RELATIVITY REVISION NOTES:

1 Time Dilation

A (proper) time interval τ (the time difference between two events *at the same place* in one frame of reference), is observed to be dilated to τ' in an inertial frame moving with velocity v relative to the first frame, where

$$\tau' = \frac{\tau}{\sqrt{1 - v^2/c^2}}$$

2 Relativistic Energy and Momentum

A particle with mass m and velocity v has total energy (including rest energy), E, given by

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

and momentum $\mathbf{p}$ (vector quantity)

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}$$

These obey the relation

$$E^2 - p^2c^2 = m^2c^4$$

or

$$E = \sqrt{p^2c^2 + m^2c^4}.$$

For a system of particles with energies E_i and momenta $\mathbf{p}_i$, the quantity

$$\left(\sum_i E_i\right)^2 - \left(\sum_i \mathbf{p}_i\right)^2 c^2$$

is invariant under Lorentz transformations.

Thus, for example, if we have two particles with masses m_1 and m_2, with energies $E_1^{C.M.}$ and $E_2^{C.M.}$ in the centre-of-mass frame for which the total momentum $\mathbf{p}_1 + \mathbf{p}_2 = 0$, this quantity is simply

$$\left(E_1^{C.M.} + E_2^{C.M.}\right)^2.$$

If we consider the same system in the rest frame of particle 2, then for particle 2

$$E_2^{LAB} = m_2c^2, \qquad \mathbf{p}_2^{LAB} = 0,$$

and for particle 1 the energy and momentum are E_1^{LAB} and $\mathbf{p}_1^{LAB}$, so that this quantity is now

$$\left(E_1^{LAB} + m_2c^2\right)^2 - (p_1^{LAB})^2c^2 \;=\; (E_1^{LAB})^2 + m_2^2c^4 + 2E_1^{LAB}m_2c^2 - (p_1^{LAB})^2c^2.$$

Using

$$(E_1^{LAB})^2 - (p_1^{LAB})^2c^2 = m_1^2c^4,$$

we get

$$m_1^2c^4 + m_2^2c^4 + 2m_2c^2E_1^{LAB} \;=\; \left(E_1^{C.M.} + E_2^{C.M.}\right)^2.$$

There are two interesting special cases

1. The masses are equal $m_1 = m_2 = m$, for which this becomes

$$m^2c^4 + mc^2E_1^{LAB} \;=\; 2\left(E_1^{C.M.}\right)^2.$$

2. Particle 1 is massless, in which case $E_1^{C.M.} = p_1^{C.M.}c$ and since $\mathbf{p}_1^{C.M.} = -\mathbf{p}_2^{C.M.}$, we have

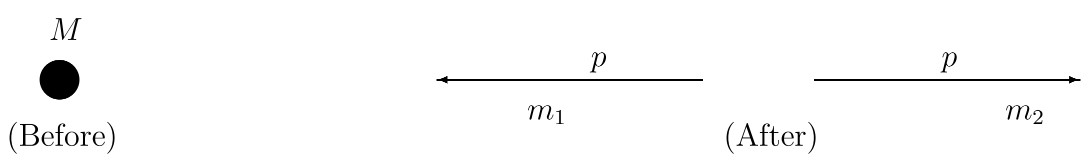
$$(E_2^{C.M.})^2 = (E_1^{C.M.})^2 + m_2^2c^4,$$

and we get

$$m_2^2c^4 + 2m_2c^2E_1^{LAB} \;=\; \left(E_1^{C.M.} + \sqrt{(E_1^{C.M.})^2 + m_2^2c^4}\right)^2.$$

PARTICLE DISINTEGRATION

In the rest frame of the decaying particle the momentum is zero therefore by momentum conservation the total momentum of the two decay particles must be zero. They must come out back to back with equal and opposite momentum p



The energy of decaying particle (which is at rest) is Mc^2. The energies of the two decay particles whose momentum both have magnitude p are $\sqrt{m_1^2c^4 + p^2c^2}$ and $\sqrt{m_2^2c^4 + p^2c^2}$ respectively. Thus conserving energy we have

$$Mc^2 = \sqrt{m_1^2c^4 + p^2c^2} + \sqrt{m_2^2c^4 + p^2c^2}$$

Case 1: $m_1 = m_2$ $(= m)$

$$Mc^2 = 2\sqrt{m^2c^4 + p^2c^2}$$

Squaring gives

$$M^2c^4 = 4(m^2c^4 + p^2c^2)$$

From which we get the momentum of the decay particles to be

$$p = \frac{c}{2}\sqrt{M^2 - 4m^2}$$

Case 2: One of the particles is massless, $(m_2 = 0,\ m_1 = m)$
Energy of massless particle with momentum p is pc. Thus we have

$$Mc^2 = \sqrt{m^2c^4 + p^2c^2} + pc$$

We rewrite this as

$$\sqrt{m^2c^4 + p^2c^2} = Mc^2 - pc$$

and squaring gives

$$m^2c^4 + p^2c^2 = M^2c^4 - 2Mc^3p + p^2c^2$$

Cancelling p^2c^2 from both sides and rearranging we have for the momentum

$$p = \frac{(M^2 - m^2)c}{2M}$$

3 Binding Energy

A bound system made up of N particles of mass m_i, $(i = 1 \cdots N)$, has a total mass which is smaller than the sum of the masses of the constituents by an amount equal to the binding energy divided by c^2

$$M = \sum_{i=i}^{N} m_i - \frac{B.E.}{c^2}$$

4 Relativistic Doppler Shift

If a photon is emitted with frequency ν according to an observer at rest relative to the source the frequency of the photon measured by an observer moving with velocity v *towards* the source is

$$\nu' = \nu\frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}} = \nu\frac{(1 + v/c)}{\sqrt{1 - v^2/c^2}} = \nu\frac{\sqrt{1 - v^2/c^2}}{(1 - v/c)}.$$

(For observer moving away from source replace v by $-v$).

In terms of wavelength λ and λ' we have (observer moving *towards* source)

$$\lambda' = \lambda \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}} = \lambda \frac{(1 - v/c)}{\sqrt{1 - v^2/c^2}} = \lambda \frac{\sqrt{1 - v^2/c^2}}{(1 + v/c)}.$$