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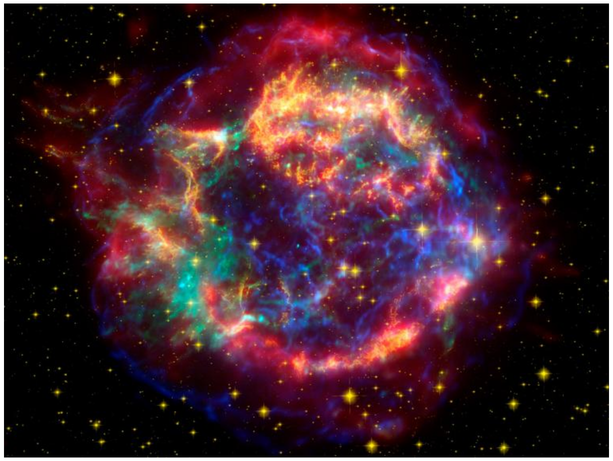
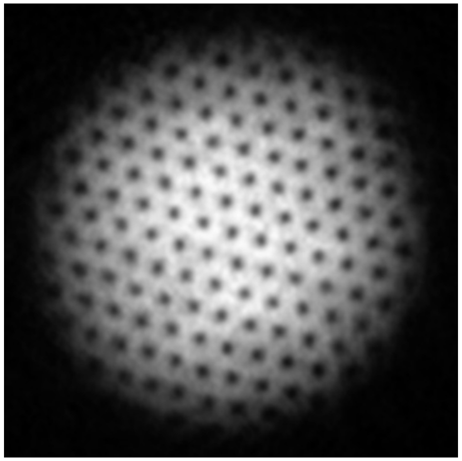
Fundamental properties and hydrodynamics of (relativistic) superfluids





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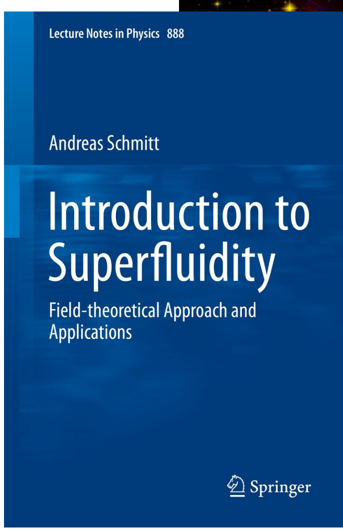
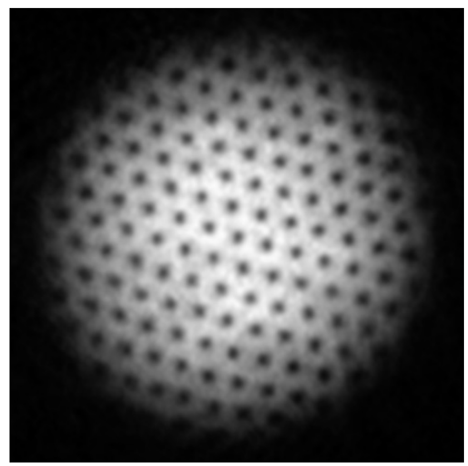
Fundamental properties and hydrodynamics of (relativistic) superfluids





lecture notes: A. Schmitt, Lect. Notes Phys. 888, 1 (2015)

Contents of lecture

- General remarks
  - Basic mechanism of superfludity, relation to superconductivity
- Microsopic model for a relativistic bosonic superfluid
  - Superfluid vortices
  - Superfluid velocity and Landau's critical velocity
  - Goldstone mode
- Two-fluid hydrodynamics of superfluids
  - First and second sound
- Superfluid density: definition and microscopic calculation
- Microscopic model for a relativistic fermionic superfluid

Brief history of superfluidity/superconductivity

1908 helium liquefied Kamerlingh Onnes

Kamerlingh Onnes superconductivity discovered in mercury 1911

1927 transition to helium II observed Wolfke/Keesom

Meissner/Ochsenfeld superconductors expel magnetic fields 1933

1938 helium II is superfluid Kapitza, Allen/Misener

1938 two-fluid model Landau, Tisza

Bardeen/Cooper/Schrieffer microscopic theory of superconductivity 1957

Bogoliubov Cooper pairing of nucleons suggested 1958

Ivanenko/Kurdgelaidze Cooper pairing of quarks suggested 1969

1972 superfluidity in ^{3}He Lee/Osheroff/ Richardson

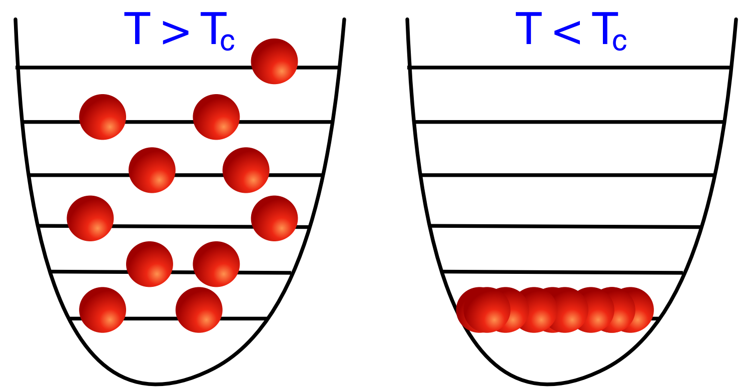
Bednorz/Müller high-T_c superconductors 1986

2005 vortex formation in fermionic gas Ketterle

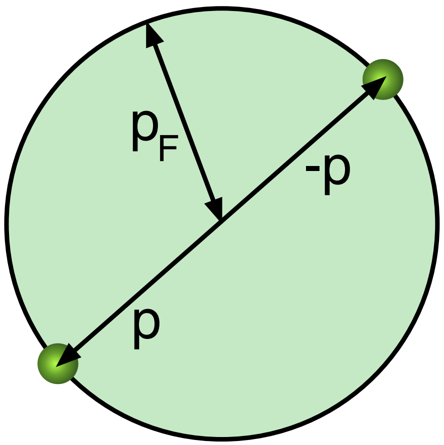
Basic mechanisms of superfluidity

- **Bosons** (^{4}He, cold bosonic atom gas, pions, kaons, ...)

Bose-Einstein condensation

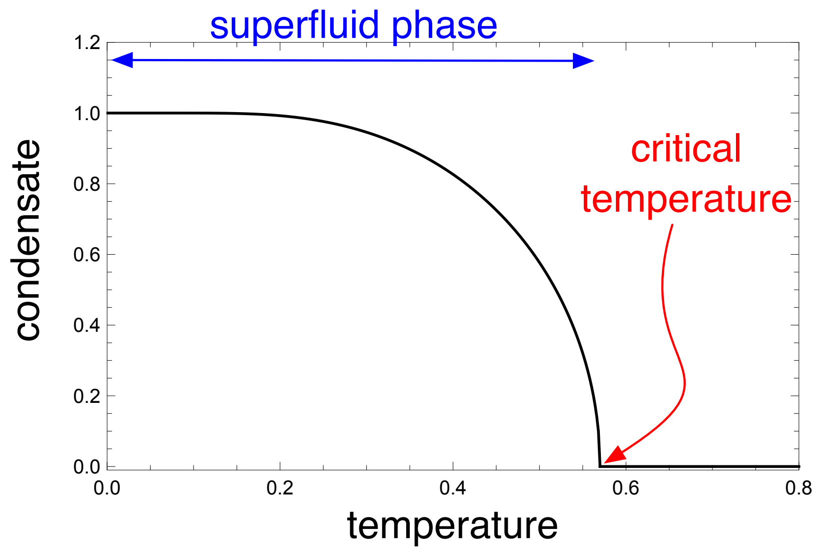


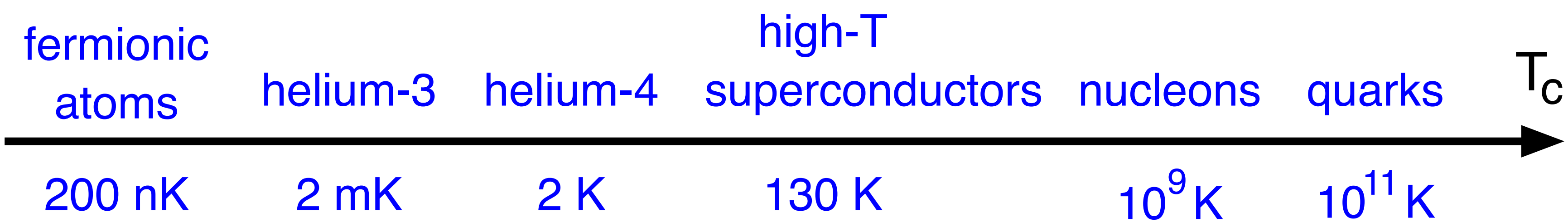
- **Fermions** (electrons in a metal, ^{3}He, cold fermionic atom gas, neutrons, protons, quarks, ...)



Cooper pair condensation
(cold fermionic system
+ arbitrarily small attraction)

Critical temperature



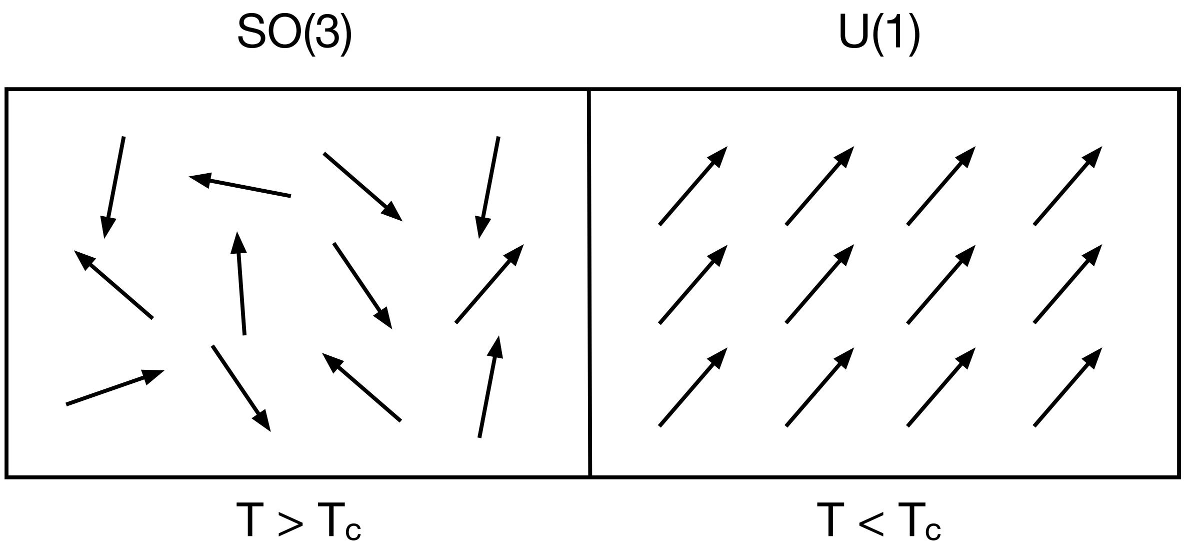


Neutron stars: temperatures are sufficiently low to allow for nucleon and quark Cooper pairs (except for very early stages and during merger)

Superfluidity is a phase transition

- condensate $\leftrightarrow$ order parameter
- superfluid and normal state have different symmetries: "symmetry is spontaneously broken"

For instance ferromagnetism: $SO(3) \to U(1)$



Superfluidity: $U(1) \to \mathbf{1}$

Comparing superfluidity to superconductivity

superfluidity	superconductivity
condensate neutral	condensate electrically charged
frictionless “charge” transport through condensate (need ”absence” of excitations)	
spontaneous breaking of *global* symmetry	spontaneous breaking of *local* symmetry
Goldstone mode (“phonon”)	Meissner effect (magnetic screening mass for gauge boson)

Cooper pairing of neutrons → superfluidity

Cooper pairing of protons → superconductivity

Cooper pairing of quarks → color superconductivity, possibly superfluidity and electromagnetic superconductivity (depends on pairing pattern)

Superfluidity from a complex scalar field (page 1/2)

chapter 3 of A. Schmitt, Lect. Notes Phys. 888, 1 (2015)

Start from Lagrangian for $\varphi \in \mathbb{C}$ with mass $m > 0$ and coupling $\lambda > 0$

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 |\varphi|^2 - \lambda |\varphi|^4$$

$\mathcal{L}$ is invariant under *global* $U(1)$ trafo $\varphi \to e^{-i\alpha}\varphi$

$$\varphi = \phi + \text{fluctuations} \qquad \Rightarrow \qquad \mathcal{L} = \mathcal{L}^{(0)} + \text{fluctuations}$$

with the condensate

$$\phi = \frac{\rho}{\sqrt{2}} e^{i\psi}$$

and

$$\mathcal{L}^{(0)} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{\rho^2}{2}(\partial_\mu \psi \partial^\mu \psi - m^2) - \frac{\lambda}{4}\rho^4$$

Euler-Lagrange eq. for ψ is current conservation (Noether's Theorem)

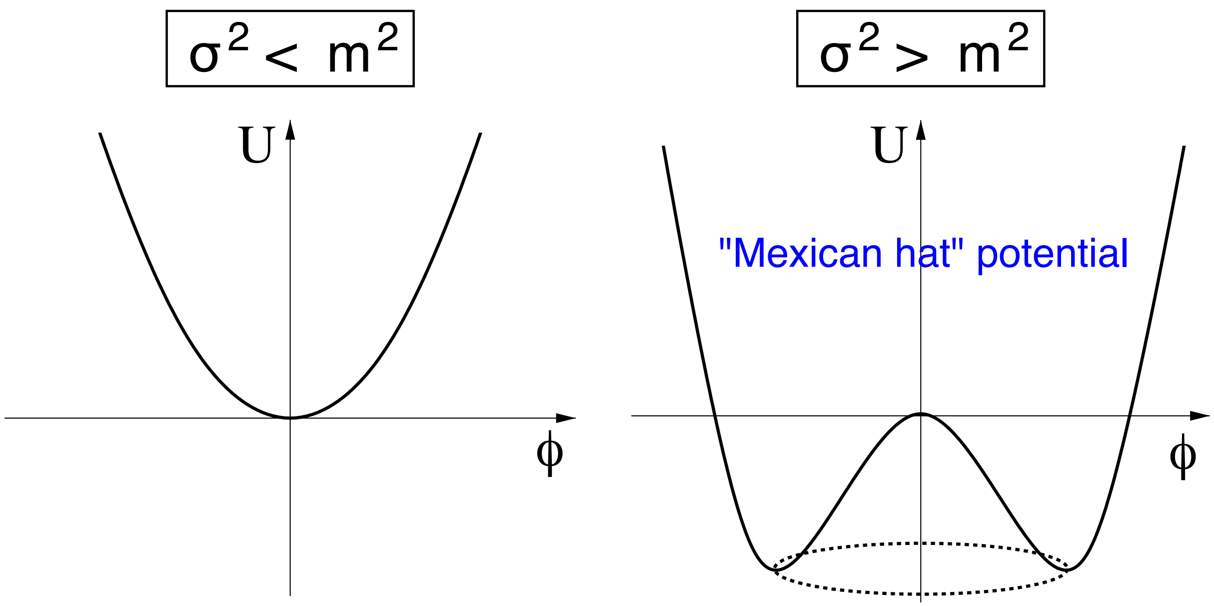
$$\partial_\mu j^\mu = 0 \qquad\qquad j^\mu = \rho^2 \partial^\mu \psi$$

Superfluidity from a complex scalar field (page 2/2)

Assume homogeneous condensate (ρ and $\partial_\mu\psi$ constant in space and time)

$$\mathcal{L}^{(0)} = -U\,, \qquad U = -\frac{\rho^2}{2}(\sigma^2 - m^2) + \frac{\lambda}{4}\rho^4$$

with $\sigma \equiv \sqrt{\partial_\mu\psi\partial^\mu\psi} = \mu\sqrt{1-v^2}$



$\rho = 0$ $\qquad\qquad$ $\rho^2 = \dfrac{\sigma^2 - m^2}{\lambda}$

Connect with hydrodynamics (1): Superfluid velocity

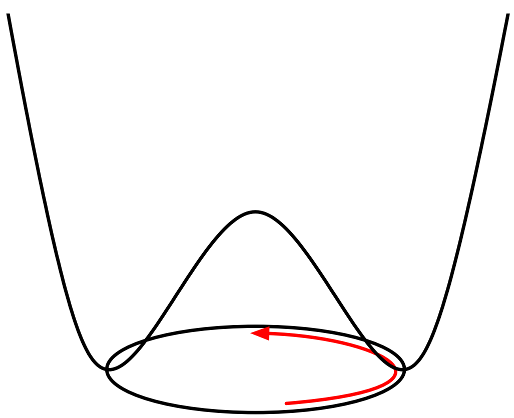
Write current in terms of superfluid four-velocity

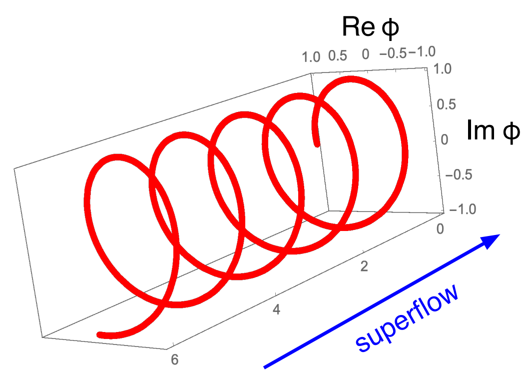
$$j^\mu = n v^\mu$$

By contracting with v_μ and j_μ we get $\rho^2\sigma = n$ and thus

$$v^\mu = \frac{\partial^\mu\psi}{\sigma} \qquad \Rightarrow \qquad \vec{v} = -\frac{\nabla\psi}{\mu}$$

→ superflow is rotating phase





(Non-relativistic: $\vec{v} = -\frac{\nabla\psi}{m} \to \nabla\times\vec{v} = 0 \to$ irrotational flow)

Connect with hydrodynamics (2): Stress-energy tensor

From microscopic theory:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} = \rho^2\partial^\mu\psi\partial^\nu\psi - g^{\mu\nu}\mathcal{L}^{(0)}$$

For hydrodynamics need:

$$T^{\mu\nu} = (\epsilon + P)v^\mu v^\nu - g^{\mu\nu}P$$

Connect both via

$$P = -\frac{1}{3}(g_{\mu\nu} - v_\mu v_\nu)T^{\mu\nu} = \mathcal{L}^{(0)} = \frac{(\sigma^2 - m^2)^2}{4\lambda}$$

$$\epsilon = v_\mu v_\nu T^{\mu\nu} = \sigma n - P = \frac{(3\sigma^2 + m^2)(\sigma^2 - m^2)}{4\lambda}$$

$\Rightarrow$ σ is chemical potential in superfluid rest frame

Speed of (first) sound

$$\frac{\partial P}{\partial \epsilon} = \frac{\sigma^2 - m^2}{3\sigma^2 - m^2}$$

Vortex solutions (page 1/2)

Euler-Lagrange equations allowing spatial dependence of ρ

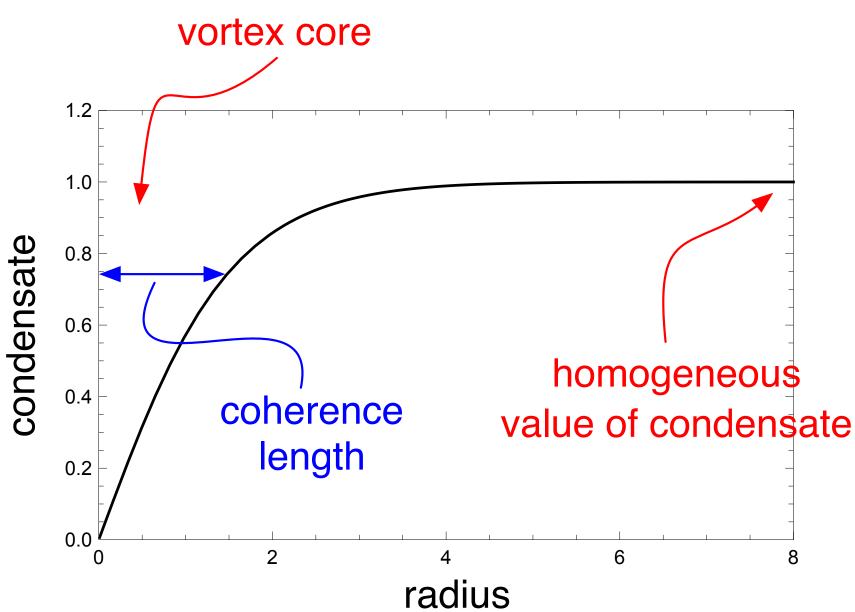
$$-\Delta\rho = \rho[\mu^2 - (\nabla\psi)^2 - m^2 - \lambda\rho^2]$$

$$\nabla\cdot(\rho^2\nabla\psi) = 0$$

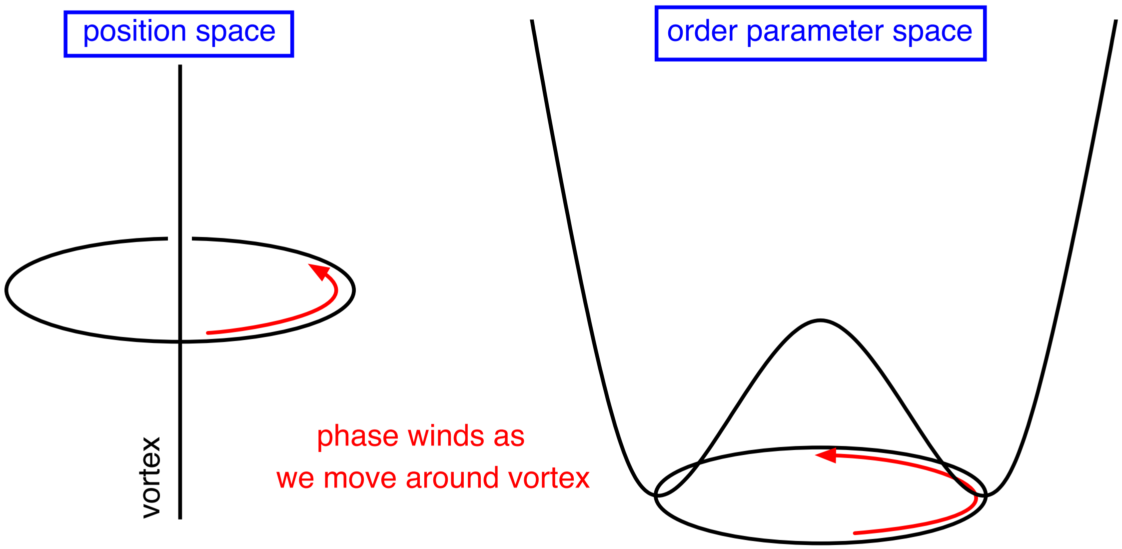
Cylindrical coordinates (r,θ,z), $\psi = n\theta$ $(n \in \mathbb{Z})$ and $\rho(\mathbf{x}) = \rho(r)$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\rho}{\partial r}\right)+\left(1-\frac{n^2}{r^2}\right)\rho-\rho^3 = 0$$

(dimensionless variables)



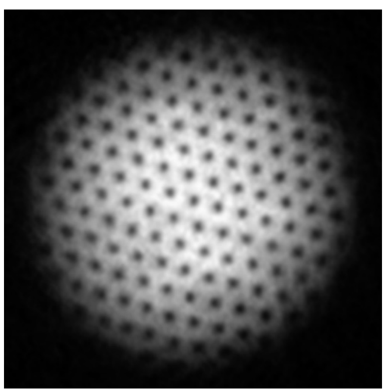
Vortex solutions (page 2/2)



$n \in \mathbb{Z}$ winding number

$\pi_1[U(1)] = \mathbb{Z}$

vortex is "topologically stable"



vortex array in rotating atomic superfluid

M. Zwierlein *et al.*, Nature 435, 7045 (2005)

Rotating neutron star: vortices from neutron superfluidity
or color-flavor locked phase

Including fluctuations and Goldstone mode (page 1/3)

Recall that so far we have dropped fluctuations

$$\mathcal{L} = \mathcal{L}^{(0)} + \text{fluctuations} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)}$$

Terms of second order in the fluctuations $\mathcal{L}^{(2)}$ give the inverse propagator

$$D^{-1}(K) = \begin{pmatrix} -K^2 + m^2 + 3\lambda\rho^2 - \sigma^2 & -2iK_\mu\partial^\mu\psi \\ 2iK_\mu\partial^\mu\psi & -K^2 + m^2 + \lambda\rho^2 - \sigma^2 \end{pmatrix}$$

which is needed for the thermodynamic potential

$$\Omega = -\frac{T}{V}\ln Z = U + \frac{1}{2}\frac{T}{V}\mathrm{Tr}\ln\frac{D^{-1}(K)}{T^2} = U + \sum_{e=\pm}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{\epsilon_k^e}{2} + T\ln\left(1 - e^{-\epsilon_k^e/T}\right)\right]$$

$\epsilon_k^\pm$ are the poles of D, $\det D^{-1} = [k_0^2 - (\epsilon_k^+)^2][k_0^2 - (\epsilon_k^-)^2]$

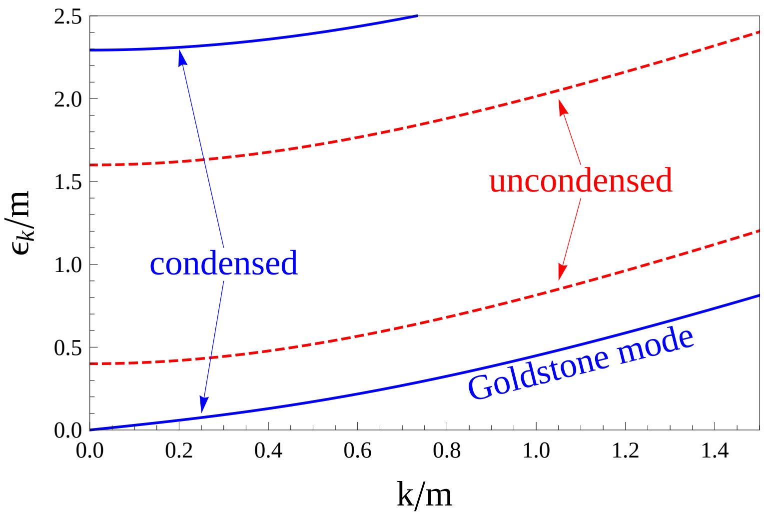
Including fluctuations and Goldstone mode (page 2/3)

Excitation energies without condensation

$$\rho = 0 : \qquad \epsilon_k^{\pm} = \sqrt{k^2 + m^2} \mp \mu$$

and with condensation

$$\epsilon_k^{+} = \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}}\, k + \mathcal{O}(k^3)$$



A massless mode has appeared!

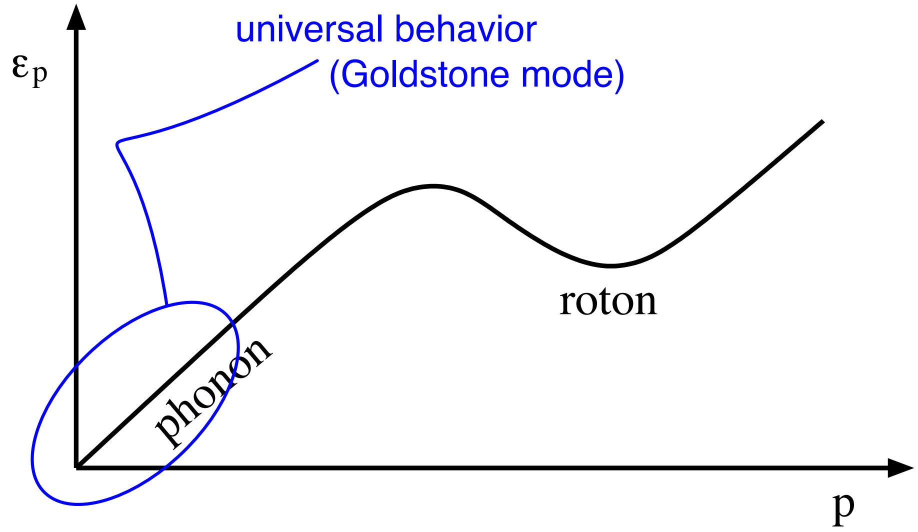
Including fluctuations and Goldstone mode (page 3/3)

> If a continuous global symmetry of the Lagrangian is spontaneously broken there exists a gapless mode. This mode is called Goldstone mode.

superfluid helium:
excitations modeled by

$\epsilon_p = cp$ (phonon)

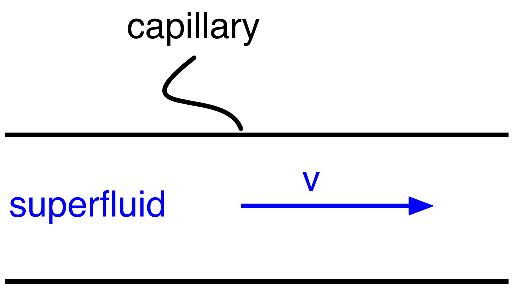
$$\epsilon_p = \Delta + \frac{(p-p_0)^2}{2m} \qquad \text{(roton)}$$



But ... superfluidity means absence of excitations ...
shouldn't the Goldstone mode destroy dissipationless flow??

Landau's critical velocity (page 1/2)

Excitations potentially lead to dissipation:



ϵ_p excitation energy in fluid rest frame

total energy in capillary frame (non-rel.)

$$E = E_{\rm kin} + \epsilon_p + \vec{p} \cdot \vec{v}$$

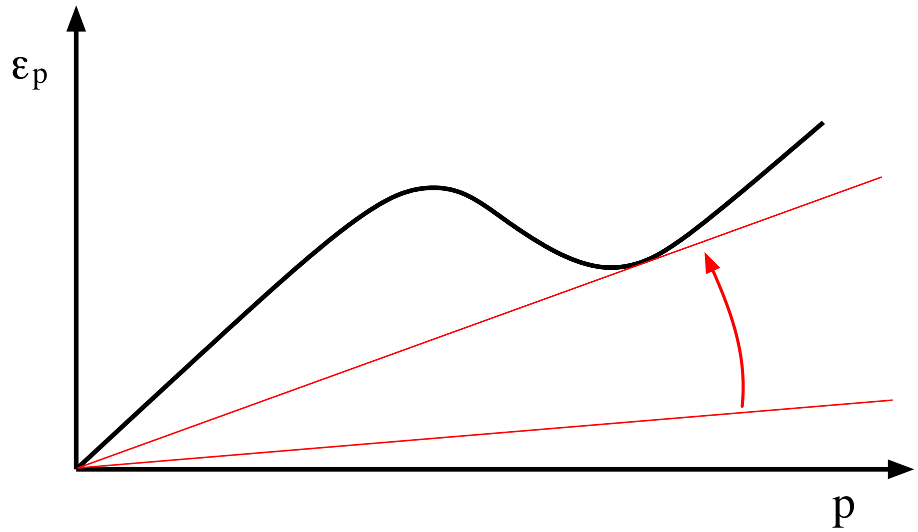
$\rightarrow$ fluid loses energy through dissipation if

$$\epsilon_p + \vec{p} \cdot \vec{v} < 0 \qquad \Rightarrow \qquad \text{superfluid for} \quad v < v_c = \min_p \frac{\epsilon_p}{p}$$

Landau's critical velocity (page 2/2)

minimum of ϵ_p/p:

$$\frac{\partial \epsilon_p}{\partial p} = \frac{\epsilon_p}{p}$$



$\rightarrow$ superfluidity = condensate + nonzero critical velocity v_c

- superfluidity exists *despite* Goldstone mode
- in the absence of roton, $v_c = c$ with $\epsilon = cp$
- if the Goldstone mode had quadratic dispersion, $v_c = 0$
  $\rightarrow$ no superfluidity

Relativistic two-fluid formalism (page 1/2)

non-rel.:London, Tisza (1938); Landau (1941)

rel: Khalatnikov, Lebedev (1982); Carter (1989)

Two fluids: conserved current j^μ and entropy current s^μ

$$T^{\mu\nu} = -g^{\mu\nu}\Psi + j^\mu \partial^\nu \psi + s^\mu \Theta^\nu$$

Conjugate momenta $\partial^\nu \psi$ and Θ^ν

$$j^\mu = \frac{\partial \Psi}{\partial(\partial_\mu \psi)} = \mathcal{B}\,\partial^\mu \psi + \mathcal{A}\,\Theta^\mu$$

$$s^\mu = \frac{\partial \Psi}{\partial \Theta_\mu} = \mathcal{A}\,\partial^\mu \psi + \mathcal{C}\,\Theta^\mu$$

$$\mathcal{B} = 2\frac{\partial \Psi}{\partial(\partial\psi)^2}\,, \qquad \mathcal{C} = 2\frac{\partial \Psi}{\partial \Theta^2}$$

$$\mathcal{A} = \frac{\partial \Psi}{\partial(\partial\psi \cdot \Theta)}$$

"entrainment coefficient"

Relativistic two-fluid formalism (page 2/2)

Hydrodynamic equations: with $d\Psi = j_\mu d(\partial^\mu\psi) + s_\mu d\Theta^\mu$ we compute

$$\partial_\mu T^{\mu\nu} = \partial^\nu\psi \underbrace{\partial_\mu j^\mu}_{=0} + j_\mu \underbrace{(\partial^\mu\partial^\nu\psi - \partial^\nu\partial^\mu\psi)}_{=0} + \Theta^\nu\partial_\mu s^\mu + s_\mu(\partial^\mu\Theta^\nu - \partial^\nu\Theta^\mu)$$

$$\Rightarrow \quad \boxed{\partial_\mu j^\mu = 0\,, \qquad \partial_\mu s^\mu = 0\,, \qquad s_\mu\omega^{\mu\nu} = 0}$$

with the vorticity $\omega^{\mu\nu} \equiv \partial^\mu\Theta^\nu - \partial^\nu\Theta^\mu$

Alternatively: two fluids from normal fluid and superfluid

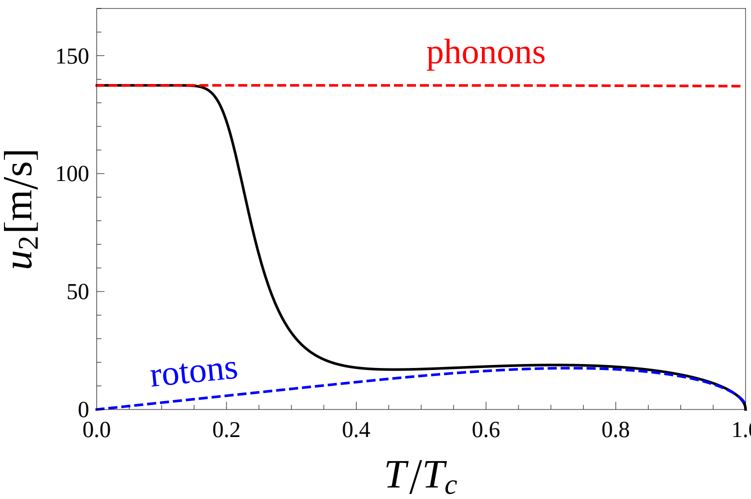
$$j^\mu = n_n u^\mu + n_s v^\mu = \frac{n_n}{s} s^\mu + \frac{n_s}{\sigma}\partial^\mu\psi$$

Superfluid density n_s needed for sound modes (next slide), pulsar glitches, ...

Second sound

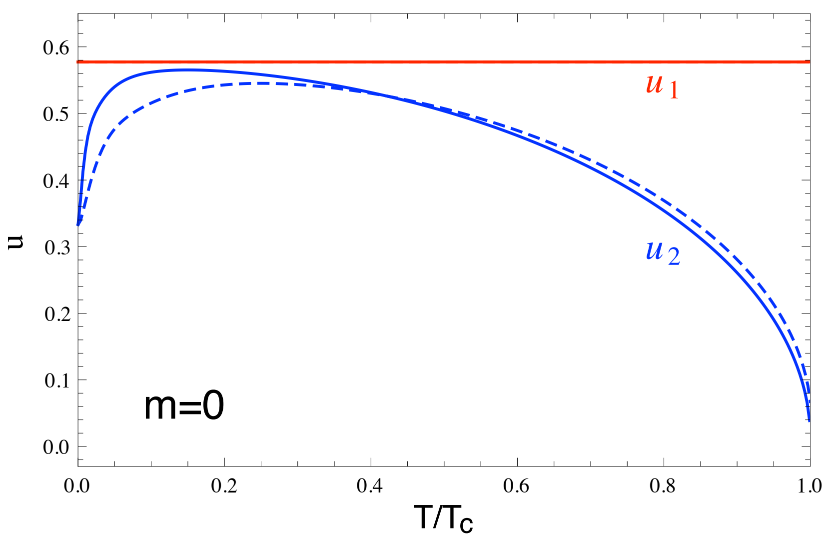
Superfluid helium

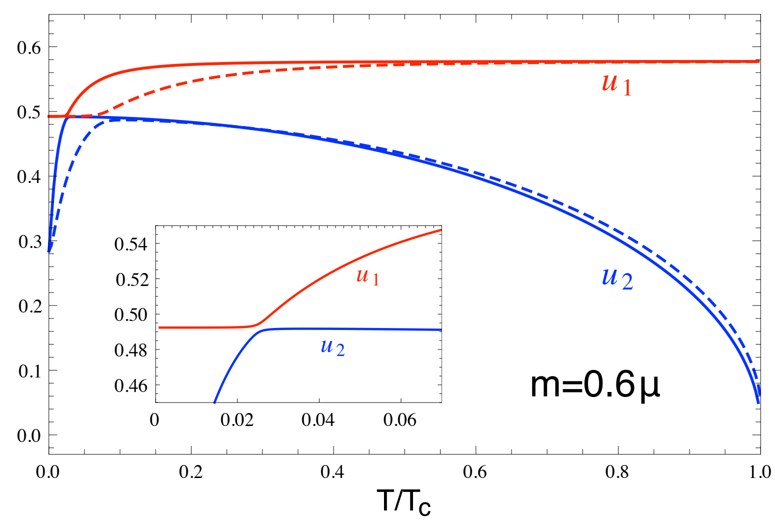
$$u_1 = \sqrt{\frac{\partial P}{\partial \rho}}, \qquad u_2 = \sqrt{\frac{s^2 T \rho_s}{\rho c_V \rho_n}}$$



Relativistic φ^4 model

M. G. Alford, S. K. Mallavarapu, A. Schmitt and S. Stetina, PRD 89, 085005 (2014)





Calculation of the superfluid density (page 1/2)

To compute superfluid density start from spatial components of

$$j^\mu = \frac{n_n}{s} s^\mu + \frac{n_s}{\sigma} \partial^\mu \psi$$

work in normal fluid rest frame, $\vec{u} = 0$, contract with $\nabla\psi$,

$$n_s = -\sigma \frac{\nabla\psi \cdot \vec{j}}{(\nabla\psi)^2}$$

Abbreviate $\vec{q} = \nabla\psi$ and work at vanishing superflow

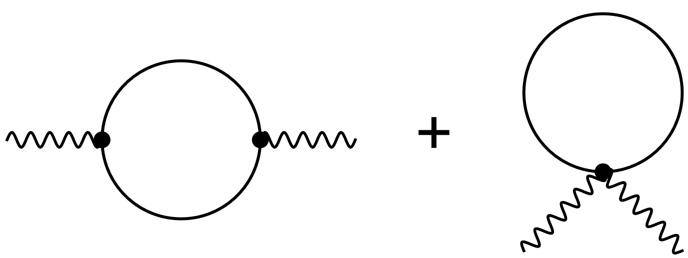
$$\boxed{n_s = \mu \hat{q}_i \hat{q}_j \left. \frac{\partial\Omega}{\partial q_i \partial q_j} \right|_{\vec{q}=0}}$$

Calculation of the superfluid density (page 2/2)

Recall thermodynamic potential from bosonic model

$$\Omega = -\frac{T}{V}\ln Z = U + \frac{1}{2}\frac{T}{V}\mathrm{Tr}\ln\frac{D^{-1}(K)}{T^2}\,, \qquad U = -\frac{(\mu^2-q^2-m^2)^2}{4\lambda}$$

$$\Rightarrow \qquad \frac{\partial\Omega}{\partial q_i\partial q_j} = \frac{\partial U}{\partial q_i\partial q_j} + \underbrace{\frac{1}{2}\frac{T}{V}\mathrm{Tr}\left[-D\frac{\partial D^{-1}}{\partial q_i}D\frac{\partial D^{-1}}{\partial q_j} + D\frac{\partial D^{-1}}{\partial q_i\partial q_j}\right]}$$



Compute Matsubara sum and momentum integral, take low-temperature limit,

$$n_s \simeq \frac{\mu(\mu^2-m^2)}{\lambda} - \frac{\pi^2T^4}{45}\frac{\mu(3\mu^2-m^2)^{1/2}(12\mu^2-m^2)}{(\mu^2-m^2)^{5/2}}$$

Microscopic theory of fermionic superfluids (brief sketch)

chapter 5 of A. Schmitt, Lect. Notes Phys. 888, 1 (2015)

Fermions with pointlike interaction ($\mathcal{L}$ invariant under $U(1)$, $\psi \to e^{i\alpha}\psi$)

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu + \gamma^0\mu - m)\psi + G(\bar{\psi}\psi)^2$$

[More realistically: interaction mediated by lattice phonons, gluons, ...]

Mean field approximation: write $\psi\psi = \langle\psi\psi\rangle$ + fluctuations, $G\psi\psi\psi\psi \to \psi\Phi\psi$

Since $\langle\psi\psi\rangle$ really is $\langle\psi_C\bar{\psi}\rangle$ → "Nambu-Gorkov space" $\Psi = (\psi, \psi_C)$

$$\mathcal{S}^{-1} = \begin{pmatrix} [G_0^+]^{-1} & \Phi^- \\ \Phi^+ & [G_0^-]^{-1} \end{pmatrix}, \qquad [G_0^\pm]^{-1} = \gamma^\mu K_\mu \pm \gamma^0\mu - m$$

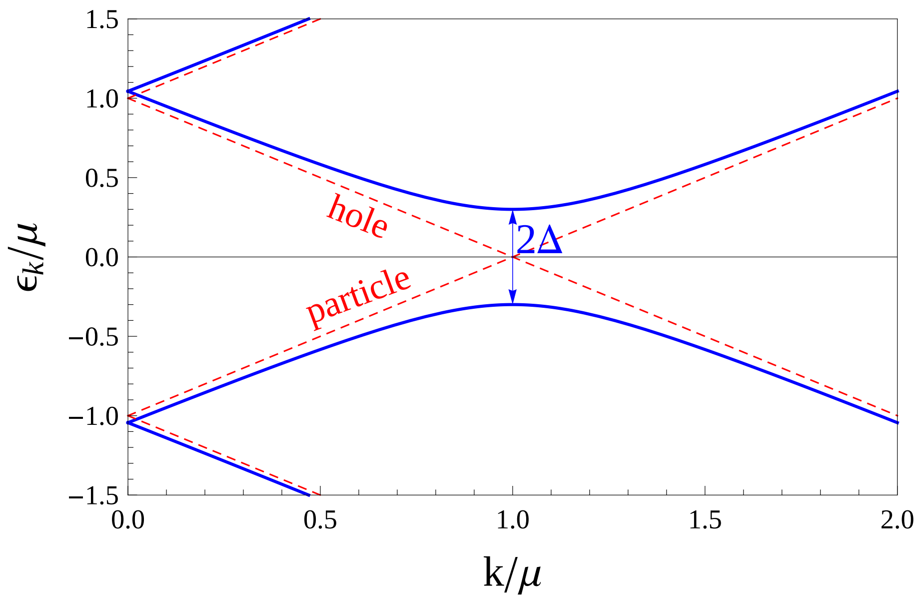
→ propagator acquires 2×2 structure

$$\mathcal{S} = \begin{pmatrix} G^+ & F^- \\ F^+ & G^- \end{pmatrix} \qquad \text{anomalous propagators } F^\pm$$

Quasi-particle excitations

Fermions acquire energy gap Δ in their spectrum

$$\epsilon_k^e \equiv \sqrt{(\mu - ek)^2 + \Delta^2}$$



For instance in QCD at ultra-high densities and $T = 0$:

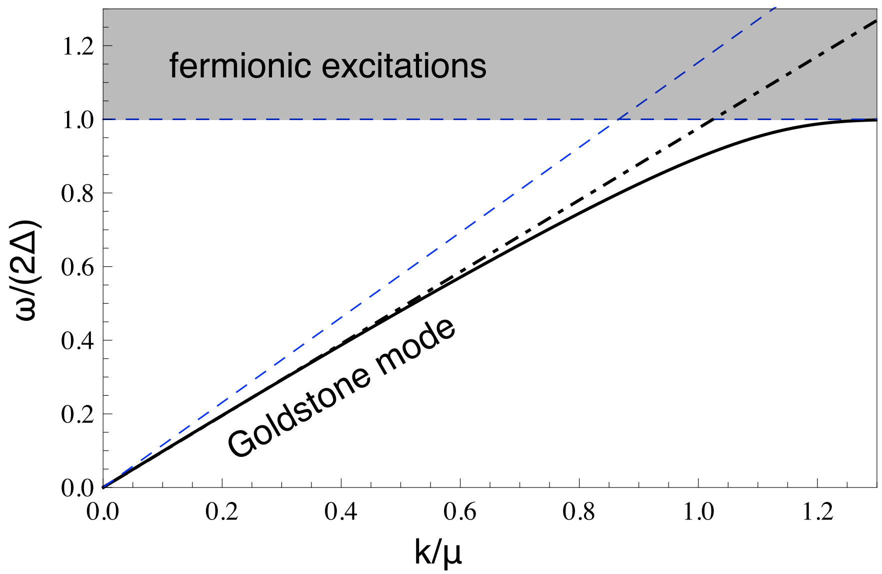
$$\Delta \simeq 2b\mu \exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right)$$

with $b \equiv 256\pi^4[2/(N_f g^2)]^{5/2}$

→ system is superfluid since fermions cannot be excited (and Goldstone mode has linear dispersion)

Superfluid density in fermionic superfluid (page 1/2)

Compute Goldstone dispersion from fluctuations

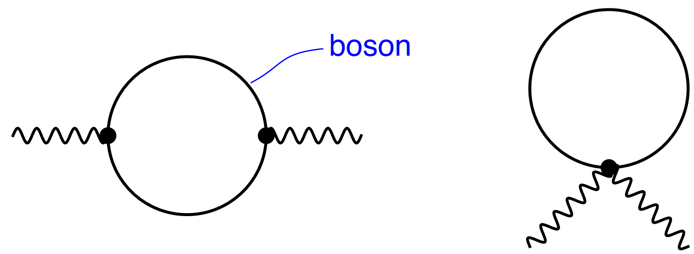


Thermodynamic potential, needed for superfluid density:

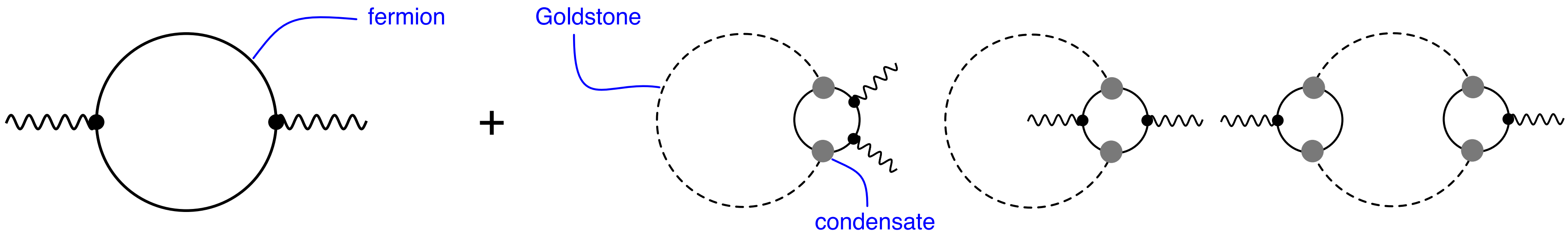
$$\Omega = \underbrace{\frac{\Delta^2}{G} - \frac{1}{2}\frac{T}{V}\sum_K \mathrm{Tr}\ln\frac{\mathcal{S}^{-1}(K)}{T}}_{\text{condensate + fermionic exc.}} + \underbrace{\frac{1}{2}\frac{T}{V}\sum_K \mathrm{Tr}\ln\frac{D^{-1}(K)}{T^2}}_{\text{Goldstone exc.}}$$

Superfluid density in fermionic superfluid (page 2/2)

Recall bosonic theory:



Fermionic theory:



$$n_s = \frac{\mu^3}{3\pi^2} + \qquad \ldots$$

Even lowest-order contribution in T very tedious to calculate

D. Müller, Master Thesis, TU Wien (2014)

Summary

- Superfluidity is a very general phenomenon and occurs on vastly different scales from low-energy physics (cold atoms) to high-energy physics (nuclear & quark matter in neutron stars)
- Superfluids and superconductors are close relatives
- Superfluidity is a phase transition:
spontaneous symmetry breaking → Goldstone mode
- Hydrodynamics of superfluids at nonzero T requires two-fluid formalism – and microscopic input is needed for instance for superfluid density

Outlook – some selected open questions

- superfluid density in fermionic relativistic superfluid

- vortices in the color-flavor locked phase: coexistence with magnetic flux tubes? continuously connected to vortices in nuclear matter?

- superfluid hydrodynamics in the presence of vortices ($\rightarrow$ quantum turbulence?)

- unconventional behavior of multi-component superfluids (cold neutron/proton matter, color-flavor locking, ...)
  A. Haber and A. Schmitt, PRD 95 116016 (2017); JPG 45, 065001 (2018)

- hydrodynamic instabilities in two-component superfluids
  A. Haber, A. Schmitt and S. Stetina, PRD 93, 025011 (2016)
  N. Andersson and A. Schmitt, in preparation