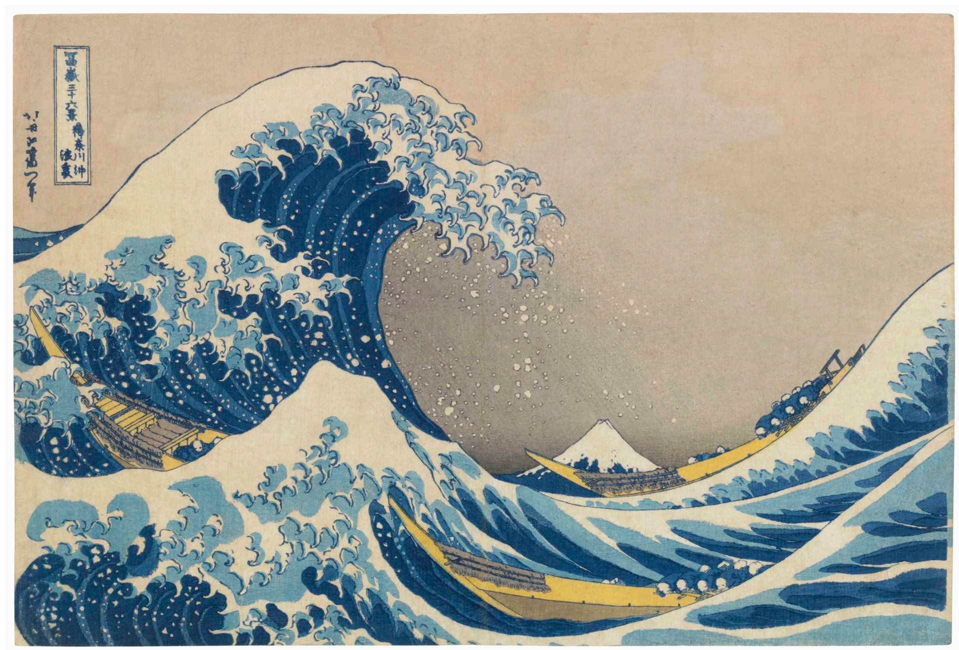
Relativistic fluid dynamics

from formulation to simulation



STAG RESEARCH CENTRE

UNIVERSITY OF Southampton

From electromagnetism to (resistive) MHD

The “magnetic" Universe:

explosions, mergers, jets, pulsars/magnetars, field configuration+evolution, and so on…

Vector potential leads to Faraday (field-strength) tensor

$$F_{ab} = \nabla_a A_b - \nabla_b A_a$$

and the dynamical equations are obtained from the Lagrangian

$$L_{\text{EM}} = -\frac{1}{4\mu_0} F_{ab} F^{ab} + j^a A_a$$

subject to the constraint (gauge invariance)

$$\nabla_a j^a = 0$$

We get

$$\nabla_b F^{ab} = \mu_0 j^a$$

while the anti-symmetry leads to

$$\nabla_{[c} F_{ab]} = 0$$

Also, introduce the Lorentz force:

$$\nabla_b T^{ba}_{\mathrm{fluid}} = -\nabla_b T^{ba}_{\mathrm{EM}} = j_b F^{ba} \equiv f^a_{\mathrm{L}}$$

So… this looks a bit “different”…

Electric/magnetic fields depend on observer.

Who measures **what**?

In general,

$$F_{ab} = 2U_{[a}E_{b]} + \epsilon_{abcd}U^c B^d = 2U_{[a}E_{b]} + \epsilon_{abd}B^d$$

leads to

$$E_a = -\,U^b F_{ba}$$

and

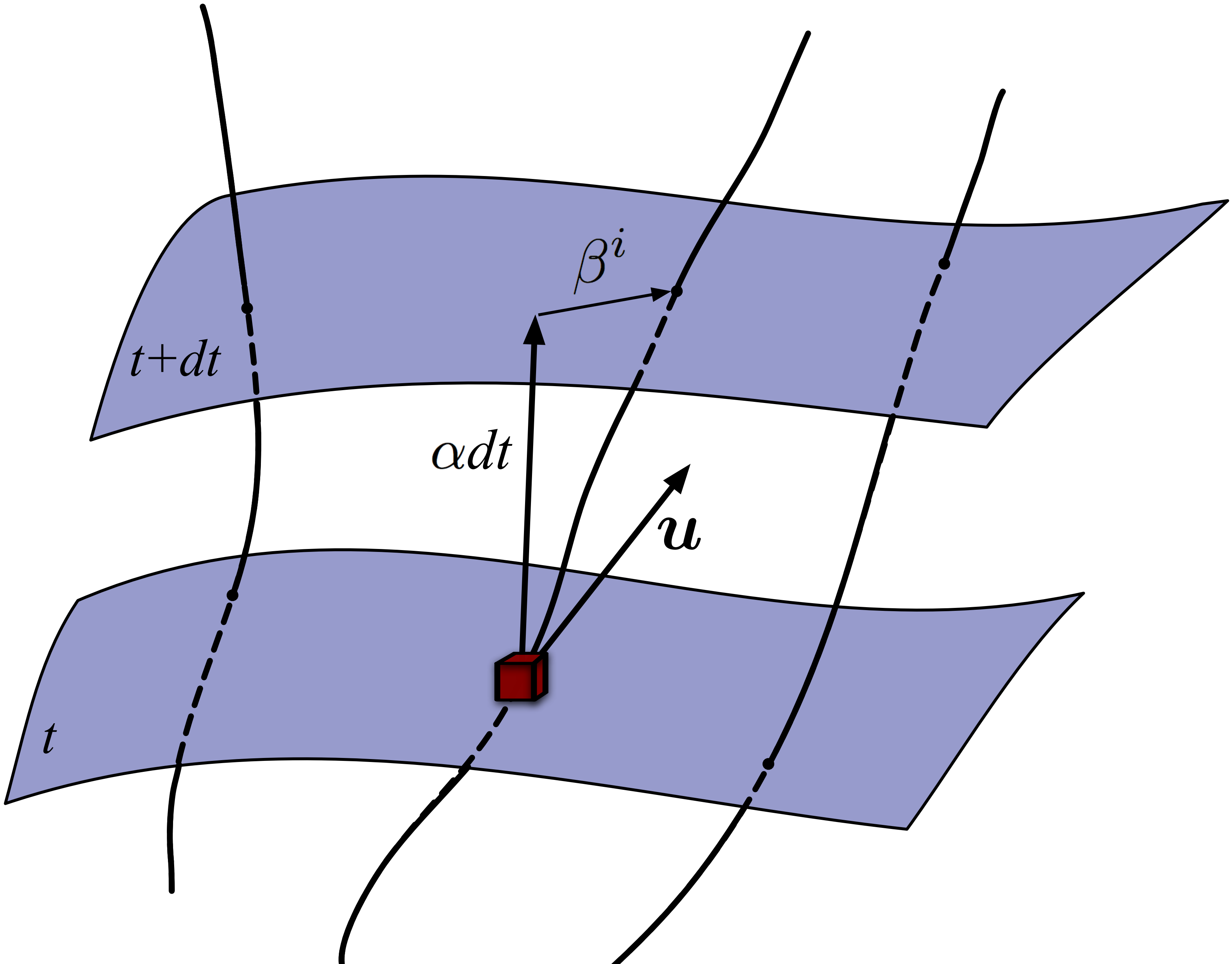
$$B_a = \frac{1}{2}\epsilon_{acd}F^{cd}$$

With the decomposition

$$j^a = \sigma U^a + J^a\ , \quad J^a U_a = 0$$

we have

$$f_{\mathrm{L}}^a = \sigma E^a - \epsilon^{abd} J_b B_d + U^a \left(J_b B^b\right)$$



β^i
$t+dt$
αdt
$\boldsymbol{u}$
t

In the fluid frame, we have Maxwell's equations

$$\perp^{ab} \nabla_b e_a - \mu_0 \sigma = 2W^a b_a$$

$$\perp_{ab} \dot{e}^b - \epsilon_{abc} \nabla^b b^c + \mu_0 J_a =$$
$$= \left(\sigma_{ab} - \varpi_{ab} - \frac{2}{3}\theta \perp_{ab} \right) e^b + \epsilon_{abc} \dot{u}^b b^c$$

$$\perp^{ab} \nabla_b b_a = -2W^a e_a$$

$$\perp_{ab} \dot{b}^b + \epsilon_{abc} \nabla^b e^c =$$
$$= \left(\sigma_{ab} - \varpi_{ab} - \frac{2}{3}\theta \perp_{ab} \right) b^b - \epsilon_{abc} \dot{u}^b e^c$$

For an inertial observer (= “special relativity”):

$$\perp^{ab} \nabla_b e_a - \mu_0 \sigma = 2W^a b_a$$

$$\perp_{ab} \dot{e}^b - \epsilon_{abc} \nabla^b b^c + \mu_0 J_a =$$

$$= \left(\sigma_{ab} - \varpi_{ab} - \frac{2}{3}\theta \perp_{ab} \right) e^b + \epsilon_{abc} \dot{u}^b b^c$$

$$\perp^{ab} \nabla_b b_a = -2W^a e_a$$

$$\perp_{ab} \dot{b}^b + \epsilon_{abc} \nabla^b e^c =$$

$$= \left(\sigma_{ab} - \varpi_{ab} - \frac{2}{3}\theta \perp_{ab} \right) b^b - \epsilon_{abc} \dot{u}^b e^c$$

For an inertial observer (tidying up):

$$\perp^{ab} \nabla_b e_a = \mu_0 \sigma$$

$$\perp_{ab} \dot{e}^b - \epsilon_{abc} \nabla^b b^c + \mu_0 J_a = 0$$

$$\perp^{ab} \nabla_b b_a = 0$$

$$\perp_{ab} \dot{b}^b + \epsilon_{abc} \nabla^b e^c = 0$$

MHD involves:

- local charge neutrality $\sigma = 0 \rightarrow e^a \approx 0$
- ignoring displacement current $\mu_0 J_a \approx \epsilon_{abc} \nabla^b b^c$

A slight as(l)ide:

electromagnetism+multi-fluids=comfortable marriage

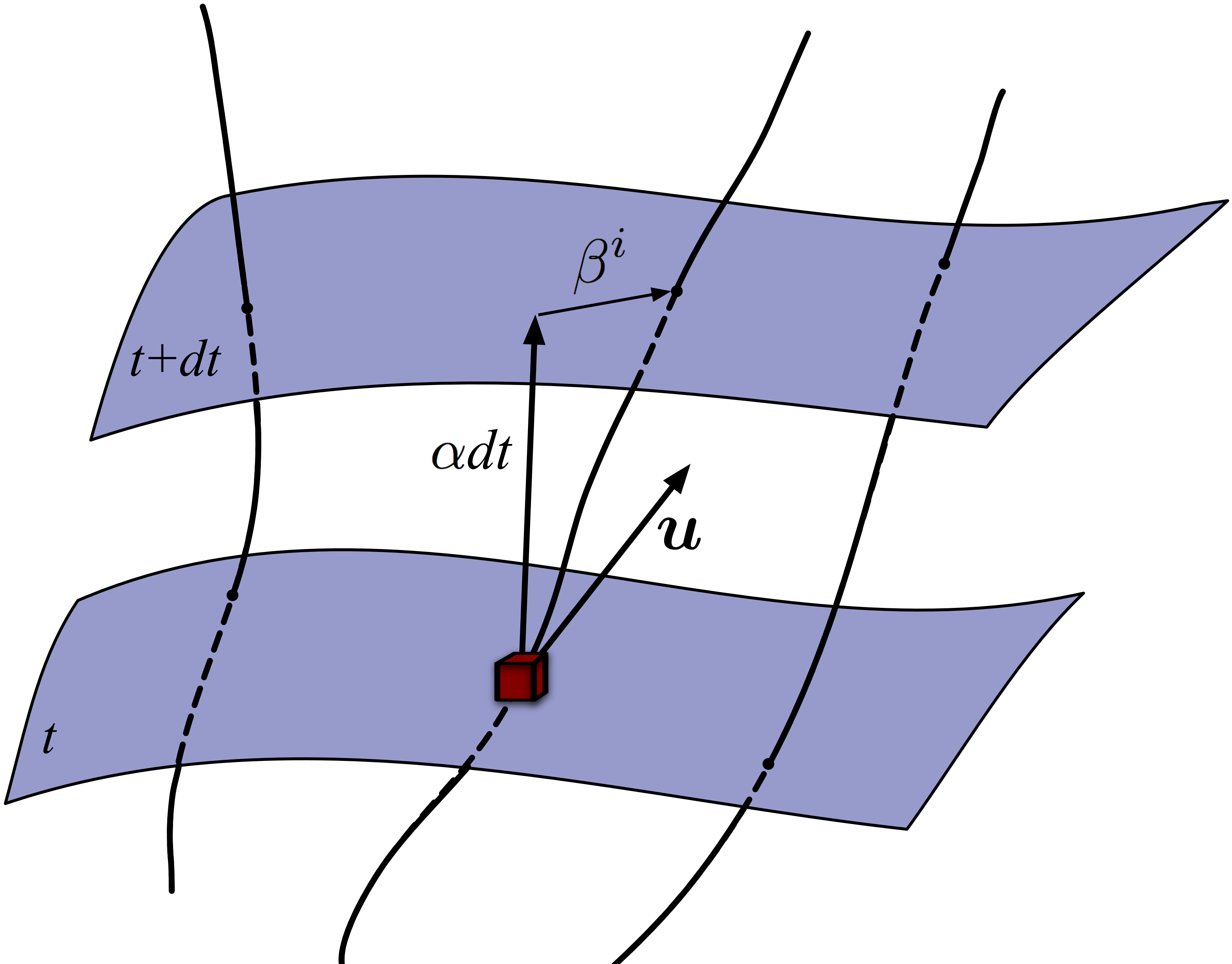
$$j^a = \sum_{\mathrm{x}} q_{\mathrm{x}} n^a_{\mathrm{x}} \quad \rightarrow \quad \nabla_a j^a = 0$$

$$L_{\mathrm{EM}} = -\frac{1}{4\mu_0} F_{ab} F^{ab} + A_a \sum_{\mathrm{x}} j^a_{\mathrm{x}}$$

$$\bar{\mu}^{\mathrm{x}}_a = \left(\frac{\partial \Lambda}{\partial n^a_{\mathrm{x}}} \right)_{n^a_{\mathrm{y}}} = \mu^{\mathrm{x}}_a + q_{\mathrm{x}} A_a$$

$$n^b_{\mathrm{x}} \bar{\omega}^{\mathrm{x}}_{ba} = 0 \rightarrow n^b_{\mathrm{x}} \omega^{\mathrm{x}}_{ba} = n^b_{\mathrm{x}} q^{\mathrm{x}} F_{ab} = j^b_{\mathrm{x}} F_{ab} \equiv f^{\mathrm{x}}_a$$

$$f^a_{\mathrm{L}} = \sum_{\mathrm{x}} f^a_{\mathrm{x}}$$



β^i
$t+dt$
αdt
$\boldsymbol{u}$
t

From the point of view of foliations…

$$F_{ab} = 2N_{[a}E_{b]} + \epsilon_{abcd}N^c B^d = 2N_{[a}E_{b]} + \epsilon_{abd}B^d$$

$$u^a = W(N^a + v^a)$$

$$e_a = -\,u^b F_{ba} = W\left[E_a + N_a(\hat{v}^b E_b)\right] + W\epsilon_{abd}\hat{v}^b B^d$$

$$e^{\parallel} = -\,e^a N_a = W\left(\hat{v}^b E_b\right)$$

$$e_a^{\perp} = W\left(E_a + \epsilon_{abc}\hat{v}^b B^c\right)$$

$$e_a^{\perp} = 0 \quad \rightarrow \quad E_a + \epsilon_{abc}\hat{v}^b B^c = 0$$

(One of) the main assumption(s) of (ideal) MHD.

According to the Eulerian observer, Maxwell's equations take the form;

$$D_i E^i = \mu_0 \hat{\sigma}$$

$$\left(\partial_t - \mathscr{L}_\beta\right) E^i - \epsilon^{ijk} D_j(\alpha B_k) + \alpha \mu_0 J^i = \alpha K E^i$$

$$D_i B^i = 0$$

$$\left(\partial_t - \mathscr{L}_\beta\right) B^i + \epsilon^{ijk} D_j(\alpha E_k) = \alpha K B^i$$

$$j^a = \hat{\sigma} N^a + J^a$$

Impact of gauge on MHD assumptions?

According to the Eulerian observer, Maxwell's equations take the form;

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$$j^a = \hat{\sigma} N^a + J^a$$

Effective MHD charge density (Goldreich-Julian):

$$\mu_0 \hat{\sigma} = - D_i \left(\epsilon^{ijk} \hat{v}_j B_k\right)$$

MHD is a "single-fluid" model.

But we (still) need to keep track of individual Lorentz factors:

$$\left(\partial_t - \mathscr{L}_\beta\right)\left(\gamma^{1/2}\hat{n}_\mathrm{x}\right) + D_i\left[\gamma^{1/2}\hat{n}_\mathrm{x}\left(\alpha\hat{v}^i_\mathrm{x} - \beta^i\right)\right] = 0$$

To avoid this, assume the relative drift is slow enough that we can linearize the relations.

$$u^a_\mathrm{x} = \gamma_\mathrm{x}\left(u^a + v^a_\mathrm{x}\right), \quad u_a v^a_\mathrm{x} = 0, \quad \gamma_\mathrm{x} = \left(1 - v^2_\mathrm{x}\right)^{-1/2}$$

Also helps make contact with the thermodynamics and the equation of state.

$$v^a_\mathrm{x} \approx W\left[\delta^a_b + W^2\hat{v}_b(N^a + \hat{v}^a)\right](\hat{v}^b_\mathrm{x} - \hat{v}^b)$$

What about charge neutrality?

$$\sigma = -u^a j_a = W\left(\hat{\sigma} - \hat{v}^a J_a\right)$$

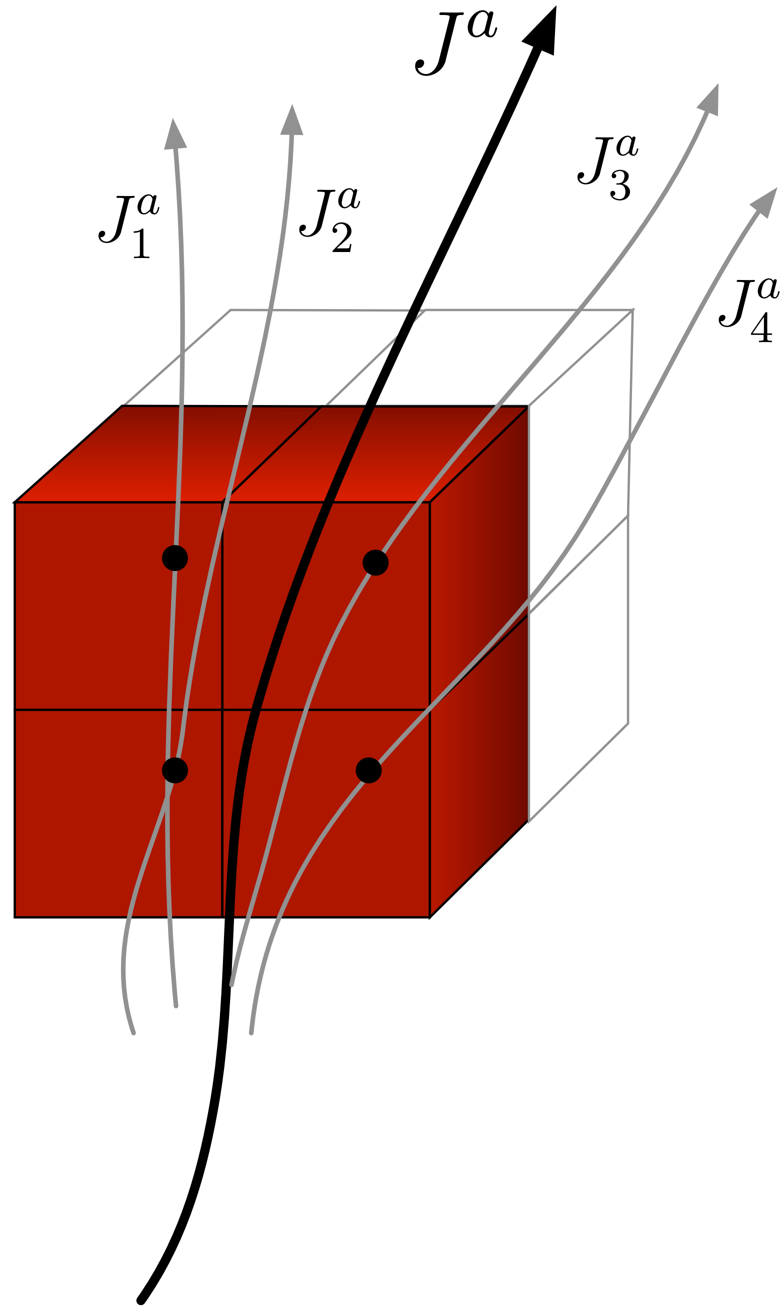
$$\hat{\sigma} = \hat{v}^a J_a$$

Is this a "useful" constraint?

Note also that we get (future reference)

$$J^a \approx \hat{\sigma}\hat{v}^a - eWn_{\mathrm{e}}(\hat{v}^a_{\mathrm{e}} - \hat{v}^a)$$

$$\hat{n}_{\mathrm{e}} = n_{\mathrm{e}} W_{\mathrm{e}} \approx n_{\mathrm{e}} W \left(1 - \frac{\hat{\sigma}}{e n_{\mathrm{e}} W}\right)$$



J^a
J_1^a
J_2^a
J_3^a
J_4^a

Still have two fluids (=expensive to evolve).

Focus on electron momentum equation;

$$\left(\partial_t + \mathscr{L}_{V_\mathrm{e}}\right) S_i^\mathrm{e} + D_i\left(\frac{\alpha\hat{\mu}_\mathrm{e}}{W_\mathrm{e}^2}\right) = \frac{\alpha}{\hat{n}_\mathrm{e}}\mathscr{F}_i^\mathrm{e}$$

with

$$V_\mathrm{e}^i = \alpha\hat{v}_\mathrm{e}^i - \beta^i$$

$$S_\mathrm{e}^i \approx \mu_\mathrm{e} W\left(\hat{v}^i - \frac{1}{en_\mathrm{e}W}J^i\right)$$

$$\mathscr{F}_i^\mathrm{e} \approx -en_\mathrm{e}W_\mathrm{e}\left(E_i + \epsilon_{ijk}\hat{v}_\mathrm{e}^j B^k\right) + \frac{\mathscr{R}}{en_\mathrm{e}}J_i$$

Ignore electron inertia and use linear drift to get Ohm's law

$$E_i + \epsilon_{ijk}\hat{v}^j_{\rm e}B^k + \frac{1}{\alpha}D_i\left[\frac{\alpha\mu_{\rm e}}{W}\left(1+\frac{\hat{\sigma}}{en_{\rm e}W}\right)\right] = \eta J_i$$

Drop chemical potential ("battery") and Hall term, to be left with

$$E_i + \epsilon_{ijk}\hat{v}^j B^k = \eta J_i \qquad\qquad \eta = \frac{\mathscr{R}}{e^2 n_{\rm e}^2}$$

Alternative: Current proportional to Lorentz force in fluid frame:

$$j_b = \bar{\eta}F_{ab}u^b \quad\rightarrow\quad \eta = 1/\bar{\eta}$$

Some kind of summary

In general relativity, MHD is more a (set of) assumption(s) than an approximation.

Need to pay attention to these assumptions, especially if we are interested in the details.

Charge neutrality provides an example (and leads to concerns about sub-grid features).

In short:

There is work to be done...