3

Gravitation and Kepler's Laws

In this chapter we will recall the law of universal gravitation and will then derive the result that a spherically symmetric object acts gravitationally like a point mass at its centre if you are outside the object. Following this we will look at orbits under gravity, deriving Kepler's laws. The chapter ends with a consideration of the energy in orbital motion and the idea of an effective potential.

3.1 Newton's Law of Universal Gravitation

For two particles of masses m_1 and m_2 separated by distance r there is a mutual force of attraction of magnitude

$$\frac{Gm_1m_2}{r^2},$$

where $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is the *gravitational constant*. If $\mathbf{F}_{12}$ is the force of particle 2 on particle 1 and vice-versa, and if $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ is the vector from particle 1 to particle 2, as shown in figure 3.1, then the vector form of the law is:

$$\boxed{\mathbf{F}_{12} = -\mathbf{F}_{21} = \frac{Gm_1m_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12}},$$

where the hat ($\hat{}$) denotes a unit vector as usual. Gravity obeys the superposition principle, so if particle 1 is attracted by particles 2 and 3, the total force on 1 is $\mathbf{F}_{12} + \mathbf{F}_{13}$.

The gravitational force is exactly analogous to the electrostatic Coulomb force if you make the replacements, $m \to q$, $-G \to 1/4\pi\varepsilon_0$ (of course, masses are always

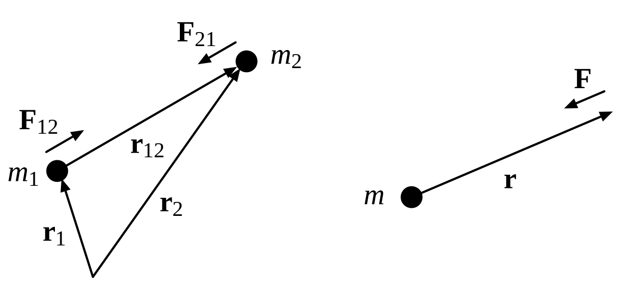


Figure 3.1 Labelling for gravitational force between two masses (left) and gravitational potential and field for a single mass (right).

positive, whereas charges q can be of either sign). We will return to this analogy later.

Since gravity acts along the line joining the two masses, it is a *central force* and therefore *conservative* (any central force is conservative — why ?). For a conservative force, you can sensibly define a *potential energy difference* between any two points according to,

$$V(\mathbf{r}_f) - V(\mathbf{r}_i) = -\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} d\cdot\mathbf{r}.$$

The definition is sensible because the answer depends only on the endpoints and not on which particular path you used. Since only *differences* in potential energy appear, we can arbitrarily choose a particular point, say $\mathbf{r}_0$, as a reference and declare its potential energy to be zero, $V(\mathbf{r}_0) = 0$. If you're considering a planet orbiting the Sun, it is conventional to set $V = 0$ at infinite separation from the Sun, so $|\mathbf{r}_0| = \infty$. This means that we can define a gravitational potential energy by making the conventional choice that the potential is zero when the two masses are infinitely far apart. For convenience, let's put the origin of coordinates at particle 1 and let $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ be the position of particle 2. Then the gravitational force on particle 2 due to particle 1 is $\mathbf{F} = \mathbf{F}_{21} = -Gm_1m_2\hat{\mathbf{r}}/r^2$ and the gravitational potential energy is,

$$\boxed{V(r) = -\int_{\infty}^{r} \mathbf{F}\cdot d\mathbf{r}' = -\int_{\infty}^{r} (-\frac{Gm_1m_2}{r'^2}) dr' = -\frac{Gm_1m_2}{r}}.$$

(The prime ($'$) on the integration variable is simply to distinguish it from the point where we are evaluating the potential energy.) It is also useful to think of particle 1 setting up a gravitational field which acts on particle 2, with particle 2 acting as a test mass for probing the field. Define the *gravitational potential*, which is the gravitational potential energy per unit mass, for particle 1 by (setting $m_1 = m$ now),

$$\boxed{\Phi(r) = -\frac{Gm}{r}}.$$

Likewise, define the *gravitational field* $\mathbf{g}$ of particle 1 as the gravitational force per unit mass:

$$\boxed{\mathbf{g}(\mathbf{r}) = -\frac{Gm}{r^2}\hat{\mathbf{r}}}.$$

The use of $\mathbf{g}$ for this field is deliberate: the familiar $g = 9.81\,\mathrm{m\,s^{-2}}$ is just the magnitude of the Earth's gravitational field at its surface. The field and potential are related in the usual way:

$$\mathbf{g} = -\nabla\Phi.$$

Gravitational Potential Energy Near the Earths' Surface If you are thinking about a particle moving under gravity near the Earth's surface, you might set the $V = 0$ at the surface. Here, the gravitational force on a particle of mass m is,

$$\mathbf{F} = -mg\hat{\mathbf{k}},$$

where $\hat{\mathbf{k}}$ is an upward vertical unit vector, and $g = 9.81\,\mathrm{m\,s^{-2}}$ is the magnitude of the gravitational acceleration. In components, $F_x = F_y = 0$ and $F_z = -mg$. Since the force is purely vertical, the potential energy is independent of x and y. We will measure z as the height above the surface. Applying the definition of potential energy difference between height h and the Earth's surface ($z = 0$), we find

$$V(h) - V(0) = -\int_0^h F_z \, dz = -\int_0^h (-mg) \, dz = mgh.$$

Choosing $z = 0$ as our reference height, we set $V(z{=}0) = 0$ and find the familiar result for gravitational potential energy,

$V(h) = mgh$	Gravitational potential energy near the Earth's surface

Note that since the gravitational force acts vertically, on any path between two given points the work done by gravity depends only on the changes in height between the endpoints. So, this force is indeed conservative.

3.2 Gravitational Attraction of a Spherical Shell

The problem of determining the gravitational attraction of spherically symmetric objects led Newton to invent calculus: it took him many years to prove the result. The answer for a thin uniform spherical shell of matter is that outside the shell the gravitational force is the same as that of a point mass of the same total mass as the shell, located at the centre of the shell. Inside the shell, the force is zero. By considering an arbitrary spherically symmetric object to be built up from thin shells, we immediately find that outside the object the gravitational force is the same as that of a point with the same total mass located at the centre.

We will demonstrate this result in two ways: first by calculating the gravitational potential directly, and then, making full use of the spherical symmetry, using the analogy to electrostatics and applying Gauss' law.

3.2.1 Direct Calculation

We consider a thin spherical shell of radius a, mass per unit area ρ and total mass $m = 4\pi\rho a^2$. Use coordinates with origin at the center of the shell and calculate the gravitational potential at a point P distance r from the centre as shown in figure 3.2.

We use the superposition principle to sum up the individual contributions to the potential from all the mass elements in the shell. All the mass in the thin annulus of width $a\,d\theta$ at angle θ is at the same distance R from P, so we can use this as our element of mass:

$$dm = \rho\, 2\pi a \sin\theta \, a d\theta = \frac{m}{2} \sin\theta \, d\theta.$$

The contribution to the potential from the annulus is,

$$d\Phi = -\frac{G\,dm}{R} = -\frac{Gm}{2}\frac{\sin\theta\, d\theta}{R}.$$

Now we want to sum all the contributions by integrating over θ from 0 to π. In fact, it is convenient to change the integration variable from θ to R. They are related using the cosine rule:

$$R^2 = r^2 + a^2 - 2ar\cos\theta.$$

From this we find $\sin\theta\, d\theta / R = dR/(ar)$, which makes the integration simple. If $r \geq a$ the integration limits are $r - a$ and $r + a$, while if $r \leq a$ they are $a - r$ and $a + r$.

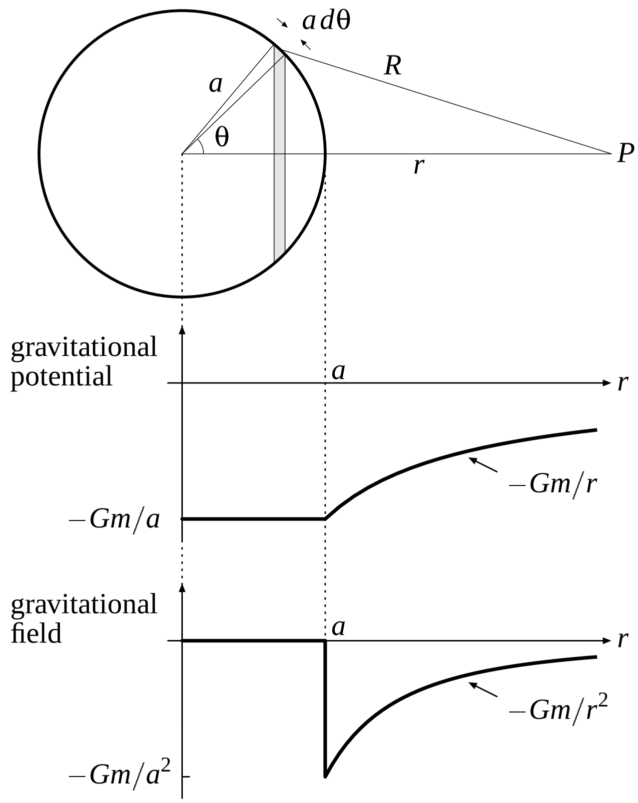


Figure 3.2 Gravitational potential and field for a thin uniform spherical shell of matter.

We can specify the limits for both cases as $|r-a|$ and $r+a$, so that:

$$\Phi(r) = -\frac{Gm}{2ar}\int_{|r-a|}^{r+a} dR = \begin{cases} -Gm/r & \text{for } r \geq a \\ -Gm/a & \text{for } r < a \end{cases}.$$

We obtain the gravitational field by differentiating:

$$\mathbf{g}(\mathbf{r}) = \begin{cases} -Gm\hat{\mathbf{r}}/r^2 & \text{for } r \geq a \\ 0 & \text{for } r < a \end{cases}.$$

As promised, outside the shell, the potential is just that of a point mass at the centre. Inside, the potential is constant and so the force vanishes. The immediate corollaries are:

- A uniform or spherically stratified sphere (so the density is a function of the radial coordinate only) attracts like a point mass of the same total mass at its centre, when you are outside the sphere;

- Two non-intersecting spherically symmetric objects attract each other like two point masses at their centres.

3.2.2 The Easy Way

Now we make use of the equivalence of the gravitational force to the Coulomb force using the relabelling summarised in table 3.1. We can now apply the integral form of Gauss' Law in the gravitational case to our spherical shell. The law reads,

$$\int_S \mathbf{g}\cdot d\mathbf{S} = -4\pi G \int_V \rho_m \, dV$$

Coulomb force		Gravitational force	
charge	q	mass	m
coupling	$1/(4\pi\varepsilon_0)$	coupling	$-G$
potential	V	potential	Φ
electric field	$\mathbf{E} = -\boldsymbol{\nabla} V$	gravitational field	$\mathbf{g} = -\boldsymbol{\nabla}\Phi$
charge density	ρ_q	mass density	ρ_m
Gauss' law	$\boldsymbol{\nabla}\cdot\mathbf{E} = \rho_q/\varepsilon_0$	Gauss' law	$\boldsymbol{\nabla}\cdot\mathbf{g} = -4\pi G\rho_m$

Table 3.1 Equivalence between electrostatic Coulomb force and gravitational force.

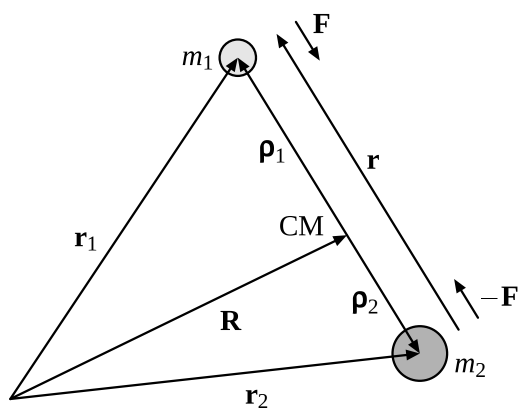


Figure 3.3 Coordinates for a two-body system.

which says that the surface integral of the normal component of the gravitational field over a given surface S is equal to $(-4\pi G)$ times the mass contained within that surface, with the mass obtained by integrating the mass density ρ_m over the volume V contained by S.

The spherical symmetry tells us that the gravitational field $\mathbf{g}$ must be radial, $\mathbf{g} = g\hat{\mathbf{r}}$. If we choose a concentric spherical surface with radius $r > a$, the mass enclosed is just m, the mass of the shell, and Gauss' Law says,

$$4\pi r^2 g = -4\pi G m$$

which gives

$$\mathbf{g} = -\frac{Gm}{r^2}\hat{\mathbf{r}} \qquad \text{for } r > a$$

immediately. Likewise, if we choose a concentric spherical surface inside the shell, the mass enclosed is zero and $\mathbf{g}$ must vanish.

3.3 Orbits: Preliminaries

3.3.1 Two-body Problem: Reduced Mass

Consider a system of two particles of masses m_1 at position $\mathbf{r}_1$ and m_2 at $\mathbf{r}_2$ interacting with each other by a conservative central force, as shown in figure 3.3. We imagine these two particle to be isolated from all other influences so that there is no external force.

Express the position $\mathbf{r}_i$ of each particle as the centre of mass location $\mathbf{R}$ plus a displacement $\boldsymbol{\rho}_i$ relative to the centre of mass, as we did in equation (1.3) in chapter 1 on page 2.

$$\mathbf{r}_1 = \mathbf{R} + \boldsymbol{\rho}_1, \qquad \mathbf{r}_2 = \mathbf{R} + \boldsymbol{\rho}_2.$$

Now change variables from $\mathbf{r}_1$ and $\mathbf{r}_2$ to $\mathbf{R}$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Since the only force acting is the internal force, $\mathbf{F} = \mathbf{F}_{12} = -\mathbf{F}_{21}$, between particles 1 and 2, the equations of motion are:

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}, \qquad m_2\ddot{\mathbf{r}}_2 = -\mathbf{F}.$$

From these we find, setting $M = m_1 + m_2$,

$$M\ddot{\mathbf{R}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0,$$

which says that the centre of mass moves with constant velocity, as we already know from the general analysis in section 1.1.1 (see page 2). For the new relative displacement $\mathbf{r}$, we find,

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F} = \frac{m_1+m_2}{m_1 m_2}\mathbf{F},$$

which we write as,

$$\boxed{\mathbf{F} = \mu\ddot{\mathbf{r}}}, \tag{3.1}$$

where we have defined the *reduced mass*

$$\boxed{\mu \equiv \frac{m_1 m_2}{m_1+m_2}}.$$

For a conservative force $\mathbf{F}$ there is an associated potential energy $V(r)$ and the total energy of the system becomes

$$E = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 + V(r).$$

This is just an application of the general result we derived for the kinetic energy of a system of particles in equation (1.4) on page 3 — we already applied it in the two-particle case on page 3. Likewise, when $\mathbf{F}$ is central, the angular momentum of the system is

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \mu\mathbf{r} \times \dot{\mathbf{r}},$$

which is an application of the result in equation (1.6) on page 7. You should make sure you can reproduce these two results.

Since the center of mass $\mathbf{R}$ moves with constant velocity we can switch to an inertial frame with origin at $\mathbf{R}$, so that $\mathbf{R} = 0$. Then we have:

$$\boxed{\begin{aligned} E &= \frac{1}{2}\mu\dot{\mathbf{r}}^2 + V(r), \\ \mathbf{L} &= \mu\mathbf{r} \times \dot{\mathbf{r}}. \end{aligned}} \tag{3.2}$$

The original two-body problem reduces to an equivalent problem of a single body of mass μ at position vector $\mathbf{r}$ relative to a fixed centre, acted on by the force $\mathbf{F} = -(\partial V/\partial r)\,\hat{\mathbf{r}}$.

It's often the case that one of the masses is very much larger than the other, for example:

$$\begin{aligned} m_{\text{Sun}} &\gg m_{\text{planet}}, \\ m_{\text{Earth}} &\gg m_{\text{satellite}}, \\ m_{\text{proton}} &\gg m_{\text{electron}}. \end{aligned}$$

If $m_2 \gg m_1$, then $\mu = m_1 m_2/(m_1+m_2) \approx m_1$ and the reduced mass is nearly equal to the light mass. Furthermore,

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \approx \mathbf{r}_2$$

and the centre of mass is effectively at the larger mass. In such cases we treat the larger mass as fixed at $\mathbf{r}_2 \approx 0$, with the smaller mass orbiting around it, and set μ equal to the smaller mass. This is sometimes called the "fixed Sun and moving planet approximation." We will use this approximation when we derive Kepler's Laws. We will also ignore interactions between planets in comparison to the gravitational attraction of each planet towards the Sun.

3.3.2 Two-body Problem: Conserved Quantities

Recall that gravity is a central force: the gravitational attraction between two bodies acts along the line joining them. In the formulation of equations 3.2 above, this means that the gravitational force on the mass μ acts in the direction $-\mathbf{r}$ and therefore exerts no torque about the fixed centre. Consequently, the angular momentum vector $\mathbf{L}$ is a constant: its magnitude is fixed and it points in a fixed direction. Since $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ (where $\mathbf{p} = \mu\dot{\mathbf{r}}$), we see that $\mathbf{L}$ is always perpendicular to the plane defined by the position and momentum of the mass μ. Alternatively stated, this means that $\mathbf{r}$ and $\mathbf{p}$ must always lie in the fixed plane of all directions perpendicular to $\mathbf{L}$, and can therefore be described using plane polar coordinates (r, θ), with origin at the fixed centre.

For completeness we quote the radial and angular equations of motion in these plane polar coordinates. We set the reduced mass equal to the planet's mass m and write the gravitational force as $\mathbf{F} = -k\hat{\mathbf{r}}/r^2$, where $k = GMm$ and M is the Sun's mass. The equations become (the reader should exercise to reproduce the following expressions):

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{k}{mr^2} && \text{radial equation,} \\ \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) &= 0 && \text{angular equation.} \end{aligned}$$

The angular equation simply expresses the conservation of the angular momentum $L = mr^2\dot{\theta}$.

The second conserved quantity is the total energy, kinetic plus potential. All central forces are conservative and in our two-body orbit problem the only force acting is the central gravitational force. We again set μ equal to the planet's mass m and write the gravitational potential energy as $V(r) = -k/r$. Then the expression for the constant total energy becomes, using plane polar coordinates,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - k/r.$$

In section 3.5 on page 33 we will deduce a good deal of information about the orbit straight from this conserved total energy.

3.3.3 Two-body Problem: Examples

Comet A comet approaching the Sun in the plane of the Earth's orbit (assumed circular) crosses the orbit at an angle of 60° travelling at $50\,\mathrm{km\,s^{-1}}$. Its closest approach to the Sun is $1/10$ of the Earth's orbital radius. Calculate the comet's speed at the point of closest approach.

Take a circular orbit of radius r_e for the Earth. Ignore the attraction of the comet to the Earth compared to the attraction of the comet to the Sun and ignore any complications due to the reduced mass.

The key to this problem is that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$ of the comet about the Sun is fixed. At the point of closest approach the comet's velocity must be tangential only (why?), so that,

$$|\mathbf{r} \times \mathbf{v}| = r_{\text{min}} v_{\text{max}}.$$

At the crossing point,

$$|\mathbf{r} \times \mathbf{v}| = r_e v \sin 30^\circ.$$

Equating these two expressions gives,

$$r_{\text{min}} v_{\text{max}} = 0.1\, r_e v_{\text{max}} = \frac{1}{2} r_e v,$$

leading to

$$v_{\text{max}} = 5v = 250\,\text{km}\,\text{s}^{-1}.$$

Cygnus X1 Cygnus X1 is a binary system of a supergiant star of 25 solar masses and a black hole of 10 solar masses, each in a circular orbit about their centre of mass with period 5.6 days. Determine the distance between the supergiant and the black hole, given that a solar mass is 1.99×10^{30} kg.

Here we apply the two-body equation of motion, equation (3.1) from page 26. Labelling the two masses m_1 and m_2, their separation r and their angular velocity ω, we have,

$$\frac{G m_1 m_2}{r^2} = \frac{m_1 m_2}{m_1 + m_2} r \omega^2.$$

Rearranging and using the period $T = 2\pi/\omega$, gives

$$\begin{aligned} r^3 &= \frac{G(m_1 + m_2)T^2}{4\pi^2} \\ &= \frac{6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2} \times (10 + 25) \times 1.99 \times 10^{30}\,\text{kg} \times (5.6 \times 86400\,\text{s})^2}{4\pi^2} \\ &= 27.5 \times 10^{30}\,\text{m}^3, \end{aligned}$$

leading to $r = 3 \times 10^{10}$ m.

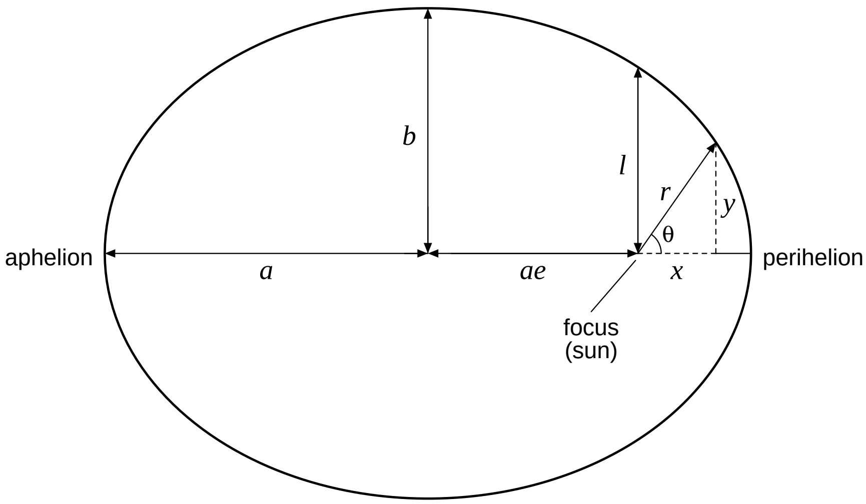
3.4 Kepler's Laws

3.4.1 Statement of Kepler's Laws

1. The orbits of the planets are ellipses with the Sun at one focus.

2. The radius vector from the Sun to a planet sweeps out equal areas in equal times.

3. The square of the orbital period of a planet is proportional to the cube of the semimajor axis of the planet's orbit ($T^2 \propto a^3$).



polar equation	$\frac{l}{r} = 1 + e\cos\theta$
cartesian equation	$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1$
semimajor axis	$a = \frac{l}{1-e^2} = -\frac{k}{2E}$
semiminor axis	$b = \frac{l}{\sqrt{1-e^2}} = a^{1/2}l^{1/2}$
eccentricity	e
semi latus rectum	$l = \frac{L^2}{mk}$
constant k	$k = GMm$
total energy	$E = -\frac{mk^2}{2L^2}(1-e^2) = -\frac{k}{2a}$

Figure 3.4 Geometry of an ellipse and relations between its parameters. In the polar and cartesian equations for the ellipse, the origin of coordinates is at the *focus*.

3.4.2 Summary of Derivation of Kepler's Laws

We will be referring to the properties of ellipses, so figure 3.4 shows an ellipse and its geometric parameters. The parameters are also expressed in terms of the dynamical quantities: energy E, angular momentum L, mass of the Sun M, mass of the planet m and the universal constant of gravitation G. The semimajor axis a is fixed by the total energy E and the semi latus rectum l is fixed by the total angular momentum L.

In general the path of an object orbiting under an inverse square law force can be any conic section. This means that the orbit may be an ellipse with $0 \leq e < 1$, parabola with $e = 1$ or hyperbola with $e > 1$. With the definition that the zero of potential energy occurs for infinite separation, the total energy of the system is negative for an elliptical orbit. When the total energy is zero the object can just escape to infinite distance, where it will have zero kinetic energy: this is a parabolic orbit. For positive energy, the object can escape to infinite separation with finite kinetic energy: this gives a hyperbolic orbit. Figure 3.5 illustrates the possible orbital

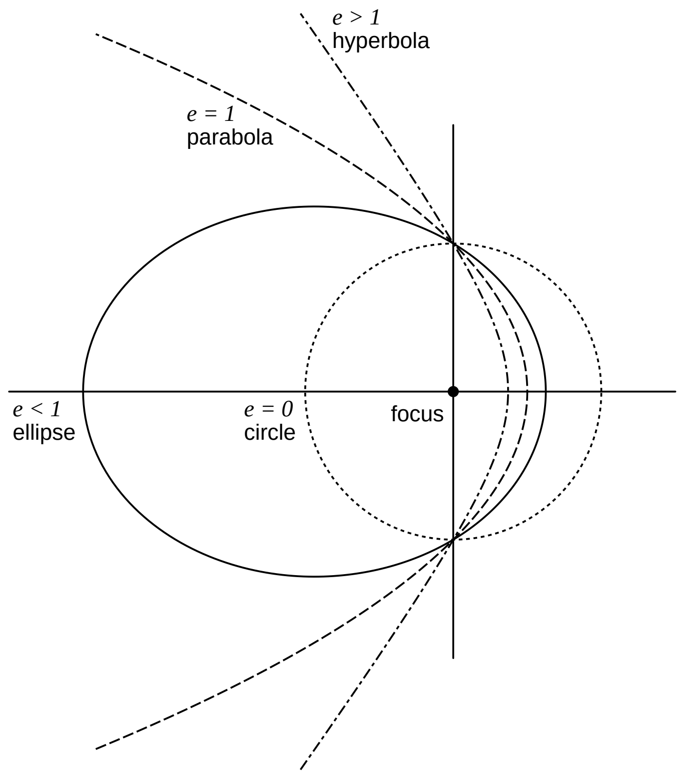


Figure 3.5 Different conic sections, showing possible orbits under an inverse square law force. The figure is drawn so that each orbit has the same angular momentum (same l) but different energy (the mass of the orbiting object is held fixed).

shapes.

2nd Law This is the most general and is a statement of angular momentum conservation under the action of the *central* gravitational force. The angular equation of motion gives:

$$r^2\dot{\theta} = \frac{L}{m} = \text{const.}$$

This immediately leads to,

$$\boxed{\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{const}}.$$

The 2nd law is illustrated in figure 3.6. An orbiting planet moves along the arc segments AB and CD in equal times, and the two shaded areas are equal.

Orbit equation The first and third laws are arrived at by finding the equation for the orbit. The fact that the orbits are ellipses is *specific* to an inverse square law for the force, and hence the first and third laws are also specific to an inverse square law force.

Proceed as follows, starting from the radial equation of motion (with $k = GMm$),

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k}{mr^2}.$$

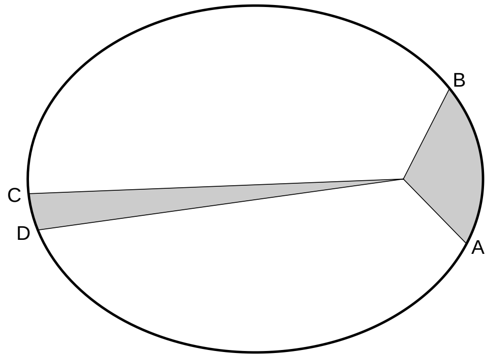


Figure 3.6 Illustration of Kepler's 2nd Law. An orbiting planet moves along the arc segments AB and CD in equal times, and the two shaded areas are equal.

(i) Eliminate $\dot{\theta}$ using angular momentum conservation, $\dot{\theta} = L/mr^2$, leading to a differential equation for r alone:

$$\ddot{r} - \frac{L^2}{m^2 r^3} = -\frac{k}{mr^2}.$$

(ii) Use the relation

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} = \frac{L}{mr^2}\frac{d}{d\theta},$$

to obtain derivatives with respect to θ in place of time derivatives. This gives a differential equation for r in terms of θ.

(iii) To obtain an equation which is easy to solve, make the substitution $u = 1/r$, to obtain the orbit equation:

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{mk}{L^2}}.$$

1st Law The solution of the orbit equation is

$$\frac{1}{r} = \frac{mk}{L^2}(1 + e\cos\theta),$$

which for $0 \le e < 1$ gives an ellipse, with semi latus rectum $l = L^2/mk$. This is the first law.

In figure 3.7 we show the orbit of a hypothetical planet around the Sun with semimajor axis 1.427×10^9 km (the same as Saturn) and eccentricity $e = 0.56$ (bigger than for any real planet — Pluto has the most eccentric orbit with $e = 0.25$). The figure also shows how the planet's distance from the Sun, speed and angular velocity vary during its orbit.

3rd Law Start with the 2nd law for the rate at which area is swept out,

$$\frac{dA}{dt} = \frac{L}{2m},$$

and integrate over a complete orbital period T, to give $T = 2mA/L$, where $A = \pi ab$ is the area of the ellipse. Substituting for b in terms of a gives the third law:

$$\boxed{T^2 = \frac{4\pi^2}{GM}a^3}.$$

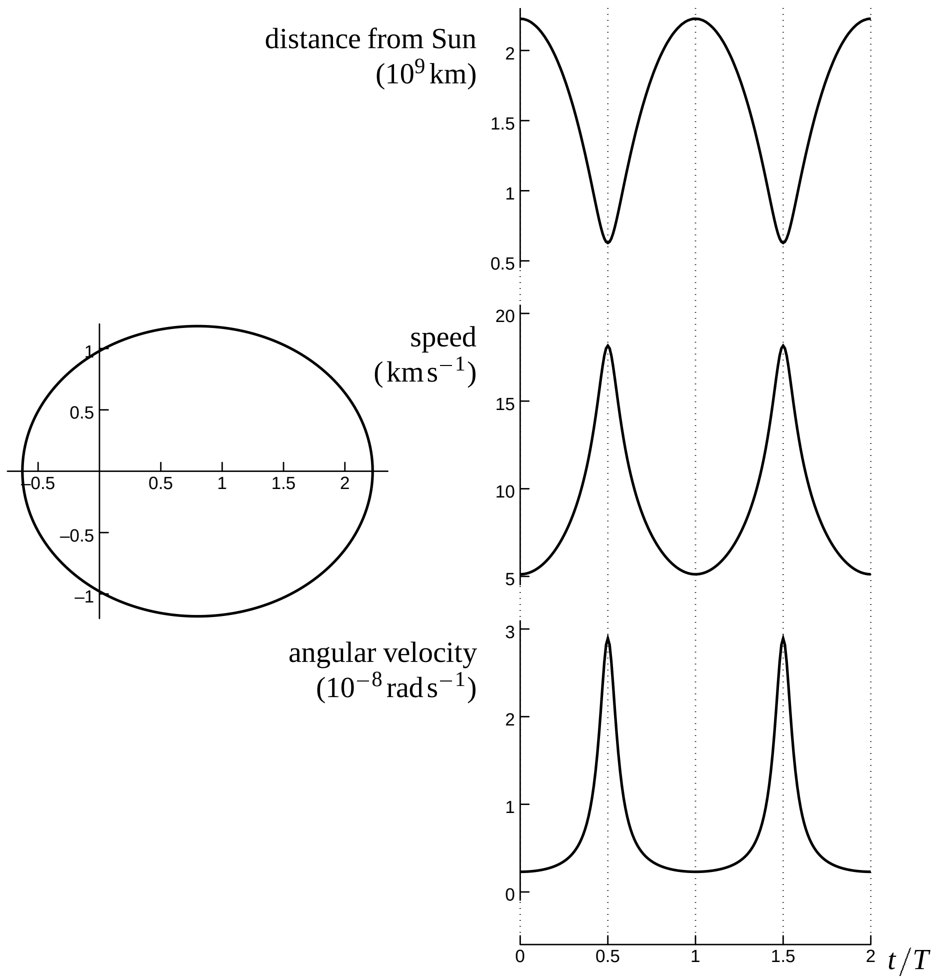


Figure 3.7 On the left is shown the orbit of a hypothetical planet around the Sun with distance scales marked in units of 10^9 km. The planet has the same semimajor axis $a = 1.427\times10^9$ km as Saturn, and hence the same period, $T = 10760$ days. The eccentricity is $e = 0.56$. The three graphs on the right show the planet's distance from the sun, speed and angular velocity respectively as functions of time measured in units of the orbital period T.

Kepler's Procedure* The solution of the orbit equation gives r as a function of θ, but if you're an astronomer, you may well be interested in knowing $\theta(t)$, so that you can track a planet's position in orbit as a function of time. You could do this by brute force by combining the angular equation of motion, $r^2\dot{\theta} = L/m$, with the equation giving the orbit, $l/r = 1 + e\cos\theta$, and integrating. This gives a disgusting integral which moreover leads to t as a function of θ: you have to invert this, by a series expansion method, to get θ as a function of t. This is tedious, and requires you to keep many terms in the expansion to match the accuracy of astronomical observations. Kepler himself devised an ingenious geometrical way to determine $\theta(t)$, and his construction leads to a much neater numerical procedure. I refer you to the textbook by Marion and Thornton[1] for a description.

[1] J B Marion and S T Thornton, *Classical Dynamics of Particles and Systems*, 3rd edition, Harcourt Brace Jovanovich (1988) p261

3.4.3 Scaling Argument for Kepler's 3rd Law

Suppose you have found a solution of the orbit equation, $\ddot{r} - r\dot{\theta}^2 = -k/mr^2$, giving r and θ as functions of t. Now scale the radial and time variables by constants α and β respectively:

$$r' = \alpha r, \qquad t' = \beta t.$$

In terms of the new variables r' and t', the left hand side of the orbit equation becomes,

$$\frac{d^2r'}{dt'^2} - r'\left(\frac{d\theta}{dt'}\right)^2 = \frac{\alpha}{\beta^2}\ddot{r} - \alpha r\left(\frac{\dot{\theta}}{\beta}\right)^2 = \frac{\alpha}{\beta^2}(\ddot{r} - r\dot{\theta}^2),$$

while the right hand side becomes,

$$-\frac{k}{mr'^2} = \frac{1}{\alpha^2}\left(-\frac{k}{mr^2}\right).$$

Comparing the two sides, you can see that we will have a new solution in terms of r' and t' provided $\beta^2 = \alpha^3$. But this says precisely that if you have orbits of similar shape, the period T and semimajor axis a (characterising the linear size of the orbit) will be related by $T^2 \propto a^3$, which is Kepler's third law.

To find the constant of proportionality and show that the orbits are conic sections, you really have to solve the orbit equation. However, the scaling argument makes clear how the third law depends on having an inverse-square force law.

3.5 Energy Considerations: Effective Potential

Since the gravitational force is conservative, the total energy E of the orbiting body is conserved. Writing $V(r)$ for the gravitational potential energy for a moment (so that we can substitute different forms for the potential energy if necessary), we find

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r).$$

Since we know that angular momentum is also conserved (the force is central), we can eliminate $\dot{\theta}$ using $r^2\dot{\theta} = L/m$, to leave,

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r)}.$$

This is just the energy equation you would get for a particle moving in one dimension in an *effective potential*

$$\boxed{U(r) = \frac{L^2}{2mr^2} + V(r)}.$$

The effective potential contains an additional *centrifugal term*, $L^2/2mr^2$, which arises because angular momentum has to be conserved. We can learn a good deal about the possible motion by studying the effective potential without having to solve the equation of motion for r.

In our case, replacing $V(r)$ by the gravitational potential energy and using $l = L^2/mk$, the effective potential becomes (see figure 3.8)

$$U(r) = \frac{kl}{2r^2} - \frac{k}{r}.$$

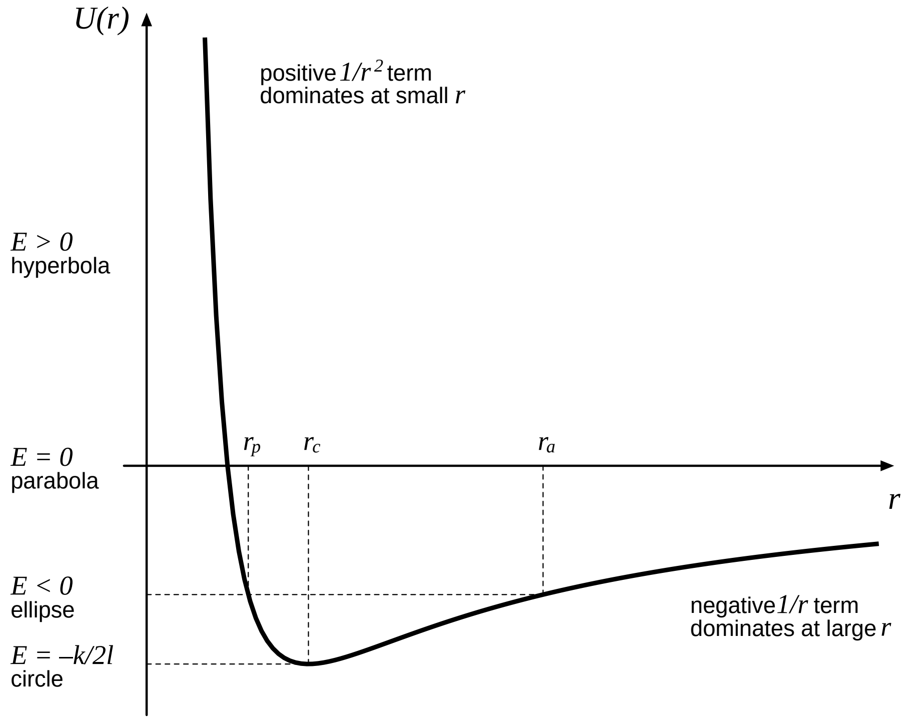


Figure 3.8 Effective potential $U(r) = kl/2r^2 - k/r$ for motion in an inverse-square law force.

The allowed motion must have $\dot{r}^2 \geq 0$, so the energy equation says

$$E \geq U(r) = \frac{kl}{2r^2} - \frac{k}{r}.$$

If we choose a value for the total energy E, we can then draw a horizontal line at this value on the graph of $U(r)$, and we know that the allowed motion occurs only where the $U(r)$ curve lies *below* our chosen value of E.

The minimum possible total energy (for a given angular momentum) is given by the minimum of the curve of $U(r)$. In this situation r is constant at

$$r_c = l = L^2/mk,$$

so the orbit is a circle and the total energy is $E = -k/2l = -mk^2/2L^2$.

If $-k/2l < E < 0$, you can see that the motion is allowed for a finite range of r, $r_p \leq r \leq r_a$. This is the case of an elliptical orbit with perihelion r_p and aphelion r_a. You can find the values of r_p and r_a by finding the roots of the equation $E = kl/2r^2 - k/r$.

If $E = 0$, you see that there is a minimum value for r, but that escape to infinity is just possible. This is the case of a parabolic orbit. For $E > 0$, escape to infinity is possible with finite kinetic energy at infinite separation. This is the case of a hyperbolic orbit.

Orbits in a Yukawa Potential We found that the orbits produced by an inverse-square law attractive force were ellipses, where the planet repeatedly traced the same path through space. Now consider a force given by the Yukawa potential,

$$V(r) = -\frac{\alpha e^{-\kappa r}}{r} \qquad (\alpha > 0, \kappa > 0).$$

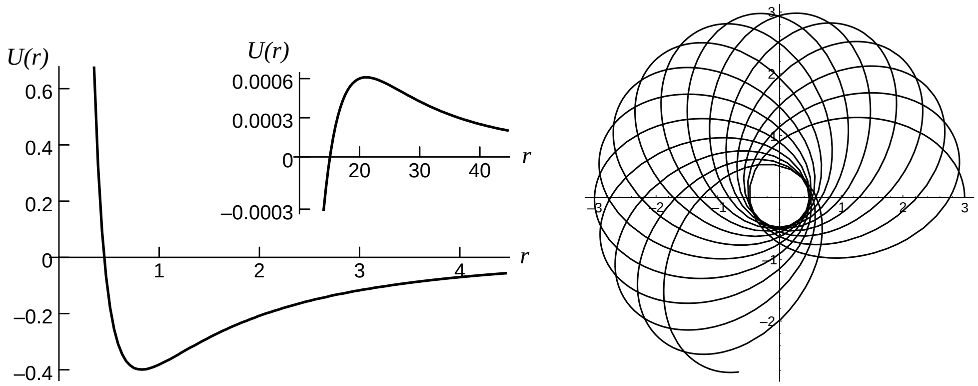


Figure 3.9 Left: effective potential $U(r) = L^2/2mr^2 - \alpha e^{-\kappa r}/r$ with $m = 1$, $\alpha = 1$, $\kappa = 0.24$ and $L = 0.9$. The inset shows $U(r)$ at large r where it has a local maximum (note the differences in scale, particularly for the value of U). Right: rosette orbit of a particle with this effective potential.

Such a potential describes, for example, the force of attraction between nucleons in an atomic nucleus. Of course, in that situation, the problem should be treated quantum mechanically, but for now, let's just look at classical orbits under the influence of this potential.

The effective potential is,

$$U(r) = \frac{L^2}{2mr^2} - \frac{\alpha e^{-\kappa r}}{r}.$$

To be specific, work in dimensionless units, setting $m = 1$, $\alpha = 1$, $\kappa = 0.24$ and choosing $L = 0.9$. The shape of the resulting effective potential as a function of r is shown in the left hand part of figure 3.9.

If the total energy E is negative but greater than the minimum of $U(r)$, then motion is allowed between a minimum and maximum value of the radius r. On the right hand side of figure 3.9 is the trajectory of a particle starting at $(x, y) = (3, 0)$ with $(v_x, v_y) = (0, 0.3)$ (so that $L = 0.9$). Here the particle's (dimensionless) energy is -0.117 and the motion is restricted to the region $0.486 \leq r \leq 3$, where 0.486 and 3 are the two solutions of the equation $U(r) = -0.117$.

Note that if $\kappa = 0$, the Yukawa potential reduces to the same form as the standard gravitational potential. So, if κr remains small compared to 1 we expect the situation to be a small perturbation relative to the gravitational case. In our example, for the "rosette" orbit on the right of figure 3.9, this is the case, and you can see that the orbit looks like an ellipse whose orientation slowly changes. This is often denoted "precession of the perihelion" and is typical of the effect of small perturbations on planetary orbits, for example those due to the effects of other planets. In fact, observed irregularities in the motion of Uranus led to the discovery of Neptune in 1846. The orientation of the major axis of the Earth's orbit drifts by about 104 seconds of arc each century, mostly due the influence of Jupiter. For Mercury, the perihelion advances by about 574 seconds of arc per century: 531 seconds of this can be explained by the Newtonian gravitational interactions of the other planets, while the remaining 43 seconds of arc are famously explained by Einstein's general relativity.

The effective potential shown in figure 3.9 displays another interesting property. At large r the $L^2/2mr^2$ dominates the exponentially falling Yukawa term, so $U(r)$ becomes positive. In our example, $U(r)$ has a local maximum near $r = 20$. If the

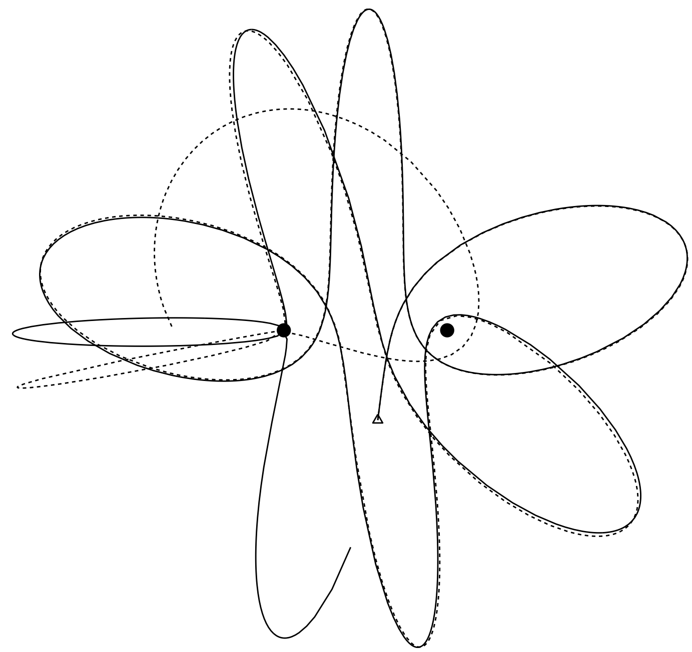


Figure 3.10 Orbital trajectories for a planet around two equal mass stars.

total energy is positive, but less than the value of U at the local maximum, there are two possibilities for orbital motion. For example, if $E = 0.0003$, we find either $0.451 \leq r \leq 16.31$ or $r \geq 36.48$. Classically these orbits are distinct, and a particle with $E = 0.0003$ which starts out in the inner region can never surmount the "barrier" in $U(r)$ and so will never be found in $r \geq 36.48$. In quantum mechanics, however, it is possible for a particle to "tunnel" through such a barrier, so that an initially bound particle has a (small) finite probability of escaping to large r. This is the case for a process like alpha decay.

3.6 Chaos in Planetary Orbits*

We have shown that a single planet orbiting the Sun follows a simple closed elliptical path. You might think that adding one more object to the system would make the equations more complicated, but that with patience and effort you might be able to figure out a solution for the trajectories. In fact, such a "three body problem" is notoriously intractable, and, even today, analytic solutions are known only in a few special cases.

In figure 3.10 is shown a numerical solution for a restricted version of the three body problem. The two black dots are stars of equal mass, held at fixed positions. This means that the total energy is conserved, but that the linear and angular momentum are not conserved since forces and torques have to be applied to hold the stars in place. The solid curve shows the trajectory of a planet which starts out with some given initial velocity at the point marked by the triangle. The stars are taken to have a finite radius and the planet is allowed to pass through them without suffering any interaction apart from the gravitational force (this avoids some numerical instability when the planet gets very close to a point mass). The complexity of the solid curve already hints at the difficulty of this problem.

In fact, the motion is chaotic in the scientific sense. One aspect of this is shown by the dashed curve. This is a second solution for a planet which also starts out at the point marked by the triangle, but has one of its initial velocity components differing by 0.5% from the corresponding component for the first case. You can see how the paths stay close together for a little while, but then rapidly diverge and show qualitatively different behaviour. This extreme (exponential) sensitivity to the initial conditions is one of the characteristics of chaotic systems. Contrast it to the two body problem, where a small perturbation to an elliptical orbit would simply result in a new slightly displaced orbit.

For an animated computer simulation of the three body problem described here, together with many other instructive examples of chaotic systems, try the program *Chaos Demonstrations* by J C Sprott and G Rowlands, available from Physics Academic Software, http://www.aip.org/pas/.