{z_1^2 + r^2}\right) dr \tag{754}$$

to obtain

$$G_1(z_1, T) = \frac{c}{2}\int_{|z_1|/c}^{\infty} \delta(T - q)\, dq. \tag{755}$$

If $|z_1|/c > T$ the integral is zero since $T - q$ is not in the integration range, and if $|z_1|/c < T$ the integral is one, since now the zero of the δ-function is inside the integration range. Thus

$$G_1(z_1, T) = \frac{c}{2}H(T - |z_1|/c), \tag{756}$$

and we have recovered (696), as expected.

10.3.2 From three to two dimensions

We can use the method of descent to deduce the two dimensional Greens function

$$G_2(x_1, x_2, t; y_1, y_2, \tau) \tag{757}$$

for the two dimensional wave equation

$$\frac{1}{c^2}u_{,tt} - (u_{,x_1x_1} + u_{,x_2x_2}) = f(x_1, x_2, t). \tag{758}$$

By starting from the three-dimensional wave equation in **cylindrical symmetry** we find that

$$G_2(x_1, x_2, t; y_1, y_2, \tau) = \int_{y_3=-\infty}^{\infty} G_3(\boldsymbol{x}, t; \boldsymbol{y}, \tau)\, dy_3. \tag{759}$$

Evaluating this integral is in principle similar to the calculation given above. Writing $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$, it can be shown that

$$G_2(\boldsymbol{x}, t; \boldsymbol{y}, \tau) = \frac{H(t - \tau - |\boldsymbol{x} - \boldsymbol{y}|/c)}{2\pi\sqrt{(t-\tau)^2 - |\boldsymbol{x} - \boldsymbol{y}|^2/c^2}}. \tag{760}$$

(In fact, the method of descent was invented not to find the one dimensional Green's function but rather, as the *easiest* way of finding the two dimensional Green's function!)

10.4 Exercises

61. **Homework 29:** Use the causal Green's function for the wave equation in one space dimension to solve the Cauchy problem with a source term

$$\begin{aligned} c^{-2}u_{,tt} - u_{,xx} &= s(x,t), && (761)\\ u(x,0) &= f(x), && (762)\\ u_{,t}(x,0) &= g(x). && (763) \end{aligned}$$

[Hint: No need to repeat the part already done in the lecture in detail, focus here on the source term that was absent in the lecture.]

62. **Homework 30:** a) Find the causal solution of

$$c^{-2}u_{,tt} - \Delta u = H(R - |\boldsymbol{x}|)H(t)\sin\omega t \tag{764}$$

in *three*-dimensional free space, where $R > 0$ and ω are constants, and simplify as much as possible (but not more). b) State the long wavelength and large distance approximations for this problem and find $u(\boldsymbol{x}, t)$ using these approximations. c) Find $u(\boldsymbol{0}, t)$ in closed form. [Hint: This is a bit fiddly. Use polar coordinates for $\boldsymbol{y}$, and in your final expression distinguish the three cases $t < 0$, $0 < t < c^{-1}R$ and $t > c^{-1}R$.]

63. Write out the causal solution of

$$c^{-2}u_{,tt} - \Delta u = f(\boldsymbol{x}, t) \tag{765}$$

in *two*-dimensional free space. Then eliminate the Heaviside function by restricting the integration domain instead. [Hint: The restricted integration domain can be described as the interior of the past lightcone of $(\boldsymbol{x}, t)$. What we are looking for is the two-dimensional equivalent of the retarded potential for the three-dimensional wave equation.]

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