Formulations of the Einstein equations for spacetime evolutions

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Overview

- 10 Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$: second-order nonlinear PDEs for 10 metric coefficients $g_{\mu\nu}$
- Gauge freedom: 4 functions of 4 variables $x^\mu \to \tilde{x}^\mu(x^\nu)$
- Why 2 polarisations of gravitational waves? Why wave equations?
- Initial data and their time evolution?
- Well-posed PDE problems?
- Notation (Wald): V^a an abstract vector, V^μ its components in coordinates $x^\mu := (t, x^i)$, $i = 1, 2, 3$
  $a \sim b$ means "something like"

Time slices and their normal vector n^a

- Time slice $t = \mathrm{const}$ has 3 tangent vectors $\partial/\partial x^i$
- Vector n^a normal to time slice is defined by

$$\left(\frac{\partial}{\partial x^i}\right)^a n_a := 0 \quad \Rightarrow \quad n_i = 0$$

- A vector X^a is purely spatial if $X^a n_a = 0$, hence if $X^0 = 0$.
- Recall $\partial/\partial t$ means "change t, keep x^i constant"

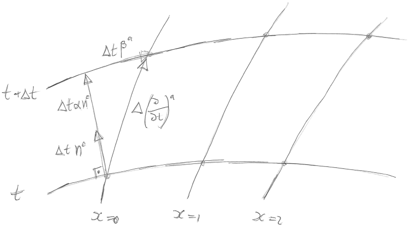
$$\left(\frac{\partial}{\partial t}\right)^a = \alpha n^a + b^a, \qquad n_a b^a := 0 \quad \Rightarrow \quad b^0 = 0$$

- n^a is defined to be future-pointing and unit length

$$n_a n^a := -1 \quad \Rightarrow \quad \left(\frac{\partial}{\partial t}\right)^a n_a = n_0 = -\alpha$$

Lapse and shift

- Starting from a point coordinates (t, x^i), the geometrical location of the point with coordinates $(t + \Delta t, x^i)$ is determined by the **lapse** α and **shift** vector b^i
- Vice versa, starting from an initial slice $t = 0$ with coordinates x^i, the coordinate system on spacetime is constructed along with the spacetime by choosing α and b^i



The spatial metric as a projection operator

- The 3-metric h_{ab} of a slice of constant t can be defined **geometrically** in terms of n^a and the spacetime metric g_{ab}:

$$h_{ab} := g_{ab} + n_a n_b \quad \Rightarrow \quad h_{ab} n^b = 0, \quad h_a{}^b h_b{}^c = h_a{}^c$$

- Hence $h_a{}^b X^a = g_a{}^b X^a = X^b$ for spatial vectors, and so we can raise and lower indices for spatial vectors with h_{ab}
- We have

$$h^{ab} n_b = 0 \quad \Rightarrow \quad h^{0i} = h^{i0} = h^{00} = 0$$

and so we define $\gamma^{ij} := h^{ij}$, then γ_{ij} as the matrix inverse of γ^{ij}.

- Recall $b^0 = 0$ and we define $\beta^i := b^i$ and then $\beta_i := \gamma_{ij}\beta^j$.

3+1 split of the 4-dimensional metric

We now have all the definitions to calculate

$$g_{00} = -\alpha^2 + \beta_i \beta^i, \quad g_{0i} = \beta_i, \quad g_{ij} = \gamma_{ij}$$

or in line element form

$$ds^2 = -\alpha^2\, dt^2 + \gamma_{ij}(dx^i + \beta^i\, dt)(dx^j + \beta^j\, dt)$$

The inverse spacetime metric is

$$g^{00} = -\alpha^{-2}, \quad g^{0i} = \alpha^{-2}\beta^i, \quad g^{ij} = \gamma^{ij} - \alpha^{-2}\beta^i\beta^j$$

Exercises: Check this against our definitions. Check $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu{}^\lambda$. Calculate $h_{\mu\nu}$, $h^{\mu\nu}$, $h_\mu{}^\nu$, n^μ, b_μ, X_μ.

The extrinsic curvature

- We could write the wave equation (in Minkowski spacetime) $\phi_{,tt} = \Delta\phi$ in first-order in time form as

$$\phi_{,t} =: \Pi, \qquad \Pi_{,t} = \Delta\phi$$

- The GR equivalent of ϕ is the 3-metric γ_{ij}
- The equivalent of Π is the **extrinsic curvature** K_{ij}
  Geometric definition (there are two conventions for the sign):

$$2K_{ab} := \mathcal{L}_n h_{ab} \quad \Rightarrow \quad K_{ab} = h_a{}^c \nabla_c n_b, \quad K_{ab} = K_{ba}, \quad K_{ab} n^b = 0,$$

  (**exercise**) so K_{ab} is a symmetric spatial tensor like h_{ab}
- In synchronous gauge $\alpha = 1$, $\beta^i = 0$ where $n^a = (\partial/\partial t)^a$

$$2K_{ij} = \mathcal{L}_{\frac{\partial}{\partial t}} \gamma_{ij} = \gamma_{ij,t}$$

- In general gauge (**exercise**)

$$\gamma_{ij,t} = 2\alpha K_{ij} + \beta^k \gamma_{ij,k} + \gamma_{ik} \beta^k{}_{,j} + \gamma_{jk} \beta^k{}_{,i}$$

3+1 split of the Einstein equations

Split the Einstein equations $E_{ab} := G_{ab} - 8\pi T_{ab}$ into

- time part E^{00} (Hamiltonian constraint) ($K := K_i{}^i$)

$$H := {}^{(3)}R^i{}_i + K - K_{ij}K^{ij} - 16\pi\rho = 0$$

- mixed part $E_i{}^0$ (momentum constraints)

$$M^i := D_j K^{ij} - D^i K - 8\pi j^i = 0$$

- spatial part E_{ij} (evolution equations)

$$\mathcal{L}_n K_{ij} = -\alpha^{-1} D_i D_j \alpha + {}^{(3)}R_{ij} + K K_{ij} - 2K_{ik}K_j{}^k + \text{matter}$$

- Definition of K_{ij} was

$$\mathcal{L}_n \gamma_{ij} = 2K_{ij}$$

- No time derivatives of α and β^i appear anywhere

Formulations of the Einstein equations

- The $6 + 6$ variables (γ_{ij}, K_{ij}) obey 4 constraints that need to be solved for the initial data, given suitable free data
- The constraints are propagated by the evolution equations

$$\dot{H} \sim M^i{}_{,i}, \quad \dot{M}_i \sim H_{,i}$$

- We need to give four **gauge conditions** (algebraic, evolution, or elliptic equations) for α and β^i
- We can **add constraints** to the evolution equations
- The resulting evolution equations need to be **well-posed**
- ...even when the constraints are violated (because of numerical error)
- Ideally, the constraints should decay in time

Solving the constraints

- Parameterize 6+6 (γ_{ij}, K_{ij}) at $t = 0$ as

$$\begin{aligned} \gamma_{ij} &= \psi^4 \tilde{\gamma}_{ij} \\ K_{ij} &= A_{ij} + \frac{1}{3}\gamma_{ij} K, \qquad A_i{}^i = 0 \\ A^{ij} &= \psi^{-10}(\tilde{A}^{ij}_{TT} + \tilde{A}^{ij}_{L}), \qquad \tilde{D}_j \tilde{A}^{ij}_{TT} = 0 \\ \tilde{A}^{ij}_{L} &= \tilde{D}^i W^j + \tilde{D}^j W^i - \frac{2}{3}\tilde{\gamma}^{ij}\tilde{D}_k W^k \end{aligned}$$

  where $\tilde{D}_k \tilde{\gamma}_{ij} := 0$
- Free data $(\tilde{\gamma}_{ij}, \tilde{A}^{ij}_{TT})$ (5+3 components)
- 4 coupled nonlinear elliptic equations for (ψ, W^i)
- Simple cases: conformally flat initial data $\tilde{\gamma}_{ij} = \delta_{ij}$, and/or time-symmetric initial data $K_{ij} = 0$

Counting degrees of freedom

- Initial data: 6+6 (g_{ij}, K_{ij}) - 4 Einstein constraints (H, M_i) $= 5+3$ $(\tilde{A}_{ij}^{TT}, \tilde{\gamma}_{ij})$
- But we can still change the 3 spatial coordinates without changing the initial data
- And we can push the initial data slice backwards and forwards in the spacetime it defines, separately at each point
- 8 $(\tilde{A}_{ij}^{TT}, \tilde{\gamma}_{ij})$ - 4 $(\Delta t, \Delta x^i) = 4$ $(h_+, h_\times, \dot{h}_+, \dot{h}_\times)$

Well-posedness of time evolution problems

- Solution exists and is unique
- Solution $u(\mathbf{x}, t)$ **depends continuously on the initial data** $u(\mathbf{x}, 0)$ (and boundary data) in suitable function norms

$$||\delta u(\cdot, t)|| \le f(t)\, ||\delta u(\cdot, 0)||$$

where $f(t)$ does **not** depend on $u(\mathbf{x}, 0)$
- Otherwise numerics do not converge with resolution
- Simple example: the flat space linear wave equation

$$\phi_{,t} =: \Pi, \qquad \Pi_{,t} = \Delta\phi$$

with $(\Pi, \phi) = 0$ at infinity (Cauchy problem) is well-posed in the energy norm

$$||(\Pi, \phi)(\cdot, t)||^2 := \int \left[\Pi^2 + (\nabla\phi)^2\right] \, d^3x$$

because $||\delta u(\cdot, t)|| = ||\delta u(\cdot, 0)||$ (**exercise**)

Testing well-posedness

- Consider first-order systems for $\mathbf{u}(\mathbf{x}, t)$

$$\mathbf{u}_{,t} = P^i(\mathbf{u}, \mathbf{x}, t)\mathbf{u}_{,i} + \mathbf{S}(\mathbf{u}, \mathbf{x}, t)$$

- Linearise about a reference solution $\mathbf{u}_0$ by setting $\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}$, then "freeze" coefficients

$$\delta\mathbf{u}_{,t} = P^i \delta\mathbf{u}_{,i} + Q\delta\mathbf{u}$$

  where P^i and Q are now constant square matrices
- This tests the high frequency, small amplitude limit
- This is the regime that potentially goes wrong: higher spatial frequencies grow faster
- Only the principal part P^i matters for well-posedness

Strong hyperbolicity in 1D

- Single first-order linear PDE with constant coefficients in 1D

$$u_{,t} + \lambda u_{,x} = 0, \quad u(x,0) = f(x) \quad \Rightarrow \quad u(x,t) = f(x - \lambda t)$$

- System of such PDEs

$$\mathbf{u}_{,t} + P\mathbf{u}_{,x} = 0, \quad \mathbf{u}(x,0) = \mathbf{f}(x)$$

- **Strong hyperbolicity in 1D**: P has a complete set of real eigenvectors with real eigenvalues $\Leftrightarrow$ it can be diagonalised $P = R\Lambda R^{-1}$ where the columns of R are the eigenvectors
- Vector of characteristic variables $\mathbf{U} := R^{-1}\mathbf{u}$

$$\mathbf{U}_{,t} + \Lambda \mathbf{U}_{,x} = 0$$

- Each characteristic variable propagates at its own speed λ

Strong hyperbolicity in 3D

- Now consider linear system in 3D with constant coefficients

$$\mathbf{u}_{,t} + P^i \mathbf{u}_{,i} = 0, \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{f}(\mathbf{x})$$

- **Strong hyperbolicity**: (1) $P^i n_i$ has a complete set of real eigenvectors with real eigenvalues for all directions n_i
(2) R and Λ depend smoothly on n_i
- Formal solution (**exercise**): Fourier transform in space, split into characteristic variables, evolve, put back together

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{2\pi} \int e^{i\mathbf{kx}} \left(R(\mathbf{k}) e^{-i\Lambda(\mathbf{k})t} R(\mathbf{k})^{-1} \int e^{-i\mathbf{kx}'} \mathbf{f}(\mathbf{x}')\, d^3x' \right) d^3k$$

- Hence Cauchy problem well-posed in L^2 norm $\sqrt{\int \mathbf{u}^2 \, d^3x}$
- Much harder: proof that the linear system with variable coefficients is well-posed (for some short finite time), and then the full nonlinear system

Symmetric hyperbolicity

- Consider again a linear(ised) system with constant (frozen) coefficients, and neglect non-principal part

$$\mathbf{u}_{,t} + P^i \mathbf{u}_{,i} = 0$$

- **Symmetric hyperbolicity**: There is a Hermitian matrix H such that $HP^i n_i$ is Hermitian for all directions n_i, with H **independent of** n_i

$$(u^\dagger H u)_{,t} + (u^\dagger H P^i u)_{,i} = 0 \qquad \Rightarrow \frac{d}{dt} \int (u^\dagger H u)\, d^3x = \text{boundary}$$

- The energy $\int (u^\dagger H u)\, d^3x$ is locally conserved (**exercise**) in the small amplitude, high frequency limit (and can be bounded in the nonlinear problem)
- $\exists$ class of boundary conditions (maximally dissipative BCs) such that the **initial-boundary value problem** is wellposed
- Symmetric hyperbolicity $\Rightarrow$ strong hyperbolicity

General considerations

- The Einstein equations written in terms of the metric are second-order in space and time
- Reducing to first order in time as in $\gamma_{ij,t} \sim K_{ij}$, $K_{ij,t} \sim \partial\partial\gamma_{kl}$ makes no difference to well-posedness
- Reducing to first order in space as in $d_{ijk} := \gamma_{ij,k}$ introduces additional constraints, and sources of numerical error
- Necessary and sufficient criteria for strong and symmetric hyperbolicity exist for general first-order in time, second-order in space systems
- Hyperbolic systems coupled to elliptic or parabolic equations through non-principal terms are also well-posed

Partly the same slide as earlier

- We need to give four **gauge conditions** (algebraic, evolution, or elliptic equations) for α and β^i
- We can introduce **new variables**
- We can **add constraints** to the evolution equations
- The resulting evolution equations need to be **well-posed**
- ...even when the constraints are violated (because of numerical error)
- The (formal) constraint evolution system should also be well-posed (constraint-preserving boundary conditions if possible)
- Free versus constrained evolution
- Ideally, the constraints should decay in time

Generalized harmonic gauge formulation

- Leading order of the vacuum Einstein equation

$$R_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \left(g_{\mu\nu,\alpha\beta} + g_{\alpha\beta,\mu\nu} - 2 g_{\alpha(\mu,\nu)\beta} \right) + \text{lower order} = 0$$

- Impose gauge condition $C^\mu := \Box x^\mu - H^\mu(\mathbf{x}, g_{\alpha\beta}) = 0$

$$\Box x^\mu = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} g^{\alpha\beta} (x^\mu)_{,\alpha} \right)_{,\beta} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} g^{\mu\beta} \right)_{,\beta}$$

- Einstein equations in GH gauge (**exercise**)

$$R_{\mu\nu} + C_{(\mu,\nu)} = -\frac{1}{2} g^{\alpha\beta} g_{\mu\nu,\alpha\beta} - H_{(\mu,\nu)} + \text{lower order} = 0$$

- Solve Einstein constraints for initial data in usual 3+1 form
- But then evolve all 10 $g_{\mu\nu}$ directly with $\Box g_{\mu\nu} \sim 0$

Z4 formulation

- Add 4 new variables Z_μ. Instead of $R_{ab} = 0$ solve
$$R_{ab} + \nabla_a Z_b + \nabla_b Z_a = 0$$
- The time derivatives of the new variables are essentially the Einstein constraints
$$\dot{Z}_\mu \sim E_\mu := (H, M_i)$$
Setting $Z_\mu = 0$ and $E_\mu = 0$ in the initial data we obtain a solution of $R_{ab} = 0$, but the new system is strongly hyperbolic in a family of useful gauges
- Modifying further
$$R_{ab} + \nabla_a Z_b + \nabla_b Z_a - \kappa(t_a Z_b + t_b Z_c - g_{ab} t^c Z_c) = 0$$
we get constraint damping
$$\dot{Z}_\mu \sim E_\mu - \kappa Z_\mu$$

BSSN formulation and some popular gauges

- Split conformal factor from γ_{ij} and trace from K_{ij}
- Add 3 new variables $\tilde{\Gamma}_i := \tilde{\gamma}^{jk}\tilde{\gamma}_{ij,k}$
- Further modifications to lower-order terms
- Strongly or symmetric hyperbolic in suitable gauges
  - Harmonic slicing $K_i{}^i = 0 \Rightarrow \Delta\alpha \sim \alpha(R + K_{ij}K^{ij})$
  - "1+log slicing" $\dot{\alpha} \sim f(\alpha)K$
  - Zero shift $\beta^i = 0$
  - "Γ-driver" shift $\dot{\beta}^i \sim \tilde{\Gamma}^i$
- Initial data for two black holes can be "puncture data" where $\gamma_{ij} \sim (M/r)\delta_{ij}$
- By contrast Z4 and harmonic gauge need black hole excision

Polar-radial coordinates in spherical symmetry

- Make the coordinate r the "area radius", meaning that the area of the 2-spheres $t = r = const$ is $4\pi r^2$
- Choose t normal to r in the sense $\nabla_a t \nabla^a r = g^{tr} = 0$

$$ds^2 = -\alpha(t,r)^2\,dt^2 + a(t,r)^2\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$

- The Hamiltonian constraint becomes an ODE for $a(r)$ at each moment t
- The polar slicing condition becomes a linear ODE for $\alpha(r)$ at each moment t
- Example of "maximally constrained evolution"
- In 3D, only "free evolution" is common

Null coordinates

- Surfaces of constant coordinate u are null

$$g^{ab}\nabla_a u \nabla_b u = g^{uu} = 0$$

- $V^a := \nabla^a u$ is null and obeys $V^a \nabla_a V^b = 0$, so u-surfaces are ruled by null geodesics with tangent vector V^a
- Coordinate choices
  - double null (u, v, θ, φ) where $g^{uu} = g^{vv} = 0$
  - Bondi (u, r, θ, φ) where r is an area radius
  - affine $(u, \lambda, \theta, \varphi)$ where λ is an affine parameter along V^a
- Constraints on constant u "time" slices can be solved by **integration** outwards (including the initial data $u = 0$)
- Another example of maximally constrained evolution
- Problems when light rays cross