Neutron Star Dynamics and Universal Relations:

Masterclasses in Relativistic Fluid Dynamics: From formulation to simulation
Southampton 16-19 July 2019

Lap-Ming Lin
Department of Physics
The Chinese University of Hong Kong

A brief overview

Our common ambitious goal:

- Neutron stars are complex objects!
- In order to understand the dynamics of neutron stars, we need….

- Cold/Hot Equation of state
- Multi-fluid dynamics
- Multi-layer structure
- Viscosity
- Elasticity
- Magnetic field
- General relativity
- Nonlinear hydrodynamics
- …..

My lecture plan:

- ~~Cold/Hot Equation of state~~
- ~~Multi-fluid dynamics~~
- ~~Multi-layer structure~~
- ~~Viscosity~~
- ~~Elasticity~~
- ~~Magnetic field~~
- ~~General relativity~~
- ~~Nonlinear hydrodynamics~~

- Newtonian gravity
- Linearized hydrodynamics (Oscillations, tidal deformations)

Learning Outcomes:

- Understand the basic concepts of Newtonian stellar perturbations
- Be able to derive the perturbation equations for oscillations and tidal deformation in the simplest settings and be able to extend to more general situations
- Have an idea on the concepts on universal relations and their potential applications

Part I: Oscillations

Newtonian Fluid Equations

We consider Newtonian ideal fluid without viscostiy, solid components, heat flow etc. The system is governed by

Mass conservation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Euler's equation: $$\rho \frac{d\vec{v}}{dt} = -\nabla P - \rho \nabla \Phi$$

Poisson's equation: $$\nabla^2 \Phi = 4 \pi G \rho$$

ρ = mass density
v = fluid velocity
P = pressure
Φ = gravitational potential

$$d/dt \equiv \partial/\partial t + \vec{v} \cdot \nabla$$

The system is completed by providing a one-parameter equation of state (EOS):

$$P = P(\rho)$$

Summary of basic concepts for perturbations:

* Eulerian perturbation: $\delta Q \equiv Q(t, \vec{x}) - Q_0(t, \vec{x})$

[Compare Q with its unperturbed value Q_0 at the same position]

* Lagrangian perturbation: $\Delta Q \equiv Q(t, \vec{x} + \vec{\xi}) - Q_0(t, \vec{x})$

ξ: Lagrangian displacement

[Compare Q at a given fluid element in the perturbed and unperturbed states]

* Relation between the two perturbations: $\Delta = \delta + \vec{\xi} \cdot \nabla$

By definition, the Lagrangian perturbation of fluid velocity is

$$\Delta \vec{v} = \frac{d}{dt}(\vec{x} + \vec{\xi}) - \frac{d\vec{x}}{dt} = \frac{d\vec{\xi}}{dt} = \frac{\partial \vec{\xi}}{\partial t} + \vec{v} \cdot \nabla \vec{\xi}$$

If the unperturbed background is static ($v_0 = 0$), to first order accuracy, we have

$$\Delta \vec{v} = \delta \vec{v} = \partial \vec{\xi} / \partial t$$

Perturbation Fluid Equations

We assume that the unperturbed solution is a nonrotating static star which satisfies

$$\nabla P_0 = -\rho_0 \nabla \Phi_0 \quad , \quad \nabla^2 \Phi_0 = 4\pi G \rho_0 \quad , \quad \vec{v}_0 = 0 \quad , \quad P_0 = P_0(\rho_0)$$

and the relevant physical variables are perturbed according to:

$$\rho = \rho_0 + \delta\rho \quad , \quad \vec{v} = \delta\vec{v} = \partial\vec{\xi}/\partial t \quad , \quad P = P_0 + \delta P \quad , \quad \Phi = \Phi_0 + \delta\Phi$$

Linearizing the original system of equations, we obtain (to first order in perturbed quantities):

$$\delta\rho + \nabla\cdot(\rho_0 \vec{\xi}) = 0$$

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{\delta\rho}{\rho_0^2} \nabla P_0 - \frac{1}{\rho_0} \nabla(\delta P) - \nabla(\delta\Phi)$$

$$\nabla^2(\delta\Phi) = 4\pi G \delta\rho$$

Homework

The EOS for the unperturbed star is governed by

$$P_0 = P_0(\rho_0)$$

In general, the EOS for the perturbed fluid can be different from the unperturbed star. Assuming no heat flow or composition change in the perturbation, we can relate the Lagrangian perturbations of density and pressure by defining an adiabatic index Γ_1:

$$\frac{\Delta P}{P} = \Gamma_1 \frac{\Delta \rho}{\rho} \qquad \text{(adiabatic condition)}$$

Using the relation between Δ and δ, we can rewrite the equation in terms of Eulerian perturbations:

$$\frac{\delta \rho}{\rho} = \frac{1}{\Gamma_1} \frac{\delta P}{P} - \vec{A} \cdot \vec{\xi}$$

where $$\vec{A} \equiv \frac{\nabla \rho}{\rho} - \frac{\nabla P}{\Gamma_1 P}$$

[Note: We have dropped the subscript "0" for unperturbed solutions]

Side note for A:

The magnitude of A is called the Schwarzschild discriminant. For a spherical background star, we only have the radial component:

$$A = \left(\frac{d \ln \rho}{dr} - \frac{1}{\Gamma_1} \frac{d \ln P}{dr} \right)$$

If the background star is described by a polytropic EOS ($P = k\rho^{\Gamma}$), we have

$$A = \left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1} \right) \frac{d \ln P}{dr}$$

In general, $A \neq 0$ and its sign is used to determine the convective stability of fluid motion. If the perturbed and unperturbed stars are described by the same EOS ($\Gamma_1 = \Gamma$), we have $A = 0$.

Note that the perturbed mass conservation equation can be written as:

$$\underbrace{\delta\rho + \vec{\xi}\cdot\nabla\rho}_{=\Delta\rho} = -\rho\nabla\cdot\vec{\xi}$$

$$\frac{\Delta P}{P} = \Gamma_1 \frac{\Delta\rho}{\rho} \quad \Rightarrow \quad \boxed{\delta P = -\vec{\xi}\cdot\nabla P - \Gamma_1 P \nabla\cdot\vec{\xi}}$$

Making use of the expressions of $\delta\rho$ and δP that we have obtained, we can now rewrite the perturbed Euler equation in the following form that is more suitable for our later discussion:

$$\boxed{\frac{\partial^2\vec{\xi}}{\partial t^2} = -\nabla\left(\frac{\delta P}{\rho} + \delta\Phi\right) + \vec{A}\,\frac{\Gamma_1 P}{\rho}\left(\nabla\cdot\vec{\xi}\right)}$$

Homework

A brief summary:

The (adiabatic) perturbation equations for a nonrotating static background star are given by

$$\delta\rho + \nabla\cdot\left(\rho\vec{\xi}\right) = 0$$

$$\frac{\partial^2\vec{\xi}}{\partial t^2} = -\nabla\left(\frac{\delta P}{\rho} + \delta\Phi\right) + \vec{A}\,\frac{\Gamma_1 P}{\rho}\left(\nabla\cdot\vec{\xi}\right)$$

$$\nabla^2\left(\delta\Phi\right) = 4\pi G\,\delta\rho$$

where we have defined

$$\frac{\Delta P}{P} \equiv \Gamma_1\frac{\Delta\rho}{\rho} \quad , \quad \vec{A} \equiv \frac{\nabla\rho}{\rho} - \frac{\nabla P}{\Gamma_1 P}$$

The Eulerian perturbations $\delta\rho$ and δP are calculated by

$$\delta P = -\vec{\xi}\cdot\nabla P - \Gamma_1 P\,\nabla\cdot\vec{\xi}$$

$$\frac{\delta\rho}{\rho} = \frac{1}{\Gamma_1}\frac{\delta P}{P} - \vec{A}\cdot\vec{\xi}$$

Vector Spherical Harmonics

Before moving on, let us introduce the concept of vector spherical harmonics which we shall need in order to solve the set of fluid perturbation equations. But let us first recall how to apply the scalar spherical harmonics Y_{lm} in solving PDE like

$$\nabla^2 \Phi(r,\theta,\phi) = G(r,\theta,\phi)$$

← A given source function

Step 1: $\Phi(r,\theta,\varphi) = \sum_{lm} f_{lm}(r) Y_{lm}(\theta,\varphi) \quad , \quad G(r,\theta,\varphi) = \sum_{lm} g_{lm}(r) Y_{lm}(\theta,\varphi)$

Step 2: Substitute into the PDE and matching coefficients of Y_{lm}:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d f_{lm}}{dr}\right) - \frac{l(l+1)}{r^2} f_{lm}(r) = g_{lm}(r)$$

where we have used $r^2 \nabla^2 Y_{lm} = -l(l+1) Y_{lm}$

Important message: The angular part is canceled and we only need to solve ODE for the radial functions!

Let us now consider the following PDE with a vector field:

$$\nabla \cdot \vec{E} = \rho(r, \theta, \phi)$$

A naive (but not helpful) approach: Expand the vector components in Y_{lm}

$$\vec{E}(r, \theta, \phi) = E^r \hat{r} + E^\theta \hat{\theta} + E^\phi \hat{\phi}$$

$$= \left(\sum_{lm} E^r_{lm}(r) Y_{lm}\right)\hat{r} + \left(\sum_{lm} E^\theta_{lm}(r) Y_{lm}\right)\hat{\theta} + \left(\sum_{lm} E^\phi_{lm}(r) Y_{lm}\right)\hat{\phi}$$

The scalar function is also expanded in Y_{lm}:

$$\rho(r, \theta, \phi) = \sum_{lm} \rho_{lm}(r) Y_{lm}(\theta, \varphi)$$

Substitute the expansions into the PDE as before, you will see that the resulting equation would involve Y_{lm} and its angular derivatives, and the angular part cannot be canceled out. The radial parts cannot be separated in the process.

Homework

We need something better and let us start off with a scalar field and try to construct vectors out of it:

$$f(r,\theta,\phi)=\sum_{lm} f_{lm}(r)\,Y_{lm}(\theta,\varphi)$$

The gradient of *f* seems to be a natural choice:

$$\nabla f(r,\theta,\phi)=\sum_{lm}\left[\frac{df_{lm}(r)}{dr}Y_{lm}\hat{r}+\underline{f_{lm}(r)\nabla Y_{lm}}\right]$$

angular part

Let us define two dimensionless vectors:

$$\vec{Y}_{lm}(\theta,\phi)\equiv Y_{lm}\hat{r} \quad , \quad \vec{Y}^{(p)}_{lm}(\theta,\phi)\equiv r\nabla Y_{lm}$$

We can also construct another vector:

$$\vec{Y}^{(a)}_{lm}(\theta,\phi)\equiv\hat{r}\times\vec{Y}^{(p)}_{lm}(\theta,\phi)=\vec{r}\times\nabla Y_{lm}$$

Note: The three vectors as defined are orthogonal

We have constructed 3 vector spherical harmonics:

$$\{\vec{Y}_{lm}, \vec{Y}^{(p)}_{lm}, \vec{Y}^{(a)}_{lm}\}$$

Claim: They form a complete set and we can expand any vector field V as

$$\vec{V}(r,\theta,\phi) = \sum_{lm}\left[V^{r}_{lm}(r)\vec{Y}_{lm} + V^{(p)}_{lm}(r)\vec{Y}^{(p)}_{lm} + V^{(a)}_{lm}(r)\vec{Y}^{(a)}_{lm}\right]$$

$$= \sum_{lm}\left[V^{r}_{lm}(r)Y_{lm}\hat{r} + r V^{(p)}_{lm}(r)\nabla Y_{lm} + V^{(a)}_{lm}(r)\vec{r}\times\nabla Y_{lm}\right]$$

This decomposition can be divided into 2 classes according to how they behave under parity transformation:

$$\vec{V}(r,\theta,\phi) = \sum_{lm}\left[V^{r}_{lm}(r)Y_{lm}\hat{r} + r V^{(p)}_{lm}(r)\nabla Y_{lm} + V^{(a)}_{lm}(r)\vec{r}\times\nabla Y_{lm}\right]$$

This part has the same parity as Y_{lm} and is said to have even parity:

$$(-1)^l$$

This part has opposite parity and is said to have odd parity:

$$(-1)^{l+1}$$

Under parity transformation: $Y_{lm} \rightarrow (-1)^l Y_{lm}$

Some useful relations:

$$\vec{Y}_{lm}(\theta,\phi) \equiv Y_{lm}\hat{r} \quad , \quad \vec{Y}^{(p)}_{lm}(\theta,\phi) \equiv r\nabla Y_{lm} \quad , \quad \vec{Y}^{(a)}_{lm}(\theta,\phi) = \vec{r}\times\nabla Y_{lm}$$

$$\nabla\cdot\left(F(r)\vec{Y}_{lm}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 F(r)\right)Y_{lm} \qquad \nabla\times\left(F(r)\vec{Y}_{lm}\right) = -\frac{F(r)}{r}\vec{Y}^{(a)}_{lm}$$

$$\nabla\cdot\left(F(r)\vec{Y}^{(p)}_{lm}\right) = -\frac{l(l+1)}{r}F(r)Y_{lm} \qquad \nabla\times\left(F(r)\vec{Y}^{(p)}_{lm}\right) = \frac{1}{r}\frac{d}{dr}\left(rF(r)\right)\vec{Y}^{(a)}_{lm}$$

$$\nabla\cdot\left(F(r)\vec{Y}^{(a)}_{lm}\right) = 0 \qquad \nabla\times\left(F(r)\vec{Y}^{(a)}_{lm}\right) = -\frac{l(l+1)}{r}F(r)\vec{Y}_{lm} - \frac{1}{r}\frac{d}{dr}\left(rF(r)\right)\vec{Y}^{(p)}_{lm}$$

$$\nabla\left(F(r)Y_{lm}\right) = \frac{d}{dr}\left(F(r)\right)\vec{Y}_{lm} + \frac{F(r)}{r}\vec{Y}^{(p)}_{lm}$$

$$\nabla^2\left(F(r)Y_{lm}\right) = \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dF}{dr}\right) - \frac{l(l+1)}{r^2}F(r)\right]Y_{lm}$$

Example:

$$\nabla \cdot \vec{E} = \rho(r,\theta,\phi)$$ ← A given source function

We now decompose the vector field E in vector spherical harmonics and the source function in Y_{lm}:

$$\vec{E}(r,\theta,\phi) = \sum_{lm} \left[E_{lm}^{r}(r)\,\vec{Y}_{lm} + E_{lm}^{(p)}(r)\,\vec{Y}_{lm}^{(p)} + E_{lm}^{(a)}(r)\,\vec{Y}_{lm}^{(a)} \right]$$

$$\rho(r,\theta,\phi) = \sum_{lm} \rho_{lm}(r)\,Y_{lm}(\theta,\varphi)$$

Using the relations given on the last page:

$$\nabla \cdot \vec{E} = \sum_{lm} \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 E_{lm}^{r}(r)\right) - \frac{l(l+1)}{r} E_{lm}^{(p)} \right] Y_{lm}$$

The angular dependence in the PDE can be canceled and we are left with

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 E_{lm}^{r}(r)\right) - \frac{l(l+1)}{r} E_{lm}^{(p)}(r) = \rho_{lm}(r)$$

To solve the problem, we need to impose an extra equation such as

$$\nabla \times \vec{E} = 0$$ Homework

Oscillation Equations

Assume that the Lagrangian displacement behaves as $\vec{\xi} \sim e^{i\omega t}$

The set of perturbation equations become

$$-\omega^2 \vec{\xi} = -\nabla\left(\frac{\delta P}{\rho} + \delta\Phi\right) + \vec{A}\,\frac{\Gamma_1 P}{\rho}\left(\nabla\cdot\vec{\xi}\right)$$

$$\delta\rho + \nabla\cdot\left(\rho\,\vec{\xi}\right) = 0 \qquad \nabla^2\left(\delta\Phi\right) = 4\pi G\,\delta\rho$$

Let us focus on a given pair of (l,m) and expand the Lagrangian displacement in vector spherical harmonics:

Polar modes / Axial modes

$$\vec{\xi} \equiv \underbrace{U(r)\,Y_{lm}\,\hat{r} + r\,V(r)\,\nabla Y_{lm}}_{\vec{\xi}^{(p)}} + \underbrace{W(r)\,\vec{r}\times\nabla Y_{lm}}_{\vec{\xi}^{(a)}}$$

The scalar fields are expanded in Y_{lm}:

$$\delta\rho \equiv \delta\tilde{\rho}(r)\,Y_{lm}\ ,\ \delta P \equiv \delta\tilde{P}(r)\,Y_{lm}\ ,\ \delta\Phi \equiv \delta\tilde{\Phi}(r)\,Y_{lm}$$

Axial modes:

Let us consider axial perturbations so that $\vec{\xi} \equiv \vec{\xi}^{(a)} = W(r)\vec{Y}_{lm}^{(a)}$

This class of modes satisfies $\nabla \cdot \vec{\xi}^{(a)} = 0$ $\quad (\vec{Y}_{lm}^{(a)} \equiv \vec{r} \times \nabla Y_{lm})$

* Perturbed mass conservation equation:

$$\delta\tilde{\rho}\, Y_{lm} + \underbrace{\nabla \cdot \left(\rho(r) W(r) \vec{Y}_{lm}^{(a)}\right)}_{=0} = 0 \quad \Rightarrow \quad \delta\tilde{\rho}(r) = 0$$

* Eulerian perturbation of P:

$$\begin{aligned}\delta P &= -\vec{\xi}^{(a)} \cdot \nabla P - \Gamma_1 P \nabla \cdot \vec{\xi}^{(a)} \\ &= -W(r) P'(r) \vec{Y}_{lm}^{(a)} \cdot \hat{r} \\ &= 0\end{aligned}$$

Together with the perturbed Euler's equation, it can be shown that

$$\boxed{\delta\rho = \delta P = \delta\Phi = \omega = 0}$$

Homework

=> *Axial modes do not exist for Newtonian non-rotating star.*

[But these modes do exist for stars with solid components or rotation]

Polar modes: Our main interest is the polar modes described by

$$\vec{\xi} \equiv \vec{\xi}^{(p)} \equiv U(r)\vec{Y}_{lm} + V(r)\vec{Y}_{lm}^{(p)}$$

$$\vec{Y}_{lm} \equiv Y_{lm}\hat{r}$$
$$\vec{Y}_{lm}^{(p)} \equiv r\nabla Y_{lm}$$

=> $$\nabla \cdot \vec{\xi}^{(p)} = \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 U(r)\right) - \frac{l(l+1)}{r}V(r)\right]Y_{lm} \equiv \alpha(r) Y_{lm}$$

Consider the perturbed Euler's equation:

$$-\omega^2\vec{\xi} = -\nabla\left(\frac{\delta P}{\rho} + \delta\Phi\right) + \vec{A}\frac{\Gamma_1 P}{\rho}\left(\nabla \cdot \vec{\xi}\right)$$

[Note: For spherical unperturbed background $\vec{A} = A\hat{r}$]

Expanding the perturbed scalar fields as before: $\delta P \equiv \delta\tilde{P}(r) Y_{lm}(\theta, \phi)$ etc

=> $$-\omega^2\vec{\xi}^{(p)} = \left[-\frac{d}{dr}\left(\frac{\delta\tilde{P}}{\rho} + \delta\tilde{\Phi}\right) + \alpha A\frac{\Gamma_1 P}{\rho}\right]\vec{Y}_{lm} - \frac{1}{r}\left(\frac{\delta\tilde{P}}{\rho} + \delta\tilde{\Phi}\right)\vec{Y}_{lm}^{(p)}$$

Homework

[Recall: ρ, P, Φ are spherical background solutions depending only on r]

Comparing the coefficients on both sides, we obtain

$$-\omega^2 U(r) = -\frac{d}{dr}\left(\frac{\delta\tilde{\rho}}{\rho} + \delta\tilde{\Phi}\right) + \alpha A \frac{\Gamma_1 P}{\rho}$$

$$\omega^2 V(r) = -\frac{1}{r}\left(\frac{\delta\tilde{\rho}}{\rho} + \delta\tilde{\Phi}\right)$$

The density perturbation can be eliminated by the perturbed mass conservation equation:

Recall: $\nabla\cdot\vec{\xi}^{(p)} \equiv \alpha(r) Y_{lm}$

$$\delta\rho = -\nabla\rho\cdot\vec{\xi}^{(p)} - \rho\nabla\cdot\vec{\xi}^{(p)} \quad => \quad \delta\tilde{\rho}(r) = -U\frac{d\rho}{dr} - \rho\alpha$$

The pressure perturbation can be eliminated by the following relation we obtained before:

$$\delta P = -\vec{\xi}\cdot\nabla P - \Gamma_1 P \nabla\cdot\vec{\xi} \quad => \quad \delta\tilde{P}(r) = -U\frac{dP}{dr} - \alpha\Gamma_1 P$$

The perturbed Poisson's equation can be handled easily:

$$\nabla^2\left(\delta\tilde{\Phi}(r) Y_{lm}\right) = 4\pi G\,\delta\tilde{\rho}(r) Y_{lm}$$

$$=> \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\tilde{\Phi}}{dr}\right) - \frac{l(l+1)}{r^2}\delta\tilde{\Phi} = -4\pi G\left(U\frac{d\rho}{dr} + \rho\alpha\right)$$

Homework

Summary: Equations for polar oscillation modes

$$\vec{\xi}^{(p)} \equiv U(r)\vec{Y}_{lm} + V(r)\vec{Y}_{lm}^{(p)} = U(r) Y_{lm}\hat{r} + r V(r) \nabla Y_{lm}$$

$$\omega^2 U = \frac{d\chi}{dr} + \alpha A \frac{\Gamma_1 P}{\rho}$$

$$\omega^2 V = \frac{\chi}{r}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\tilde{\Phi}}{dr}\right) - \frac{l(l+1)}{r^2}\delta\tilde{\Phi} = -4\pi G\left(U\frac{d\rho}{dr} + \rho\alpha\right)$$

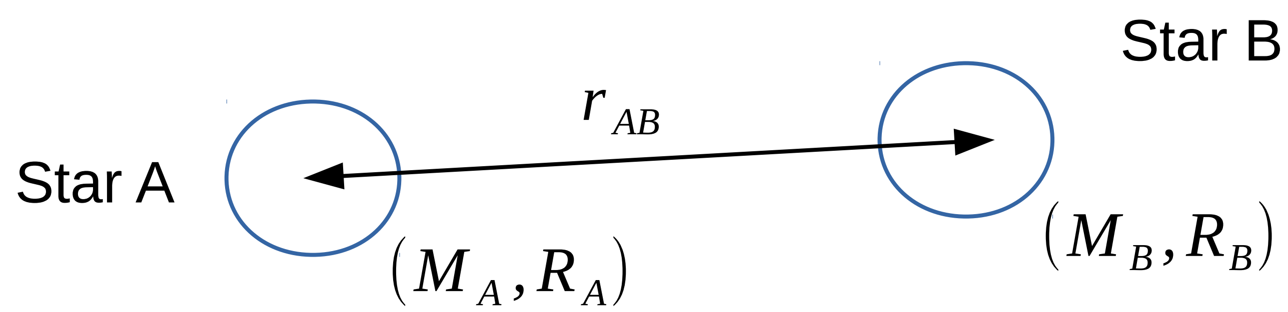
$$\alpha \equiv \frac{1}{r^2}\frac{d}{dr}(r^2 U) - \frac{l(l+1)}{r}V$$

$$\chi \equiv \frac{-1}{\rho}\frac{dP}{dr}U - \frac{\Gamma_1 P}{\rho}\alpha + \delta\tilde{\Phi}$$

The above equations have to been solved with appropriate boundary conditions at the center and surface. [see McDermott et al. (1988) and Cox (1980)]

Part II: Tidal deformation

Tidal Deformation



Circular orbital period: $T_{\text{orbit}} \sim \sqrt{\dfrac{r_{AB}^3}{G(M_A + M_B)}}$

Internal dynamical timescale of A: $T_{\text{int}} \sim \sqrt{\dfrac{R_A^3}{G M_A}}$

$$\left(\frac{T_{\text{orbit}}}{T_{\text{int}}}\right) \sim \left(\frac{r_{AB}}{R_A}\right)^3$$

If r_{AB} >> R_A, then the orbital timescale is much longer than the internal dynamical timescale of A, and so the gravitational effects due to B can be considered as "static" as viewed by A.

Assume that the unperturbed state of star A is non-rotating (spherical), the perturbed fluid inside A is described by

$$0 \equiv \frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{\delta\rho}{\rho^2} \nabla P - \frac{1}{\rho} \nabla(\delta P) - \nabla(\delta \Phi)$$

(Recall: variables without δ are background solutions)

The perturbation due to the companion B is time-independent (static tides)

The perturbed potential has 2 contributions:

$$\delta\Phi = \delta\Phi_A + \delta\Phi^{(\mathrm{ext})}$$

"Self" part:
due to the deformation of A
(away from spherical background)

"External" part:
due to the companion star B

characterized by mass multipoles of A: Q_{lm}

$$Q_{lm} \sim \varepsilon_{lm}$$

Q_{lm} is created by the external tidal field moments ε_{lm}

The gravitational field outside star A ($r > R_A$), but far away from B, is given by

$$\Phi_{\text{total}} = \Phi_A^{(0)} + \delta\Phi_A + \delta\Phi^{(\text{ext})}$$

$$\Phi_A^{(0)} = -\frac{GM_A}{r}$$ (spherical background monopole solution)

The “self” part is given by the standard multipole expansion:

$$\delta\Phi_A = -G \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{Q_{lm}}{r^{l+1}} Y_{lm}$$

[The l=1 term vanishes by choosing the origin at the center of mass of A]

where the mass multipole moments are defined by:

$$Q_{lm} \equiv \int_A \bar{r}^{l+2} \rho_{lm}(t, \bar{r})\, d\bar{r}$$

As we are interested in the neighborhood of star A, we can Taylor expand the "external" field (generated by B) about star A's center of mass:

$$\delta\Phi^{(\mathrm{ext})}(t,\vec{x}) = \delta\Phi^{(\mathrm{ext})}(t,0) + x^j \partial_j \delta\Phi^{(\mathrm{ext})}(t,0) + \frac{1}{2} x^j x^k \partial_j \partial_k \delta\Phi^{(\mathrm{ext})}(t,0) + \dots$$

Constant (no contribution)

This term is canceled by a corresponding term due to the acceleration of A's center of mass [see Poisson & Will (2014)]

It is standard to express the tidal field using multi-index notation:

$$\delta\Phi_A = \sum_{l=2} \frac{1}{l!} \varepsilon_L x^L$$

$$x^L \equiv x^{i_1} x^{i_2} \dots x^{i_l}$$

$$\partial_L \equiv \partial_{i_1} \partial_{i_2} \dots \partial_{i_l}$$

where the tidal moment ε_L is a symmetric tracefree (STF) tensor:

$$\varepsilon_L \equiv \partial_L \delta\Phi^{(\mathrm{ext})}(t,0)$$

** symmetric and tracefree with respect to any pair of indices
[Note: ε_L is tracefree because $\delta\Phi^{(\mathrm{ext})}$ satisfies the Laplace's equation]

The tidal field ε_L is defined naturally as a STF tensor, while the fluid perturbations inside star A are expanded in Y_{lm}. In order to relate ε_L to the fluid perturbations, we expand ε_L in STF tensor $\hat{Y}^{k_1 k_2 \ldots k_l}_{lm}$ which is defined by

$$Y_{lm}(\theta,\phi) = \hat{Y}^{k_1 k_2 \ldots k_l}_{lm} n_{k_1} n_{k_2} \ldots n_{k_l} \qquad n_i \equiv x_i / r$$

(see Appendix for more details)

Although initially defined as STF tensor, the tidal field can be re-expressed in Y_{lm} expansion via the above connection.

Claim: The total perturbed field is

$$\delta\Phi = -G \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{Q_{lm}}{r^{l+1}} Y_{lm} + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{(2l+1)!!} \varepsilon_{lm} r^l Y_{lm}$$
$$\equiv \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \delta\tilde{\Phi}_{lm}(r) Y_{lm}$$

[We follow the normalization used in Poisson & Will (2014), but we use a different sign convention for the potential.]

The mass multipole moments of A is a linear response of the tidal field, it is conventional to define the so-called tidal deformability λ_l:

$$G\,Q_{lm} = -\lambda_l\,\varepsilon_{lm}$$

For a given pair of (l, m), we relabel: $H(r) \equiv \delta\tilde{\Phi}_{lm}(r)$

Outside star A ($r > R_A$), and making use of the definition of λ_l, we have

$$H_{(\mathrm{out})}(r) = \frac{4\pi}{2l+1}\,\varepsilon_{lm}\left[\frac{\lambda_l}{r^{l+1}} + \frac{r^l}{(2l-1)!!}\right]$$

Static Tides

When the separation between the two stars is much larger than the radius of star A ($r_{AB} >> R_A$), the fluid perturbation inside A is determined by the set of polar oscillation equations in the zero-frequency limit ($\omega = 0$):

$$\omega^2 U = 0 = \frac{d\chi}{dr} + \alpha A \frac{\Gamma_1 P}{\rho}$$

$$\omega^2 V = 0 = \frac{\chi}{r}$$

$$\Rightarrow \quad \alpha = \chi = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dH}{dr}\right) - \frac{l(l+1)}{r^2} H = -4\pi G \left(U \frac{d\rho}{dr} + \rho\alpha\right)$$

$$\alpha \equiv \frac{1}{r^2}\frac{d}{dr}\left(r^2 U\right) - \frac{l(l+1)}{r} V \qquad \chi \equiv \frac{-1}{\rho}\frac{dP}{dr} U - \frac{\Gamma_1 P}{\rho}\alpha + H$$

From the definition of χ : $U = \left(\frac{1}{\rho} \frac{dP}{dr} \right)^{-1} H$

The perturbed Poisson's equation becomes:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dH}{dr} \right) - \frac{l(l+1)}{r^2} H = -4\pi G \left(\frac{1}{\rho} \frac{dP}{d\rho} \right)^{-1} H$$

This is the main Newtonian equation for determining the tidal deformation of a non-rotating (spherical) star in the static-tide limit.
[Regularity condition at $r = 0$: $H(r) \sim r^l$]

In principle, the tidal deformation is determined by integrating the equation for $H(r)$ inside the star and then match the solution at the star surface $H(R)$ to the exterior solution:

$$H_{(\mathrm{out})}(r) = \frac{4\pi}{2l+1} \varepsilon_{lm} \left[\frac{\lambda_l}{r^{l+1}} + \frac{r^l}{(2l-1)!!} \right]$$

To extract λ_l, we consider the log derivative: $y(r) \equiv \frac{r H'(r)}{H(r)}$

For the exterior solution: $r H_{\text{out}}'(r) = \frac{4\pi}{2l+1} \varepsilon_{lm} \left[\frac{-(l+1)}{r^{l+1}} \lambda_l + \frac{l}{(2l-1)!!} r^l \right]$

This motivates the definition of the (dimensionless) Love number k_l:

$$\lambda_l \equiv \frac{2}{(2l-1)!!} k_l R^{2l+1}$$

This exterior solution of y at $r = R$ is then given by: $y_{\text{out}}(R) = \frac{-2(l+1)k_l + l}{2k_l + 1}$

Once the interior solution has been solved $y_{\text{in}}(r)$, we can match the two solutions at the surface to determine the Love number k_l:

$y_{\text{in}}(R) = y_{\text{out}}(R)$ => $$k_l = \frac{l - y_{\text{in}}(R)}{2[y_{\text{in}}(R) + l + 1]}$$

Uniform density model:

The background density and pressure profiles are

$$\rho(r) \equiv \rho_0 \Theta(R-r) \qquad P(r) = \frac{2\pi G \rho_0^2}{3}(R^2 - r^2)$$

Master equation: $$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dH}{dr}\right) - \frac{l(l+1)}{r^2} H = -4\pi G \left(\rho \frac{d\rho}{dP}\right) H$$

Note: $\frac{d\rho}{dP} \frac{3}{4\pi G \rho_0} \frac{1}{r} \delta(R-r)$ => $H'(r)$ and hence $y(r)$ has a jump at the surface

The boundary condition at $r = R$ needs to be considered carefully!

Outline of the solution:

Step 1: Show that the interior solution satisfies $y_{\text{in}}(r) = l$

Step 2: We cannot simply match $y_{\text{out}}(R) = y_{\text{in}}(R) = l$ because of the delta function. Instead, show that the appropriate matching condition is

$$\boxed{y_{\text{out}}(R) = y_{\text{in}}(R) - 3}$$

Homework

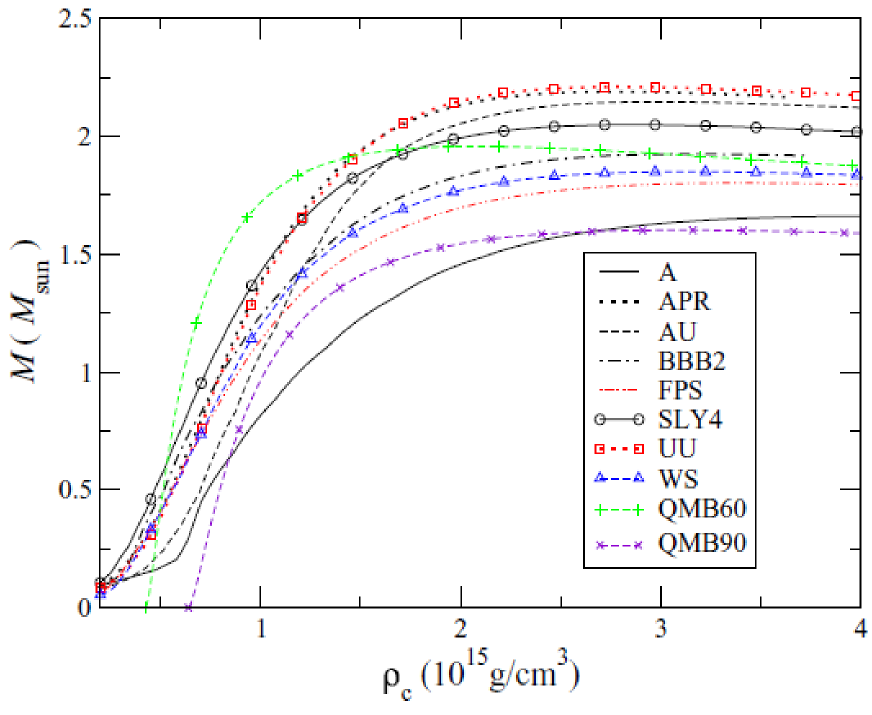
Step 3: Solve for the Love number: $k_l = 3/4(l-1)$

Part III: Universal Relations

The following results are done in the framework of general relativity

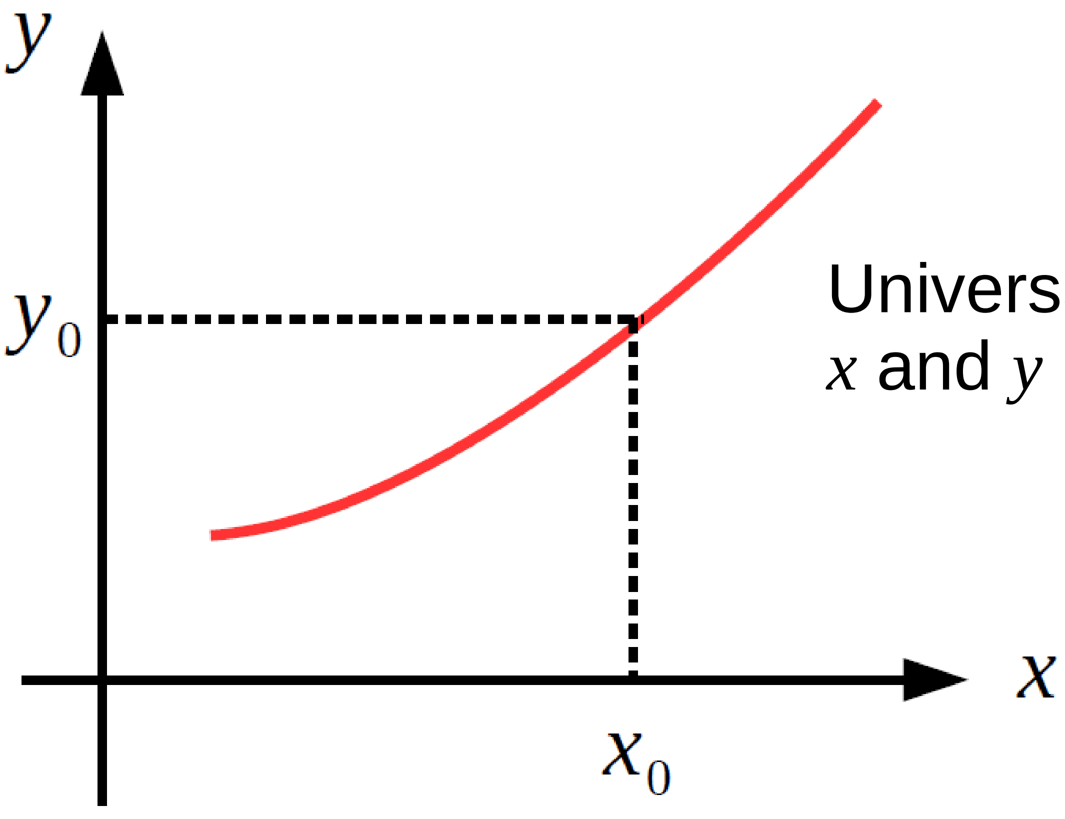
Universal Relations

- It is well known that many physical quantities of neutron stars depend sensitively on EOS (Good for constraining EOS)
- It is also known that there exist various approximately EOS-insensitive relations connecting different quantities of neutron stars. We simply call them universal relations in this lecture
  [Definition? EOS-insensitive to ~O(1%) level?]



Potential applications:

- If one of the quantities can be measured, the other one can be inferred from the relation
- If both quantities can be measured together, then we can test for GR…..or may be some exotic microphysics?

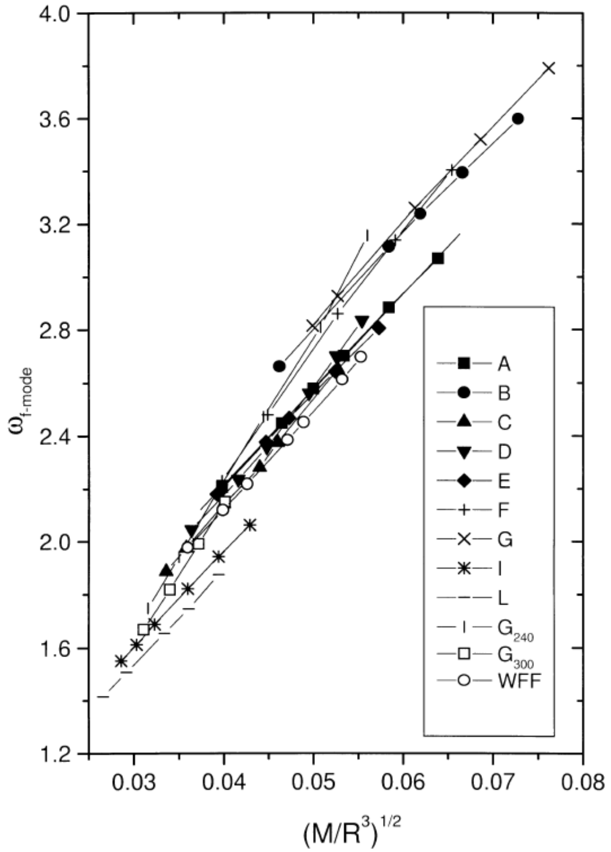


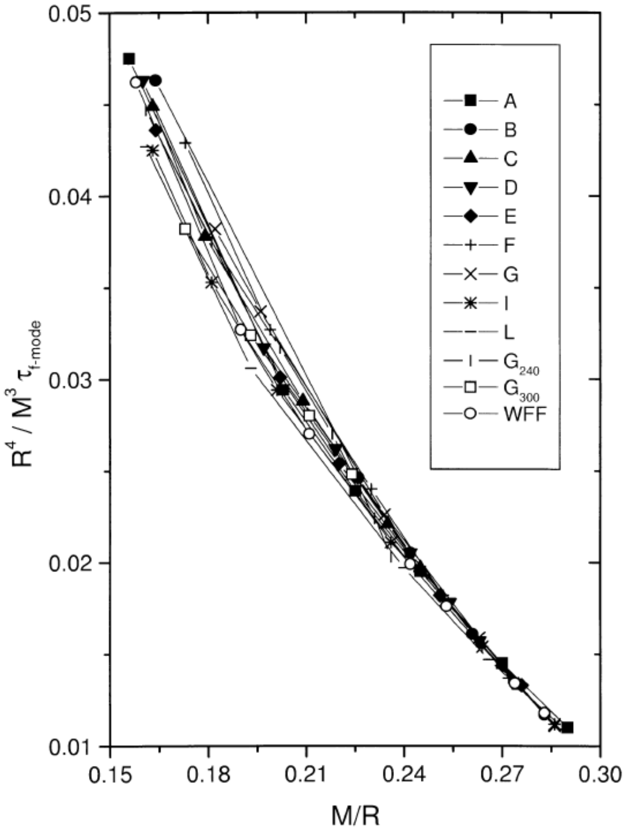
Universal relation between x and y (in GR)

f-mode universal relations

In 1998, Andersson & Kokkotas(1998) proposed the following empirical relations for the f-mode oscillation frequency and damping time.

[Note: In GR, nonradial oscillations emits gravitational waves and so the oscillations are damped.]





$$M_{1.4} = M/(1.4\,M_{\mathrm{sun}})$$
$$R_{10} = R/(10\ \mathrm{km})$$

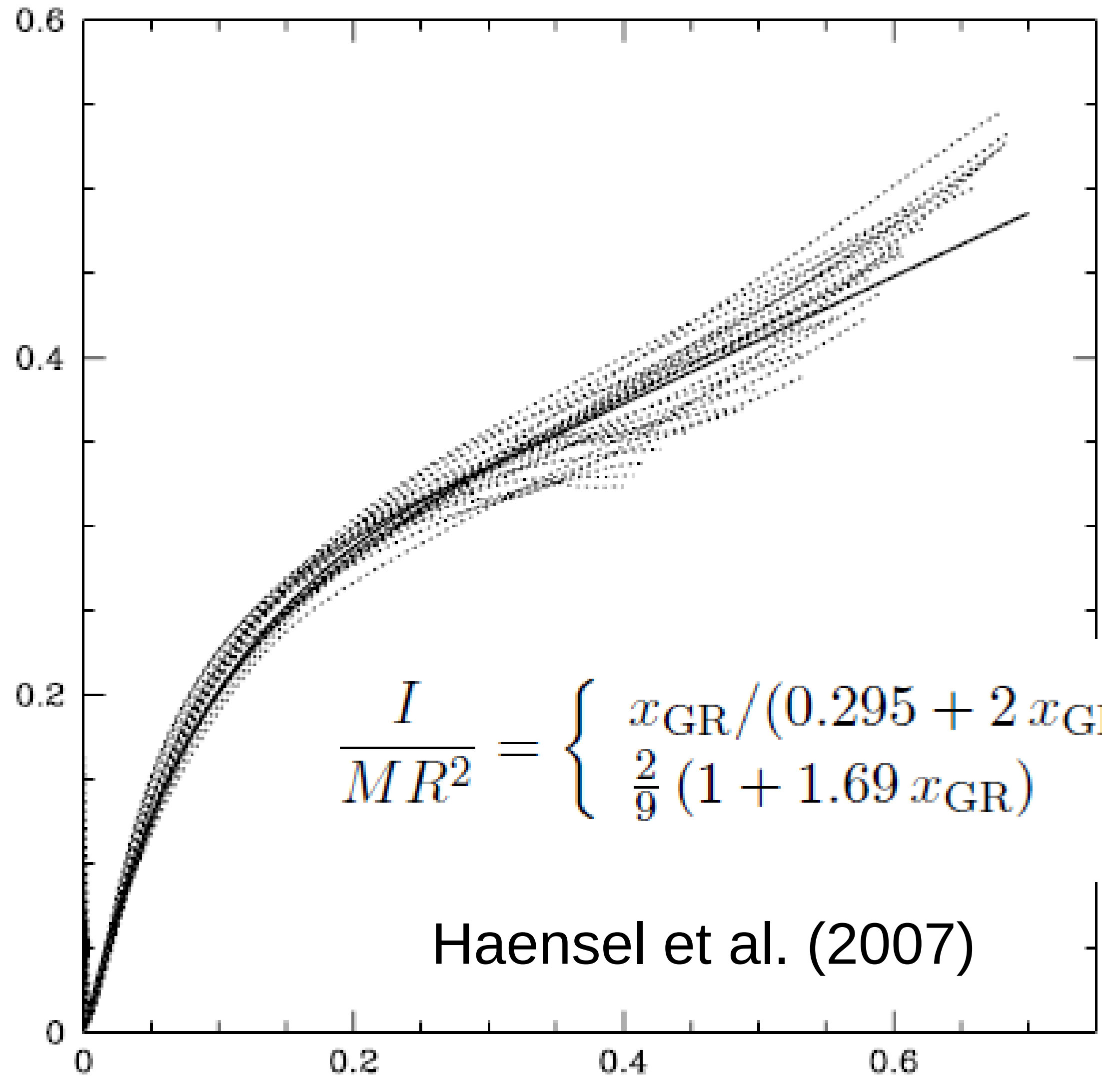
$$f\,(\mathrm{kHz}) \approx 0.78 + 1.635\left(\frac{M_{1.4}}{R_{10}^3}\right)^{1/2}$$

$$\frac{1}{\tau}(\mathrm{s}) \approx \frac{M_{1.4}^3}{R_{10}^4}\left[22.85 - 14.65\left(\frac{M_{1.4}}{R_{10}}\right)\right]$$

Andersson & Kokkotas (1998)

- Empirical relations between NS's moment of inertia (I) and compactness (M/R) [Bejger & Haensel (2002); Lattimer & Schutz (2005)]

$$\frac{I}{MR^2}$$



$$\frac{I}{MR^2} = \begin{cases} x_{\rm GR}/(0.295 + 2\,x_{\rm GR}) & \text{for } x_{\rm GR} \le 0.3 \,, \\ \frac{2}{9}\,(1 + 1.69\,x_{\rm GR}) & \text{for } x_{\rm GR} > 0.3 \,. \end{cases}$$

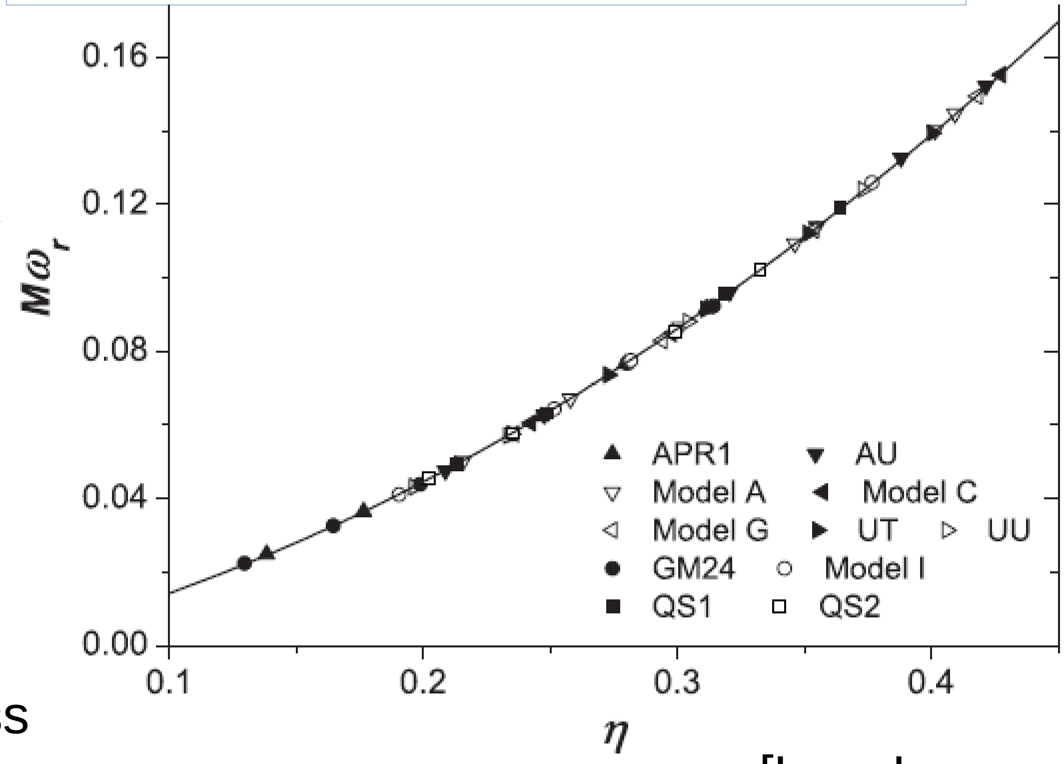
Haensel et al. (2007)

$$x_{\rm GR} = \frac{M/M_{\rm sun}}{R/\rm km}$$

In Andersson & Kokkotas (1998), the compactness M/R is used to connect the f-mode frequency. Motivated by the fact that I carries richer information about the mass distribution, it turns out that a more robust universal relation for the f-mode can be obtained by replacing R by I:

$$M\omega_r = -0.0047 + 0.133\,\eta + 0.575\,\eta^2$$

Scaled f-mode oscillation frequency



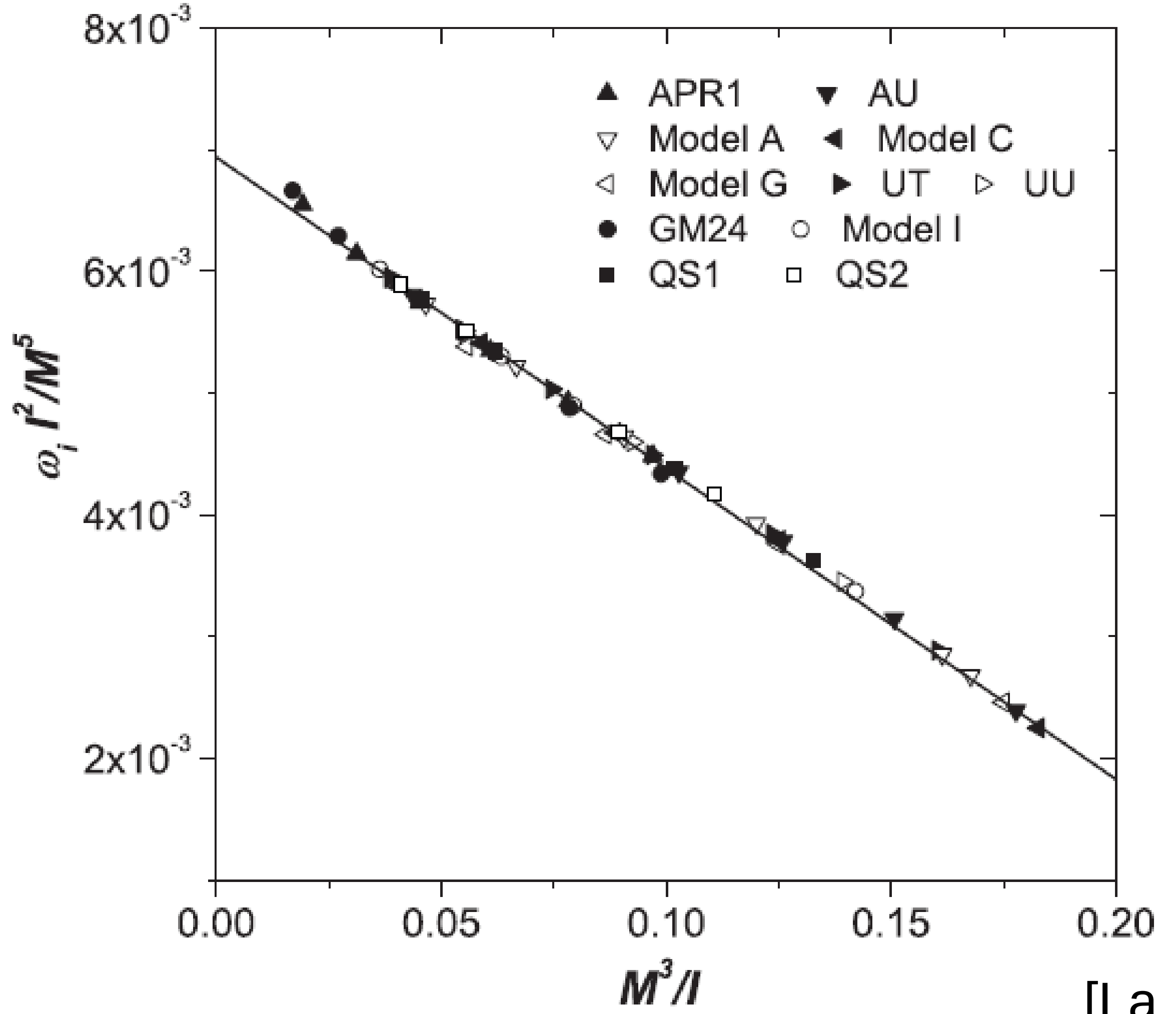
[Lau, Leung and LML (2010)]

Remark:
We set $G = c = 1$ and so $M\omega$ is dimensionless

Effective compactness: $\eta \equiv \sqrt{\frac{M^3}{I}}$ ← Moment of inertia

A similar universal relation exists for the imaginary part ω_i :

$$I^2 \omega_i / M^5 = 0.00694 - 0.0256\,\eta^2$$



[Lau, Leung and LML (2010)]

$= \eta^2$

Potential application: Inversion of physical parameters

If the f-mode frequency (real & imaginary parts) is measured, M and I can be obtained.

* If we additionally assume the empirical relation between I and M/R proposed by Lattimer and Schutz (2005), we can also obtain the radius R:

$$I/MR^2 = 0.237\left(1+4.2x+90x^4\right) \quad , \quad x = \left(M/M_{\text{sun}}\right)\left(\text{km}/R\right)$$

Percentage Error of Our Inversion Scheme [%]

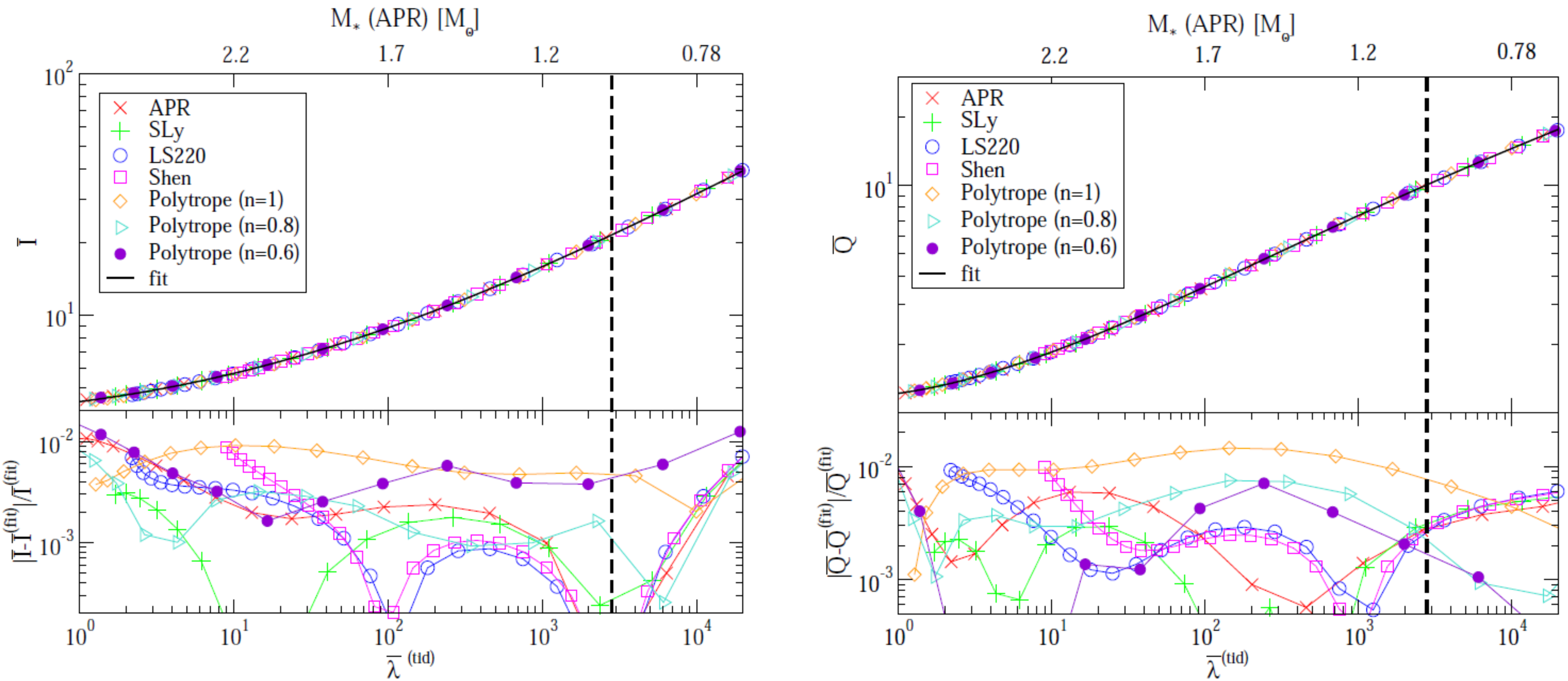
EOS	M	$\delta M/M$	$\delta R/R$	$\delta I/I$
AU	0.8	−0.056	−2.6	−0.031
AU	1.0	−0.13	−0.67	−0.16
AU	1.6	4.0	2.1	6.9
APR1	0.8	−0.066	−5.1	−0.23
APR1	1.2	−0.21	−1.6	−0.19
APR1	1.6	0.50	−0.11	0.78
EOS A	1.535	0.40	−1.2	0.27
EOS A	1.328	1.1	−0.46	1.8
EOS B	1.405	−0.17	−3.5	0.54
EOS B	0.971	1.4	−4.1	2.7
GM24	1.536	−2.2	−4.9	−2.6
GM24	1.405	2.5	−1.1	5.1

I-Love-Q universal relations

In 2013, Yagi and Yunes discovered the so-called I-Love-Q relations for

$$\bar{I} \equiv \frac{I}{M^3} \quad , \quad \bar{Q} \equiv -\frac{Q}{M^3 j^2} \quad , \quad \bar{\lambda} \equiv \frac{\lambda}{M^5}$$

- Q = rotation-induced quadrupole moment
- λ = tidal deformability
- j = spin parameter

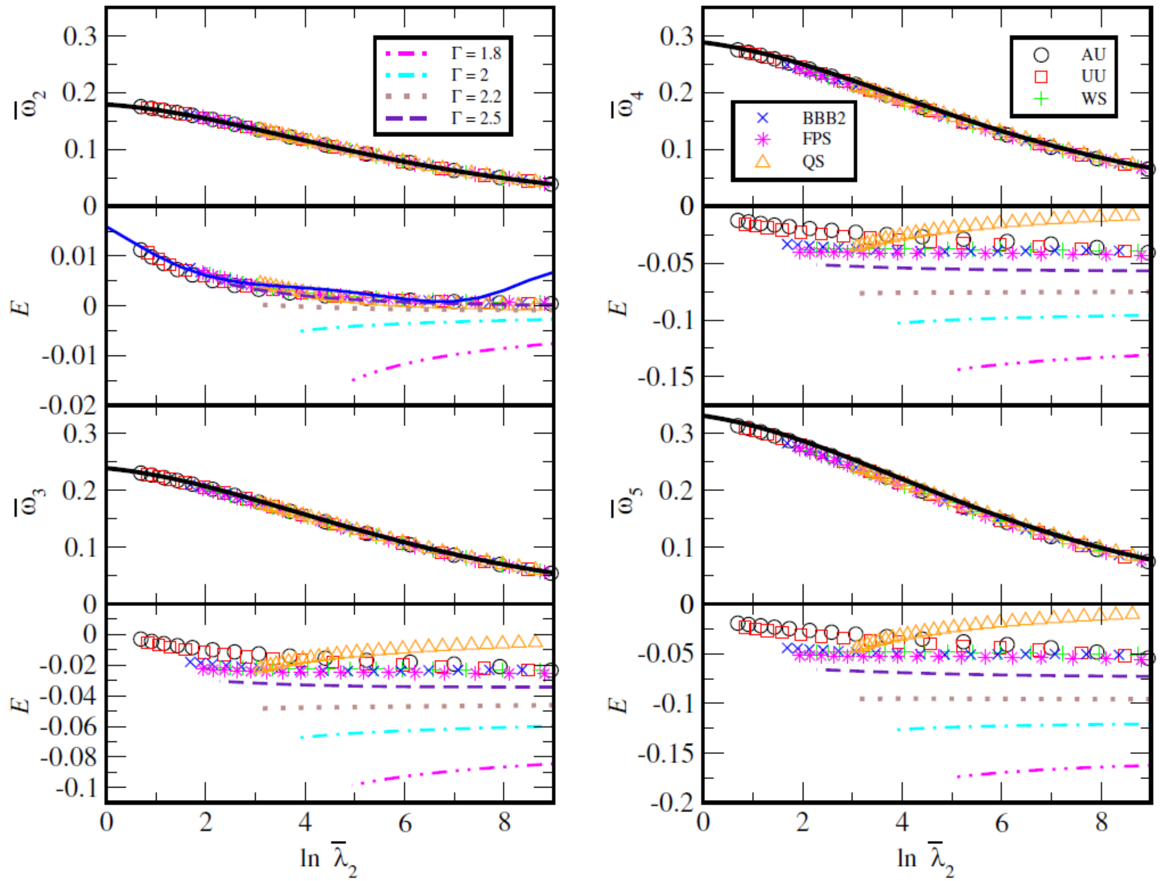


Yagi & Yunes (2013)

f-mode-Love universal relations

The f-mode relation of Lau et al. (2010) and the I-Love-Q relation of Yagi and Yunes (2013) both involve the effective compactness parameter, and hence it is natural to expect that a universal relation between ω and λ should exist.

[Chan, Sham, Leung, and LML (2014)]



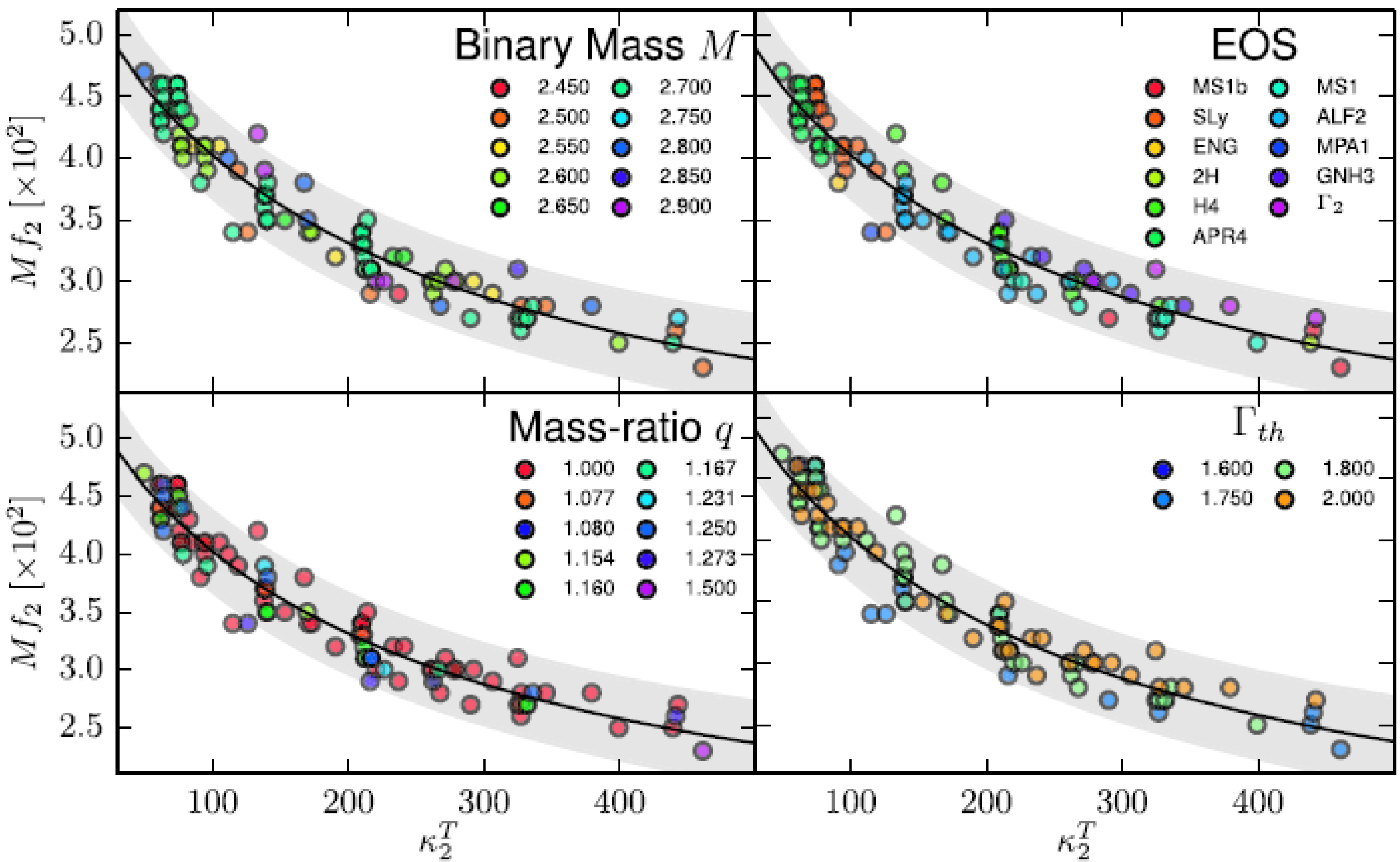
$$\bar{\omega}_l \equiv M\,\omega_l$$

$$\bar{\lambda}_l \equiv \frac{\lambda_l}{M^{2l+1}}$$

In Chan et al. (2014), such a relation is proposed and it is also found that the relation becomes more EOS-sensitive for ω and λ with different values of l. The reason behind this observation is also studied in Newtonian gravity in Chan et al. (2014).

"Universal" relation in binary neutron star simulations

Bernuzzi, Dietrich, and Nagar (2015)



f_2 = post-merger GW peak frequency

$$\kappa_2^T = 2\left(\frac{q^4}{(1+q)^5}\frac{k_2^A}{C_A^5} + \frac{q}{(1+q)^5}\frac{k_2^B}{C_B^5}\right),$$

k_2 = Love number

C = compactness

q = mass ratio

Learning Outcomes:

- Understand the basic concepts of Newtonian stellar perturbations
- Be able to derive the perturbation equations for oscillations and tidal deformation in the simplest settings and be able to extend to more general situations
- Have an idea on the concepts on universal relations and their potential applications

Some references for beginners:

** Oscillations
- P. N. McDermott et al., ApJ 325, 725 (1988)
- J. P. Cox, “The theory of stellar pulsation”, Princeton University Press (1980)
- K. D. Kokkotas and B. G. Schmidt, gr-qc/9909058
- K. S. Thorne and A. Campolattaro, ApJ 149, 591 (1967)
- S. Detweiler and L. Lindblom, ApJ 292, 12 (1985)

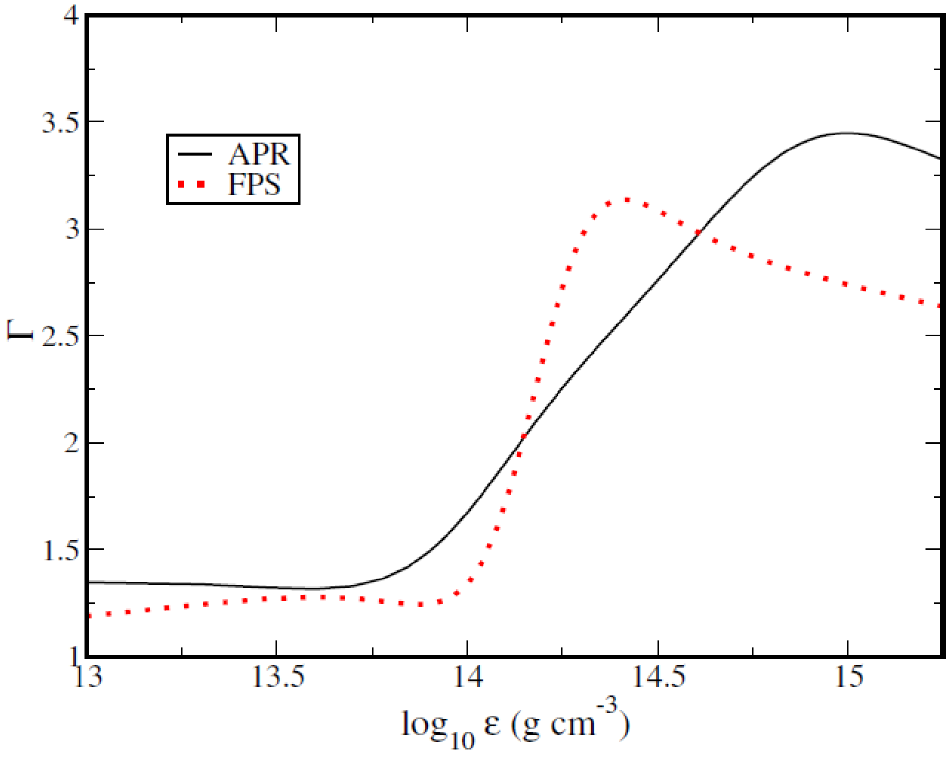
** Tidal deformations and universal relations
- T. Hinderer, ApJ 677, 1216 (2008)
- H. K. Lau, P. T. Leung, and L. M. Lin, ApJ 714, 1234 (2010)
- Yagi and Yunes, PRD 88, 023009 (2013)
- Chapters 1 & 2 in E. Poisson and C. M. Will, “Gravity: Newtonian, Post-Newtonian, Relativistic”, Cambridge University Press (2014)
- Yagi and Yunes, Physics Reports 681, 1 (2017)

Appendix

Why I-Love-Q?

Our proposal:
NS EOSs are stiff enough!

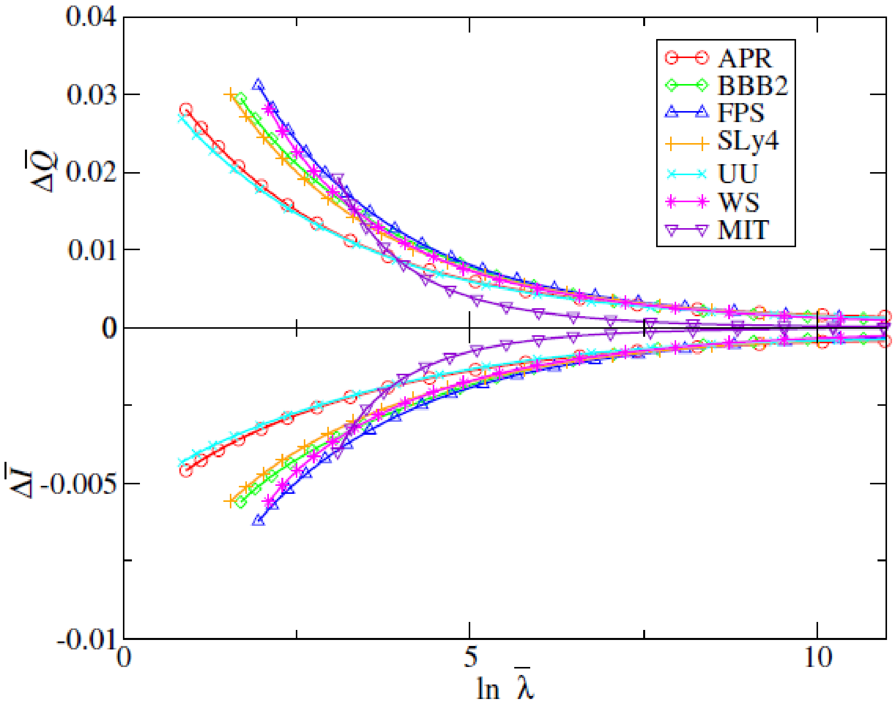
- We propose that the stiffness of modern EOS is the key!
- Realistic EOS models typically have effective adiabatic index $\Gamma \geq 2$

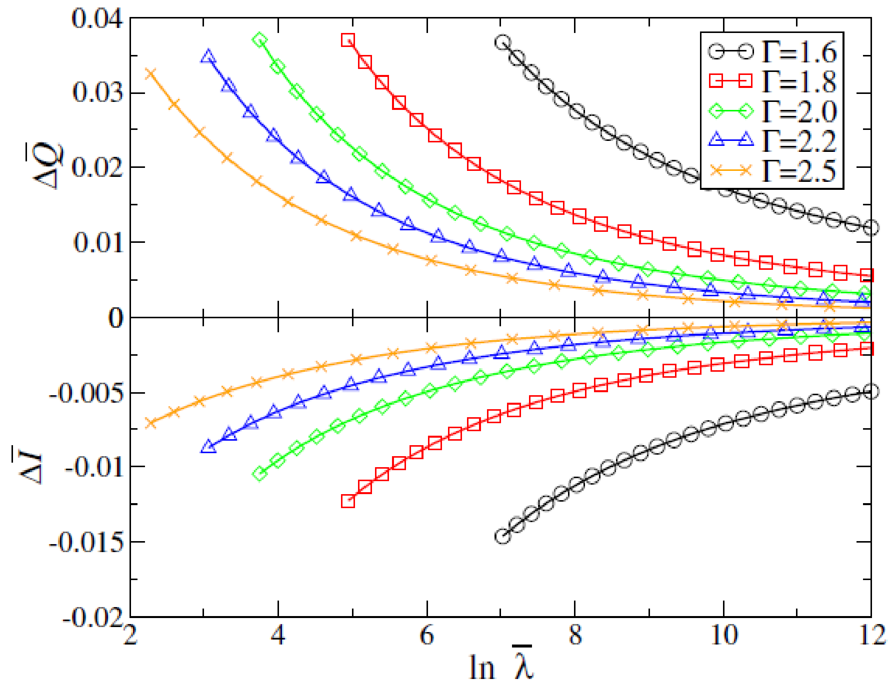


(above nuclear density)

[Sham, Chan, Lin, & Leung (2015)]

- The EOS models are so stiff that the I-Love-Q relations are well modeled by the incompressible limit





$$\Delta \bar{I} \equiv \frac{\bar{I} - \bar{I}_{incom}}{\bar{I}_{incom}}$$

For incompressible model

similarly for $\Delta \bar{Q}$ and $\Delta \bar{\lambda}$

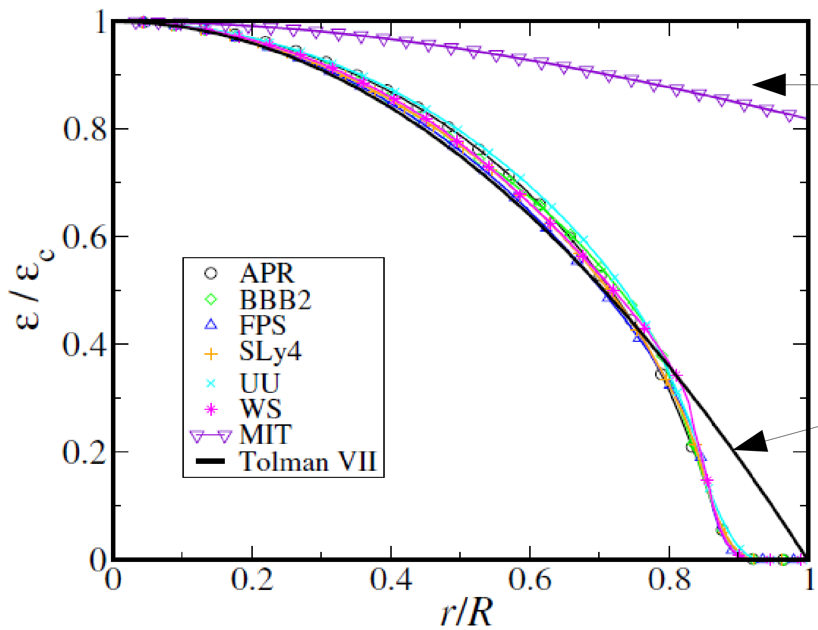
Sham, Chan, Lin, & Leung
ApJ, 798, 121 (2015)

Analytical study

- Here we study the I-Love relation analytically in Newtonian gravity and show that the incompressible limit is a key point to the universal relations

- Neutron stars are modeled well by the density profile:

$$x \equiv \frac{r}{R} \qquad \rho = \rho_0 \left(1 - \delta x^2\right) \qquad \delta = \text{free parameter}$$



Quark star (MIT bag model) $\delta \approx 0$

$\delta = 1$

- Scaled moment of inertia

$$\bar{I} = \frac{\int_0^R \rho\, r^4\, dr}{24\,\pi^2 \left(\int_0^R \rho\, r^2\, dr \right)^3}$$

With the density profile $\rho = \rho_0 \left(1 - \delta\, x^2\right)$

$$\bar{I} = \frac{2(7 - 5\,\delta)}{7(5 - 3\,\delta)} C^{-2}$$

$$C \equiv \frac{M}{R}$$

- Tidal deformability λ (in Newtonian gravity)

$$\frac{d^2 h}{dr^2}+\frac{2}{r}\frac{dh}{dr}-\left(\frac{6}{r^2}-4\pi\rho\frac{d\rho}{dP}\right)h=0$$

$$\bar{\lambda}=\frac{\lambda}{M^5}$$

$$\bar{\lambda}=\frac{2-y(R)}{3[3+y(R)]}C^{-5}$$

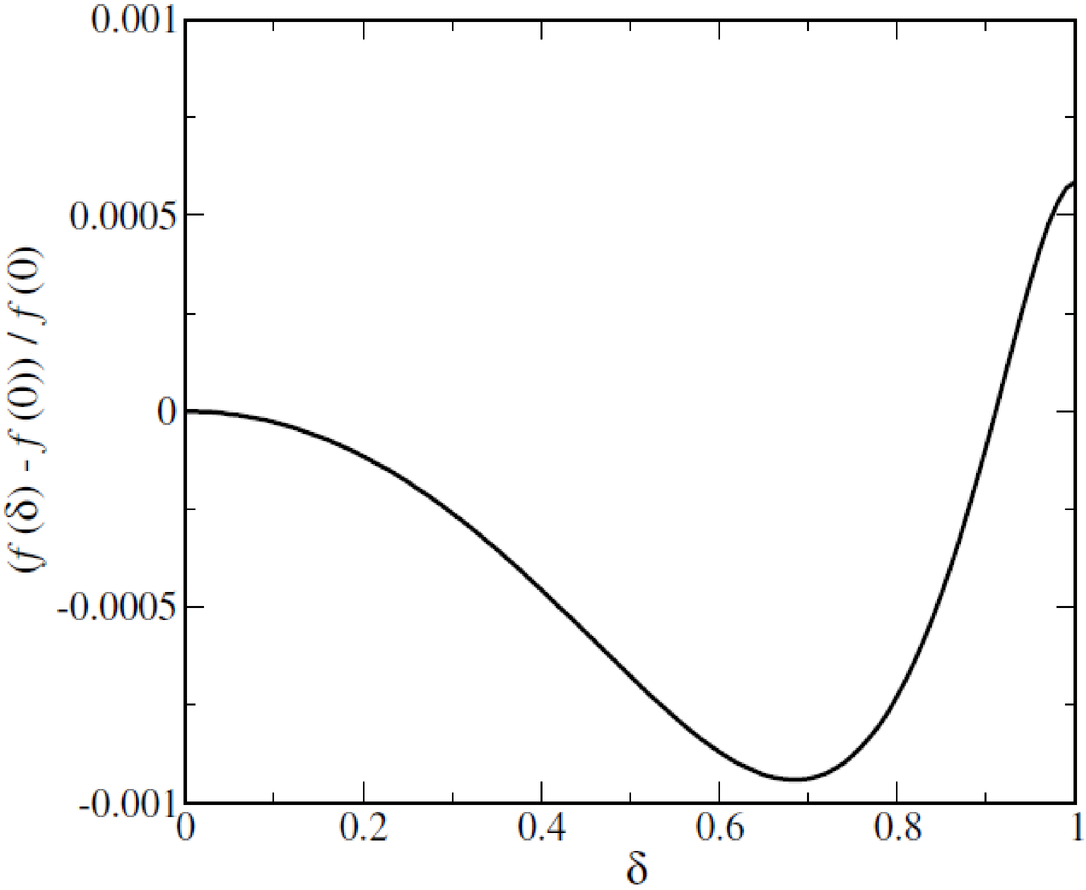
$$y(R)\equiv\frac{R\,h'(R)}{h(R)}$$

- By eliminating the compactness C, we obtain the I-Love relation

$$f(\delta)\equiv\bar{\lambda}\,\bar{I}^{-5/2}=\frac{2-y(R)}{3(3+y(R))}\left[\frac{2(7-5\delta)}{7(5-3\delta)}\right]^{-5/2}$$

- The I-Love relation $f(\delta)$ depends weakly on δ

$$f(\delta) \equiv \bar{\lambda}\,\bar{I}^{-5/2} = \frac{2-y(R)}{3(3+y(R))}\left[\frac{2(7-5\delta)}{7(5-3\delta)}\right]^{-5/2}$$



$$\rho = \rho_0 (1 - \delta x^2)$$

$$x \equiv \frac{r}{R}$$

$\delta = 0$ incompressible limit

- We can expand the I-Love relation about the incompressible limit ($\delta=0$)

$$\bar{\lambda}\,\bar{I}^{-5/2}=5\sqrt{\frac{5}{2}}\left(\frac{5}{8}-\frac{1}{588}\delta^2+\ldots\right)$$

2nd order

- The incompressible stellar model is a stationary point for the I-Love relation

Remark: $y(R)$ has a formal solution given by the hypergeometric function

$$f(\delta)\equiv\bar{\lambda}\bar{I}^{-5/2}=\frac{2-y(R)}{3(3+y(R))}\left[\frac{2(7-5\delta)}{7(5-3\delta)}\right]^{-5/2}$$

$$y(R)=2-\frac{6\delta}{7}\frac{{}_2F_1\left(\frac{9-\sqrt{65}}{4},\frac{9+\sqrt{65}}{4},\frac{9}{2},\frac{3\delta}{5}\right)}{{}_2F_1\left(\frac{5-\sqrt{65}}{5},\frac{5+\sqrt{65}}{4},\frac{7}{2},\frac{3\delta}{5}\right)}-\frac{15(1-\delta)}{5-3\delta}.$$

- Brief summary

$$\rho=\rho_0\left(1-\delta x^2\right)$$

δ is used to model the stellar structure and EOS

$$\bar{I}=\frac{2\left(7-5\delta\right)}{7\left(5-3\delta\right)}C^{-2}$$

$$\bar{\lambda}=\frac{2-y\left(R\right)}{3\left[3+y\left(R\right)\right]}C^{-5}$$

They depend non-trivially on δ

- However, the I-Love relation depends weakly on δ (and hence EOS)

$$\bar{\lambda}\,\bar{I}^{-5/2}=\frac{2-y\left(R\right)}{3\left(3+y\left(R\right)\right)}\left[\frac{2\left(7-5\delta\right)}{7\left(5-3\delta\right)}\right]^{-5/2}$$

$$=a+b\,\delta^2+....$$

Construction of $\hat{Y}^{lm}_{k_1 k_2 \ldots k_l}$

$$Y^{lm}(\theta,\varphi) = \hat{Y}^{lm}_{k_1 k_2 \ldots k_l} n_{k_1} n_{k_2} \ldots n_{k_l}$$

Here we shall obtain the explicit form of $\hat{Y}^{lm}_{k_1 k_2 \ldots k_l}$ by starting with standard results of Y^{lm}:

$$Y^{lm}(\theta,\varphi) \equiv C^{lm} e^{im\varphi} P^{lm}(\cos\theta) \qquad C^{lm} = (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}$$

$$(0 \le m \le l)$$

Associated Legendre function: $P^{lm}(x) \equiv (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_l(x)$

$$P_l(x) = \sum_{j=0}^{[l/2]} a_{lj} x^{l-2j} \qquad a_{lj} = (-1)^j \frac{(2l-2j)!}{2^l j!(l-j)!(l-2j)!}$$

[l/2] = largest integer less than or equal to l/2

Note: For negative m, we use $Y^{lm}(\theta,\varphi) = (-1)^m \left(Y^{l|m|}(\theta,\varphi)\right)^*$

=> $$P^{lm}(x)=(1-x^2)^{m/2}\sum_{j=0}^{[(l-m)/2]} a^{lmj}\, x^{l-m-2j} \qquad a^{lmj}\equiv a_{lj}\frac{(l-2j)!}{(l-2j-m)!}$$

Hence, Y^{lm} can be written as $$Y^{lm}(\theta,\varphi)=C^{lm}(e^{i\varphi}\sin\theta)^m\sum_{j=0}^{[(l-m)/2]} a^{lmj}(\cos\theta)^{l-m-2j}$$

Recall: Unit vector in the direction (θ,φ)

$$n_x+i\,n_y=e^{i\varphi}\sin\theta \quad , \quad n_z=\cos\theta$$

$$Y^{lm}(\theta,\varphi)=C^{lm}(n_x+i\,n_y)^m\sum_{j=0}^{[(l-m)/2]} a^{lmj}(n_z)^{l-m-2j}$$

Note that we can express

$$n_x=\delta^1_{k_1}n_{k_1}=\delta^1_{k_2}n_{k_2}\ldots\text{etc}$$
$$n_y=\delta^2_{k_1}n_{k_1}\ldots\text{etc}$$
$$n_z=\delta^3_{k_1}n_{k_1}\ldots\text{etc}$$

$$(n_x+in_y)^m=(\delta^1_{k_1}+i\delta^2_{k_1})(\delta^1_{k_2}+i\delta^2_{k_2})....(\delta^1_{k_m}+i\delta^2_{k_m})n_{k_1}n_{k_2}...n_{k_m}$$

$$(n_z)^{l-m-2j}=(\delta^3_{k_{m+1}}\delta^3_{k_{m+2}}...\delta^3_{k_{l-2j}})(n_{k_{m+1}}n_{k_{m+2}}...n_{k_{l-2j}})$$

$$Y^{lm}(\theta,\varphi)=C^{lm}\sum_{j=0}^{[(l-m)/2]}a^{lmj}(\delta^1_{k_1}+i\delta^2_{k_1})(\delta^1_{k_2}+i\delta^2_{k_2}).....(\delta^1_{k_m}+i\delta^2_{k_m})$$
$$\times(\delta^3_{k_{m+1}}\delta^3_{k_{m+2}}.....\delta^3_{k_{l-2j-1}}\delta^3_{k_{l-2j}})$$
$$\times(\underbrace{n_{k_1}n_{k_2}.....n_{k_m}n_{k_{m+1}}.....n_{k_{l-2j}}}_{l-2j\text{ factors}})$$

We need $2j$ more factors of n_j in order to extract the factor $(n_{k_1}n_{k_2}....n_{k_l})$

n is unit vector: $n^{a_1}n^{a_1}=1=(\delta^{a_1}_{k_p}n_{k_p})(\delta^{a_1}_{k_n}n_{k_n})$

We can insert the following factor in the summation:

$$1=(\delta^{a_1}_{k_{l-2j+1}}\delta^{a_1}_{k_{l-2j+2}})(\delta^{a_2}_{k_{l-2j+2}}\delta^{a_2}_{k_{l-2j+3}})....(\delta^{a_j}_{k_{l-1}}\delta^{a_j}_{k_l})(\underbrace{n_{k_{l-2j+1}}n_{k_{l-2j+2}}....n_{k_{l-1}}n_{k_l}}_{2j\text{ of them}})$$

$$Y^{lm}(\theta,\varphi)=C^{lm}\sum_{j=0}^{[(l-m)/2]} a^{lmj}(\delta^1_{k_1}+i\delta^2_{k_1})(\delta^1_{k_2}+i\delta^2_{k_2}).....(\delta^1_{k_m}+i\delta^2_{k_m})$$
$$\times\left(\delta^3_{k_{m+1}}\delta^3_{k_{m+2}}.....\delta^3_{k_{l-2j-1}}\delta^3_{k_{l-2j}}\right)\left(\delta^{a_1}_{k_{l-2j+1}}\delta^{a_1}_{k_{l-2j+2}}\right)......\left(\delta^{a_j}_{k_{l-1}}\delta^{a_j}_{k_l}\right)$$
$$\times\left(n_{k_1}n_{k_2}.....n_{k_m}n_{k_{m+1}}.....n_{k_{l-1}}n_{k_l}\right)$$

Hence, we have $Y^{lm}(\theta,\varphi)=\hat{Y}^{lm}_{k_1k_2\ldots k_l}n_{k_1}n_{k_2}\ldots n_{k_l}$

$$\hat{Y}^{lm}_{k_1k_2\ldots k_l}\equiv C^{lm}\sum_{j=0}^{[(l-m)/2]} a^{lmj}(\delta^1_{(k_1}+i\delta^2_{(k_1})(\delta^1_{k_2}+i\delta^2_{k_2}).....(\delta^1_{k_m}+i\delta^2_{k_m})$$
$$\times\left(\delta^3_{k_{m+1}}\delta^3_{k_{m+2}}.....\delta^3_{k_{l-2j-1}}\delta^3_{k_{l-2j}}\right)\left(\delta^{a_1}_{k_{l-2j+1}}\delta^{a_1}_{k_{l-2j+2}}\right)......\left(\delta^{a_j}_{k_{l-1}}\delta^{a_j}_{k_l)}\right)$$
$$(\text{for } 0\leq m\leq l)$$
$$\hat{Y}^{lm}_{k_1k_2\ldots k_l}\equiv(-1)^m\left(\hat{Y}^{l|m|}_{k_1k_2\ldots k_l}\right)^* \qquad (\text{for } -l\leq m<0)$$

In the above definition, we have included a symmetrization operation. It can also be shown that the object so defined is completely tracefree. So, it is a STF tensor.

Remark: For an arbitrary tensor S_{abc},

$$S_{abc}n_an_bn_c=S_{(abc)}n_an_bn_c$$

(contracted and summed over the indices)

Example: Explicit expressions for l = 2

$$\hat{Y}^{lm}_{k_1 k_2 \ldots k_l} \equiv C^{lm} \sum_{j=0}^{[(l-m)/2]} a^{lmj} \left(\delta^1_{(k_1} + i\,\delta^2_{(k_1}\right)\left(\delta^1_{k_2} + i\,\delta^2_{k_2}\right) \ldots\ldots \left(\delta^1_{k_m} + i\,\delta^2_{k_m}\right)$$

$$\times \left(\delta^3_{k_{m+1}}\,\delta^3_{k_{m+2}} \ldots\ldots \delta^3_{k_{l-2j-1}}\,\delta^3_{k_{l-2j}}\right)\left(\delta^{a_1}_{k_{l-2j+1}}\,\delta^{a_1}_{k_{l-2j+2}}\right) \ldots\ldots \left(\delta^{a_j}_{k_{l-1}}\,\delta^{a_j}_{k_l)}\right)$$

$$C^{lm} = (-1)^m \left[\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}\right]^{1/2} \qquad a^{lmj} \equiv \frac{(-1)^j}{2^l\, j!\,(l-j)!}\,\frac{(2l-2j)!}{(l-m-2j)!}$$

We need to consider m = 0, 1, 2 only. Negative values of m (−1, −2) are obtained by

$$\hat{Y}^{lm}_{k_1 k_2 \ldots k_l} \equiv (-1)^m \left(\hat{Y}^{l|m|}_{k_1 k_2 \ldots k_l}\right)^* \qquad (\text{for } -l \le m < 0)$$

For l = 2, m = 0, there are two terms (j = 0, 1) in the sum:

$$\text{For } j=0: \qquad \delta^3_{(k_1}\,\delta^3_{k_2)} = \delta^3_{k_1}\,\delta^3_{k_2}$$

$$\text{For } j=1: \qquad \delta^{a_1}_{(k_1}\,\delta^{a_1}_{k_2)} = \delta^1_{k_1}\,\delta^1_{k_2} + \delta^2_{k_1}\,\delta^2_{k_2} + \delta^3_{k_1}\,\delta^3_{k_2}$$

$$\hat{Y}^{20}_{k_1 k_2} = \sqrt{\frac{5}{16\pi}}\left(2\,\delta^3_{k_1}\,\delta^3_{k_2} - \delta^1_{k_1}\,\delta^1_{k_2} - \delta^2_{k_1}\,\delta^2_{k_2}\right)$$

For l = 2, m = 1, there is only one term (j = 0) in the sum:

$$\text{For } j=0: \qquad \left(\delta^1_{(k_1}+i\,\delta^2_{(k_1}\right)\delta^3_{k_2)}$$

$$\hat{Y}^{21}_{k_1k_2}=-\sqrt{\frac{15}{32\pi}}\left(\delta^1_{k_1}\delta^3_{k_2}+\delta^1_{k_2}\delta^3_{k_1}+i\,\delta^2_{k_1}\delta^3_{k_2}+i\,\delta^2_{k_2}\delta^3_{k_1}\right)$$

For l = 2, m = 2, there is only one term (j = 0) in the sum:

$$\text{For } j=0: \qquad \left(\delta^1_{(k_1}+i\,\delta^2_{(k_1}\right)\left(\delta^1_{k_2}+i\,\delta^2_{k_2)}\right)$$

$$\hat{Y}^{22}_{k_1k_2}=\sqrt{\frac{15}{32\pi}}\left(\delta^1_{k_1}\delta^1_{k_2}-\delta^2_{k_1}\delta^2_{k_2}+i\,\delta^1_{k_1}\delta^2_{k_2}+i\,\delta^2_{k_1}\delta^1_{k_2}\right)$$

Summary:

$$\hat{Y}^{20}_{k_1k_2}=\sqrt{\frac{15}{16\pi}}\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{bmatrix} \qquad \hat{Y}^{21}_{k_1k_2}=-\sqrt{\frac{15}{32\pi}}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & i\\ 1 & i & 0\end{bmatrix} \qquad \hat{Y}^{22}_{k_1k_2}=\sqrt{\frac{15}{32\pi}}\begin{bmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\hat{Y}^{2,-1}_{k_1k_2}=-\left(\hat{Y}^{21}_{k_1k_2}\right)^* \qquad \hat{Y}^{2,-2}_{k_1k_2}=\left(\hat{Y}^{22}_{k_1k_2}\right)^*$$