Study of the Nonlinear Mode-tide Coupling of Coalescing Binary Neutron Stars in Relativistic Formalism

Abhishek Das, Fatemeh Hossein-Nouri, Sukanta Bose, Matt Duez

The Brief History of Mode-tide Coupling Instabilities in NSNS Systems

- **Weinberg et al. (ApJ, 769, 121, 2015)** proposed p -g tidal instability, in which the tidal bulge excites a low-frequency g-mode and a high frequency p-mode, can extract orbital energy, and cause a phase shift in the GW signal from the early inspiral phase, observable by gravitational wave detectors.
- **Venumadhav et al. (ApJ, 781, 23, 2015)** showed in second order perturbation, assuming a *static tide*, near-exact *cancellation* occurs between the three and four-mode couplings, which reduced the growth rate and implied that the instability cannot affect the inspiral significantly.
- **Weinberg (ApJ, 819, 109, 2016)** relaxed simplifying assumptions, allowing the stars become compressible under the influence of non-static linear tides. As a result, the near-exact cancellation is undone, and the instability becomes important once again.
- **Zhou & Zhang (ApJ, 849,114, 2017)** computed mode-tide coupling strength (MTCS) for TOV models using six different equations of state for both static (time-independent) and non-static tides.

Zhou & Zhang's MTCS studies (2017)

Mode-tide coupling strength (MTCS) computed applying the same formalism introduced in Weinberg's and Venumadhav's works using perturbation theory, to study the effects of EOS on the MTCS.

1) Confirm the near-exact cancellation happens between three and four mode couplings for static tides.
2) Stronger mode-tide coupling for non-static tides, trigger the instability in late inspiral phase.

TABLE 3
THE MTCS EVALUATED WITH THE $l_g = 4$, $n = 32$ EXAMPLE G-MODES AT $A = 95$ KM (NO RESONANCE OCCURS AT THIS BINARY SEPARATION), TOGETHER WITH THE INSTABILITY THRESHOLD AND THE DURATION OF THE INSTABILITY GROWTH WINDOW.

EOS	f_g (Hz)	MTCS ($A = 95$ km) Static tide	MTCS ($A = 95$ km) Non-static tide	Instability Threshold (km)	t_{mg} (ms)
SLy4	2.71	1.75×10^{-3}	2.01	144	1583
Shen	27.7	1.60×10^{-4}	1.23×10^{-2}	57.0	35.4
APR1	23.3	4.53×10^{-5}	9.71×10^{-2}	41.2	10.1
APR2	23.7	2.99×10^{-5}	5.27×10^{-2}	46.8	16.9
APR3	15.9	5.58×10^{-5}	1.91×10^{-2}	58.9	42.8
APR4	18.8	3.95×10^{-5}	4.10×10^{-2}	54.4	30.7

Zhou & Zhang (ApJ, 849,114, 2017)

MTCS equations:

g-mode frequency shift resulting from tidal perturbations. Keeping the leading order, we have:

$$\frac{{\omega_-}^2}{{\omega_g}^2} = 1 - \epsilon \underbrace{\left|U_{gg} + 2\kappa_{\chi gg}\right|}_{\text{MTCS}} + \mathcal{O}(\epsilon^2)$$

- $\kappa_{\chi gg}$: coupling coefficient between g-mode and the tide.
- U_{gg} : defined as $-\frac{1}{E_0}\int d^3\mathbf{x}\rho\boldsymbol{\xi}_a \cdot (\boldsymbol{\xi}_b \cdot \nabla)\nabla U$
- ϵ : tidal strength defined as R^3/A^3
- ω_- : is the perturbed g-mode frequency
- ω_0 : characteristic frequency $\sqrt{GM/R^3}$

Static tide:
The tidal field (ϵU) is time-independent:

$$U \approx -\omega_0^2 r^2 P_2(cos\theta)$$

Non-static tide:
The tidal field (ϵU) is time-dependent:

$$U_{\text{full}} \approx -\omega_0^2 r^2 \sum_{m=-2}^{2} W_{2m} Y_{2m}(\theta, \phi) e^{-im\Omega t}$$

The dimensionless MTCS value determines how much the g-mode frequency is shifted due to the coupling with the tide.

$$\frac{{\omega_-}^2}{{\omega_g}^2} \approx 1 - \epsilon|MTCS|$$

When MTCS > 1, the perturbed frequency ω_- becomes imaginary and exponentially drives the mode to large amplitudes.

New stuff added to our work:

Zhou & Zhang (2017):

- ADIPLS package used to derive eigen–modes of neutron stars, which means Newtonian adiabatic oscillation equations for a relativistic star!!!
- Some hydro variables are defined in Newtonian way, which are not consistent for a relativistic system.
- The Cowling approximation (neglecting the perturbation to gravitational potential from g-mode) was used to compute MTCS, but has not been applied to the rest of the computations, i.e. eigenmodes.

Our work:

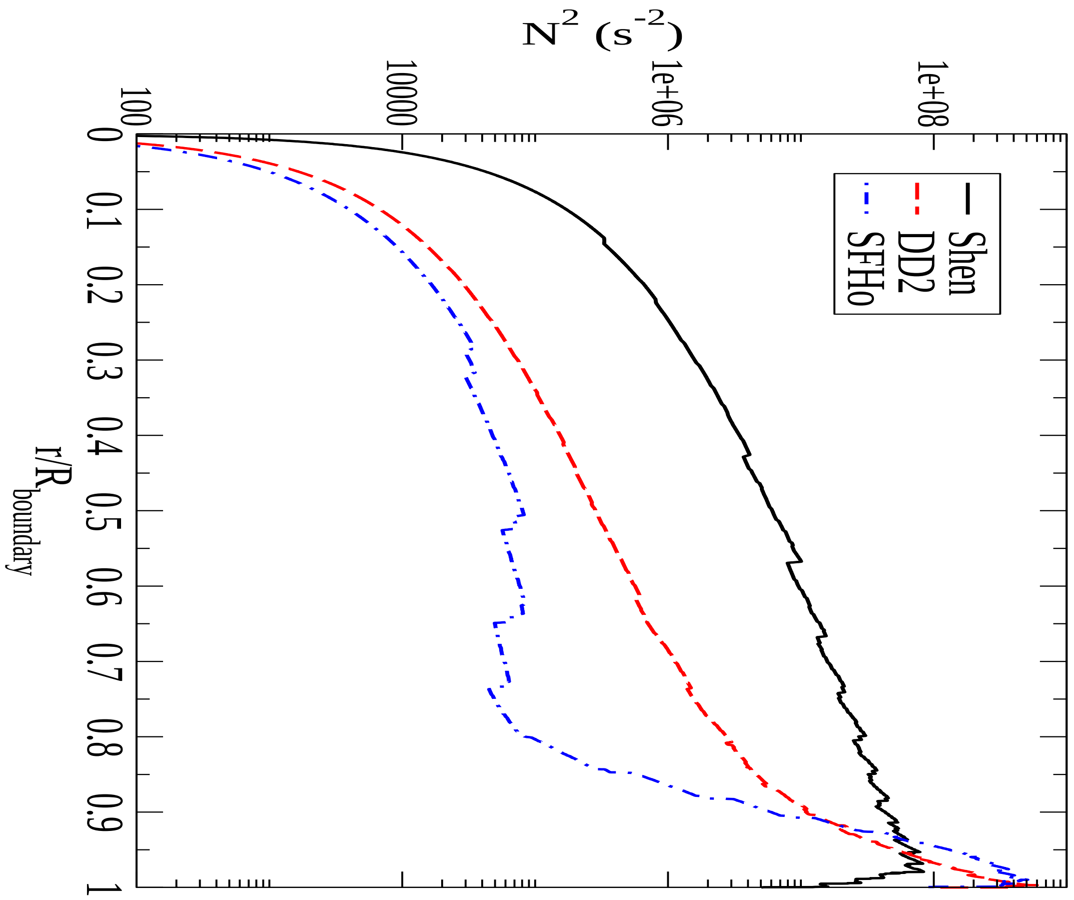
- We add relativistic corrections to ZZ’s MTCS computation by:
- 1- Developing our own code to compute relativistic eigenmodes in the Cowling approximation.
- 2- Using the relativistic definition of hydro variables for consistency.
- 3- Using more updated relativistic nuclear equations of state (such as SFHo, DD2)
- For a better comparison and study how significant these relativistic components are, we compute MTCS in pure Newtonian formalism as well (the background star is still relativistic).

Equations of State

- The central density adjusted to have equal mass for all the TOV models, $M = 1.4_{\odot}$

EOS	SFHo	DD2	Shen
R (km)	11.25	13.22	14.53

- We consider only core g-modes by resetting the Brunt-Vaisala frequency to zero in the crust region.
- Crust-core boundary is determined by choosing the position where Y_e hits its minimum.
- The neutron stars in the binary system are identical (equal mass and radius)



Brunt-Vaisala frequency for different EOS

Results: Static Tide

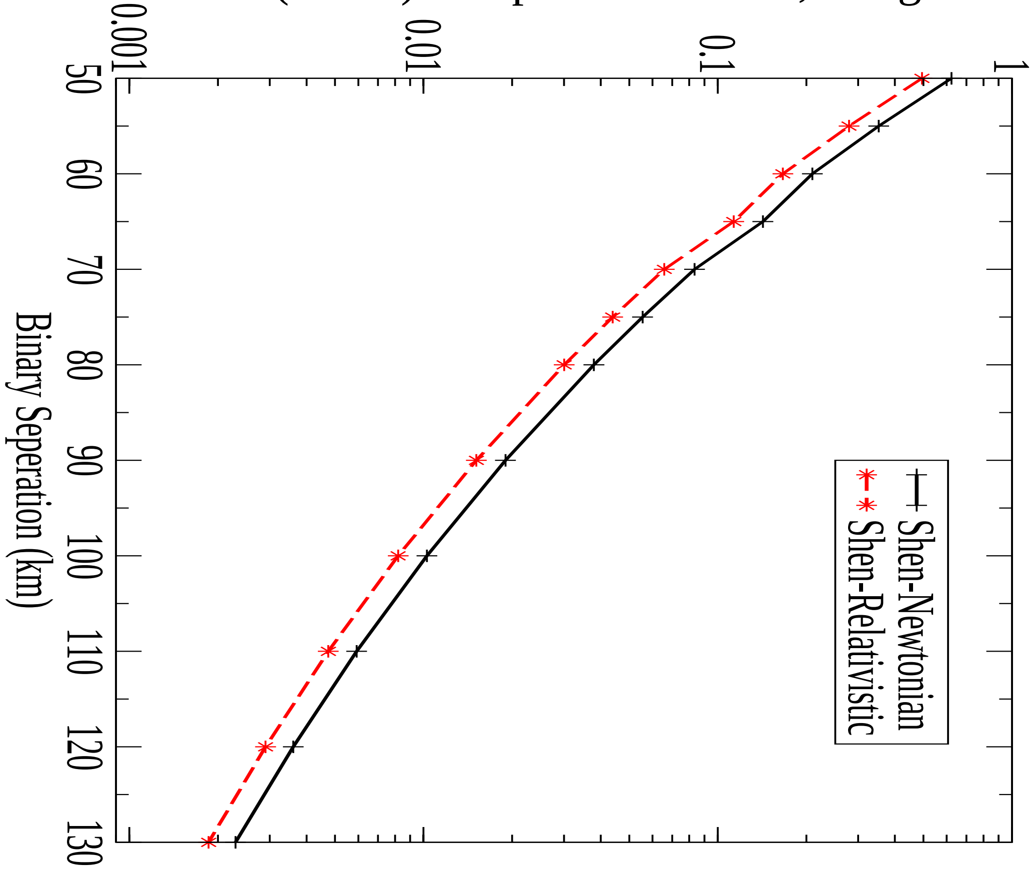
MTCS computed for static tide for two binary separations, A=100 km, and A=2R, for g-mode, n=32, l_g=4

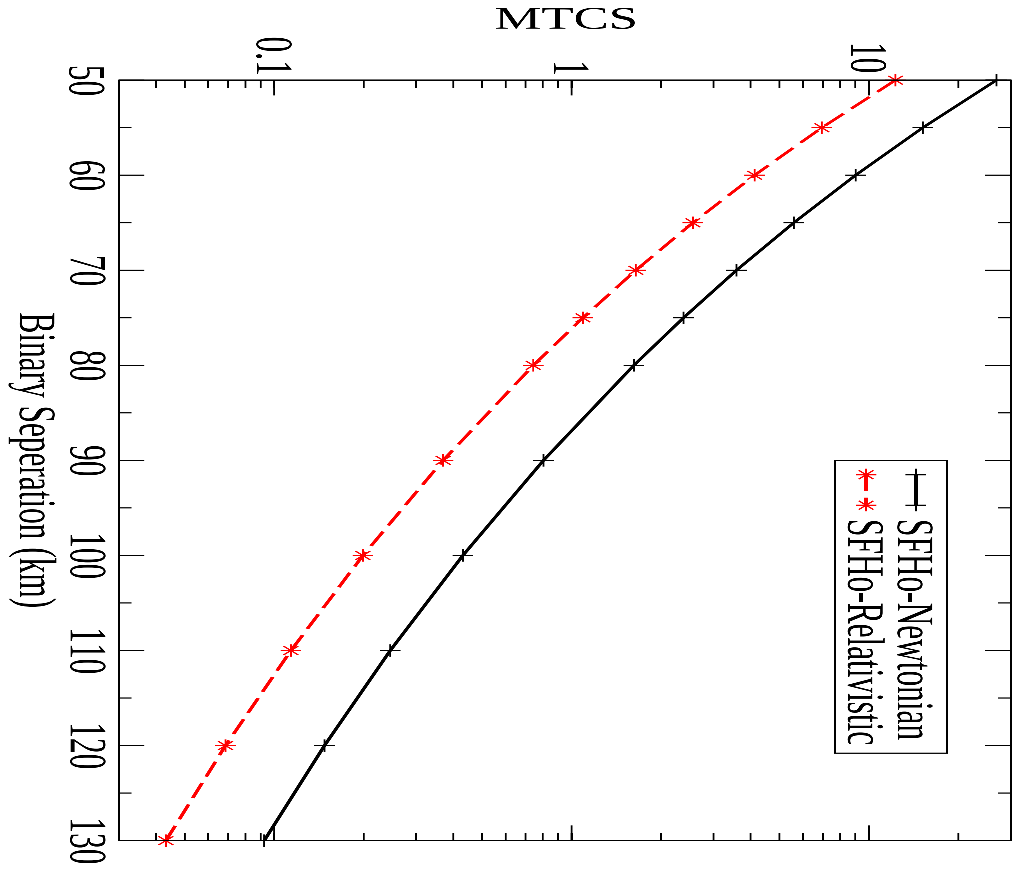
- The relativistic corrections change the MTCS only by a few percent.
- MTCS is still too small to trigger the instability during inspiral.

EOS	MTCS (Newtonian) (A = 2R)	MTCS (Relativistic) (A = 2R)	MTCS (Newtonian) (A = 100 km)	MTCS (Relativistic) (A = 100 km)
Shen	0.2466	0.2621	0.000258	0.000282
DD2	0.2955	0.3901	0.000349	0.000418
SFHo	1.19	1.07	0.000312	0.000335

Results: Non-static (Time-dependent) Tide

Newtonian and relativistic MTCS for non-static tide verses binary separation for a stiff EOS (Shen), and for a soft EOS (SFHo) computed for n=32, l=4 g-mode.





Results: Non-static tide and Instability Threshold

The Newtonian and relativistic MTCS evaluated for g-mode n=32, l=4, for non-static tides with binary separation A=95 km. The instability threshold happens at the binary separation where MTCS exceeds 1.

EOS	$f_{g\ (N)}$ (Hz)	$f_{g\ (R)}$ (Hz)	$MTCS_{(N)}$	$MTCS_{(R)}$	$Threshold_{(N)}$ (km)	$Threshold_{(R)}$ (km)
Shen	36.06	27.46	0.01384	0.01103	48.3*	47.2*
DD2	14.15	10.91	0.01866	0.01212	54*	52*
SFHo	11.45	8.91	0.5837	0.2688	86.8	75.9

* These values are estimated based on other data points, and not very accurate.

Conclusions (so far)

- The relativistic corrections have the major effect in the softer equation of state, decrease the MTCS for non-static tide by more than 50% for SFHo.
- The relativistic corrections could only shift the MTCS for static tide by a few percent, the magnitude of the MTCS is still too low to cause any instability.
- For non-static tide, the relativistic corrections make the MTCS smaller over a wide range of binary separations, this would shift the instability threshold to smaller binary separations (late inspiral phase).
- EOS with lower Brunt-Vaisala frequency has stronger MTCS.

Future Work

- Consider crust g-modes (we should probably add a suitable crust treatment first, i.e. adding a shear stress term, etc)
- Compute MTCS for Higher order g-modes. Based on what proposed by Weinberg et al. (2015), the coupling is stronger for higher modes, which can trigger the instability in earlier time of the inspiral phase.
- Consider other EOS, to study the effects of stiffness on the MTCS.

MTCS main equations:

Static tide:
Eq(33) from Zhou & Zhang (2017)

$$
\begin{aligned}
\mathrm{MTCS} \equiv\ & \epsilon^2 |2\kappa_{gg\sigma} + V_{gg}| \\
\approx & -\frac{1}{E_0} \int dr \rho g_r^2 c^2 \left\{ \left[\Gamma_1 + 1 + \left(\frac{\partial \ln \Gamma_1}{\partial \ln \rho} \right)_s \right] \right. \\
& \times \left(r \frac{dX}{dr} + 3X \right) \frac{r^2 \mathrm{g}_1^2}{c_{s1}^2} - 4X \frac{r^2 \mathrm{g}_1^2}{c_{s1}^2} \\
& - 2r^2 \left(r \Lambda_g^2 \frac{\omega_g^2}{c^2} \frac{g_h^2}{g_r^2} + 2\mathrm{g}_1 \right) \frac{dX}{dr} - \mathrm{g}_1 r^3 \frac{d^2 X}{dr^2} \\
& \times \left[-r^2 \mathrm{g}_1 \frac{dX}{dr} - r \mathrm{g}_1 X + r^2 X \frac{d \mathrm{g}_1}{dr} \right. \\
& \left. - \frac{(6-n)(3-n) r^3}{10 A^6 \mathrm{g}_1} \left(\frac{GM}{c^2} \right)^2 \right] \\
& \left. \times \left(\frac{2r \mathrm{g}_1}{c_{s1}^2} + \frac{d \ln \rho}{d \ln r} \right) \right\},
\end{aligned}
$$

Non-static tide:
Eq.(43-45) from Zhou & Zhang (2017)

$$
\begin{aligned}
\mathrm{MTCS} \equiv\ & \epsilon |U_{gg} + 2\kappa_{\chi^{(1)} gg,I} + 2\kappa_{\chi^{(1)} gg,H}| \\
\approx\ & \epsilon \left| \frac{1}{E_0} \int dr \{ T \rho r^2 c_s^2 [\Gamma_1 + 1 \right. \\
& + \left(\frac{\partial \ln \Gamma_1}{\partial \ln \rho} \right)_s \bigg] \nabla \cdot \chi^{(1)} (\nabla \cdot g)_r^2 \\
& + T \rho r c_s^2 [2 \nabla \cdot \chi^{(1)} (\nabla \cdot g)_r (g_h \Lambda_g^2 - 4 g_r) \\
& + (\nabla \cdot g)_r^2 (\chi_h^{(1)} \Lambda_\chi^2 - 4 \chi_r^{(1)})] \\
& + T \rho \frac{d \ln \rho}{d \ln r} \left(4\mathrm{g} + r \frac{d\mathrm{g}}{dr} \right) \chi_r^{(1)} g_r^2 \\
& + T \rho r \left(4\mathrm{g} + r \frac{d\mathrm{g}}{dr} \right) [\nabla \cdot \chi^{(1)} g_r^2 \\
& + 2 (\nabla \cdot g)_r g_r \chi_r^{(1)}]
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
& - \rho r \chi_h^{(1)} g_h g_h (\omega_\chi^2 G_\chi + \omega_g^2 G_g + \omega_g^2 G_g) \\
& - \rho r \chi_r^{(1)} g_h g_h [(\omega_\chi^2 - 3\omega_g^2 - 3\omega_g^2) F_\chi - 2(\omega_g^2 F_g + \omega_g^2 F_g)] \\
& - \rho r \chi_h^{(1)} g_r g_h [(\omega_g^2 - 3\omega_g^2 - 3\omega_\chi^2) F_g - 2(\omega_g^2 F_g + \omega_\chi^2 F_\chi)] \\
& - \rho r \chi_h^{(1)} g_r g_h [(\omega_g^2 - 3\omega_\chi^2 - 3\omega_g^2) F_g - 2(\omega_\chi^2 F_\chi + \omega_g^2 F_g)] \\
& + \rho r \chi_h^{(1)} g_r g_r (\omega_g^2 F_g + \omega_g^2 F_g - 6\omega_\chi^2 T) \\
& + \rho r \chi_r^{(1)} g_r g_h (\omega_g^2 F_g + \omega_\chi^2 F_\chi - 6\omega_g^2 T) \\
& + \rho r \chi_r^{(1)} g_r g_h (\omega_\chi^2 F_\chi + \omega_g^2 F_g - 6\omega_g^2 T) \}
\end{aligned}
\tag{44}
$$

$$
- W_{lm} \frac{T(l+2)}{M \mathcal{R}^l} \int dr \rho r^l \left[\frac{\partial \ln \rho}{\partial \ln r} g_r^2 + 2 r g_r (\nabla \cdot g)_r \right] \Bigg|, \tag{45}
$$